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ON ABSOLUTE NÖRLUND SUMMABILITY OF FOURIER SERIES

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Abstract. We obtain two theorems on the absolute Nörlund summability of Fourier series and factored Fourier series.

In 1990 Sulaiman [3] obtained sufficient conditions for a Fourier series and the derived Fourier seriese to be absolutely Nörlund summable of order $k \geq 1$. However, he used an incorrect definition of absolute summability. (See, e.g., [2].)

In this paper we obtain the comparable results of [3] using the correct definition.

Let A be a lower triangular matrix, $\sum a_k$ a series with partial sums s_n,

$$T_n := \sum_{k=0}^{n} a_{nk} s_k.$$

Then the series $\sum a_k$ is absolutely summable A of order k, written $|A|_k$, if

$$\sum_{n=1}^{\infty} n^{k-1} |T_n - T_{n-1}|^k < \infty.$$

Let f be a 2π-periodic function, Lebesgue integrable over $(-\pi, \pi)$.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) := \sum_{n=0}^{\infty} A_n(x). \tag{1}$$

The derived series for f is denoted by

$$\sum_{n=1}^{\infty} nB_n(x) := \sum_{n=1}^{\infty} n(b_n \cos nx - a_n \sin nx). \tag{2}$$

We shall write

$$\phi(u) = f(x+u) + f(x-u) - 2f(x)$$
$$\psi(u) = f(x+u) - f(x-u) - 2f'(x)$$
$$\phi_1(u) = \frac{1}{t}\int_0^t \phi(u)du = \frac{1}{t}\Phi(t)$$
$$\psi_1(t) = \frac{1}{t}\int_0^t d\psi(u)du = \frac{1}{t}\psi_2(t).$$

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For any sequence $\{u_n\}$, $\Delta u_n := u_n - u_{n+1}$ and $\nabla u_n := u_n - u_{n-1}$. Let h denote a positive function such that, for some $\beta, 0 < \beta < 1$, $u\beta h(u^{-1})$ is nondecreasing.

A Nörlund matrix is a lower trianglular matrix with entries p_{n-k}/P_n, where $\{p_k\}$ is a positive sequence and $P_n := \sum_{k=0}^{n} p_k$. The condition $\lim p_n/P_n = 0$ is necessary and sufficient for (N, p_n) to be regular.

We shall prove the following two theorems.

Theorem 1. *Let $\{p_n\}$ be a positive sequence such that $\{\nabla p_n\}$ is bounded, nonincreasing, $\{1/p_n^{k-1}P_n\}$ is nonincreasing, and*

$$\frac{np_n}{P_n} = O(1). \tag{3}$$

If

$$\Phi_1(t) := \int_t^{\delta} u^{-1}\phi_1(u)du = O(h(t^{-1})), \tag{4}$$

and

$$\sum \frac{n^k[h(n)]^k}{p_n^{1-k}P_n} < \infty, \tag{5}$$

then the series (1) *is summable* $|N, p_n|_k$, $k \geq 1$.

Theorem 2. *Let $\{p_n\}$ satisfy the conditions of Theorem* 1*. If*

$$\Psi(t) := \int_t^{\delta} u^{-1}\Psi_1(u)du = O(h(t^{-1})) \tag{6}$$

then the series (2) *is summable* $|N, p_n|_k$, $k \geq 1$, *provided* (4) *exists.*

The proof of Theorem 1 requires the following lemma.

Lemma 1. *Let $s_n^{(1)} := \sum_{k=0}^{n} s_k$. If $\{p_n\}$ is a sequence satisfying* (3), *$\{\nabla p_n\}$ and $\{1/(p_n^{k-1}P_n)\}$ are nonincreasing, and*

$$\sum_{n=0}^{\infty} \frac{|s_n^{(1)}|^k}{p_n^{k-1}P_n} < \infty,$$

then the series $\sum a_k$ is summable $|N, p_n|_k$, $k \geq 1$.

Proof. As in [3] we may write

$$\begin{aligned} t_{n-1} - t_n &= \Delta\left(\frac{1}{P_{n-1}}\right)\sum_{v=0}^{n-2}\nabla_n(p_{n-1-v})s_v^{(1)} + \Delta\left(\frac{1}{P_{n-1}}\right)p_0 s_{n-1}^{(1)} \\ &\quad + \frac{1}{P_n}\sum_{v=0}^{n-2}\nabla_n(p_{n-1-v} - p_{n-v})s_v^{(1)} + \frac{1}{P_n}(p_0 - p_1)s_{n-1}^{(1)} - \frac{p_0 s_n^{(1)}}{P_n} + \frac{p_0 s_{n-1}^{(1)}}{P_n} \\ &= T_1 + T_2 + T_3 + T_4 + T_5 + T_6, \qquad \text{say.} \end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k-1}|T_1|^k &= \sum_{n=1}^{\infty} n^{k-1}\left|\frac{p_n}{P_nP_{n-1}}\sum_{v=0}^{n-2}\nabla_n(p_{n-1-v})s_v^{(1)}\right|^k \\ &\leq \sum_{n=1}^{\infty} n^{k-1}\frac{p_n^k}{P_n^kP_{n-1}}\sum_{v=0}^{n-2}p_v^{1-k}|\nabla_n p_{n-1-v}|^k|s_v^{(1)}|^k\left(\frac{1}{P_{n-1}}\sum_{v=0}^{n-2}p_v\right)^{k-1} \\ &= O(1)\sum_{n=1}^{\infty}\frac{n^{k-1}p_n^k}{P_n^kP_{n-1}}\sum_{v=0}^{n-2}p_v^{1-k}|s_v^{(1)}|^k \\ &= O(1)\sum_{v=0}^{\infty}p_v^{1-k}|s_v^{(1)}|^k\sum_{n=v+1}^{\infty}\frac{n^{k-1}p_n^k}{P_n^kP_{n-1}}.\end{aligned}$$

Using (3),

$$\sum_{n=v+1}^{\infty}\frac{n^{k-1}p_n^k}{P_n^kP_{n-1}} = O\left(\frac{1}{P_v}\right).$$

Thus

$$\sum_{n=1}^{\infty} n^{k-1}|T_1|^k = O(1)\sum_{v=0}^{\infty}p_v^{1-k}|s_v^{(1)}|^k\frac{1}{P_v} = O(1).$$

Using (3),

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k-1}|T_2|^k &= \sum_{n=1}^{\infty} n^{k-1}\left|\Delta\left(\frac{1}{P_{n-1}}\right)p_0s_{n-1}^{(1)}\right|^k \\ &= \sum_{n=1}^{\infty} n^{k-1}\left(\frac{p_n}{P_nP_{n-1}}\right)^k p_o^k|s_{n-1}^{(1)}|^k \\ &= O(1)\sum_{n=1}^{\infty}\left(\frac{np_n}{P_n}\right)^{k-1}\frac{p_n}{P_nP_{n-1}^k}|s_{n-1}^{(1)}|^k \\ &= O(1)\sum_{n=1}^{\infty}\frac{1}{P_{n-1}^k}|s_{n-1}^{(1)}|^k \\ &= O(1)\sum_{n=1}^{\infty}\left(\frac{p_{n-2}}{P_{n-2}}\right)^{k-1}\left(\frac{P_{n-2}}{P_{n-1}}\right)^{k-1}\frac{|s_{n-1}^{(1)}|^k}{p_{n-2}^{k-1}P_{n-1}} \\ &= O(1)\sum_{n=1}^{\infty}\frac{|s_{n-1}^{(1)}|^k}{p_{n-2}^{k-1}P_{n-1}} = O(1).\end{aligned}$$

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k-1}|T_3|^k &= \sum_{n=1}^{\infty} n^{k-1}\left|\frac{1}{P_n}\sum_{v=0}^{n-2}\nabla_n(p_{n-1-v}-p_{n-v})s_v^{(1)}\right|^k \\ &\leq \sum_{n=1}^{\infty}\frac{n^{k-1}}{P_n^k}\sum_{v=0}^{n-2}|\nabla_n(p_{n-1-v}-p_{n-v})||s_v^{(1)}|^k\left(\sum_{v=0}^{n-2}|\nabla_n(p_{n-1-v}-p_{n-v})|\right)^{k-1}\end{aligned}$$

$$= O(1)\sum_{n=1}^{\infty}\frac{n^{k-1}}{P_n^k}\sum_{v=0}^{n-2}|\nabla_n(p_{n-1-v}-p_{n-v})||s_v^{(1)}|^k$$

$$= O(1)\sum_{b=0}^{\infty}|s_v^{(1)}|^k\sum_{v=1}^{\infty}\frac{n^{k-1}}{P_n^k}(\nabla_n(p_{n-1-v}-p_{n-v})).$$

Using (3),

$$\frac{n^{k-1}}{P_n^k} = \left(\frac{np_n}{P_n}\right)^{k-1}\frac{1}{p_n^{k-1}P_n} = \frac{O(1)}{p_n^{k-1}P_n}.$$

Therefore

$$\sum_{n=1}^{\infty}n^{k-1}|T_3|^k = O(1)\sum_{v=0}^{\infty}|s_v^{(1)}|^k\sum_{n=v+1}^{\infty}\frac{1}{p_n^{k-1}P_n}\nabla_n(p_{n-1-v}-p_{n-v})$$

$$= O(1)\sum_{v=0}^{\infty}\frac{|s_v^{(1)}|^k}{p_{v+1}^{k-1}P_{v+1}}\sum_{n=v+1}^{\infty}\nabla_n(p_{n-1-v}-p_{n-v})$$

$$= O(1)\sum_{v=0}^{\infty}\frac{|s_v^{(1)}|^k}{p_v^{k-1}P_v} = O(1).$$

$$\sum_{n=1}^{\infty}n^{k-1}|T_4|^k = \sum_{n=1}^{\infty}n^{k-1}\left|\frac{1}{P_n}(p_0-p_1)s_{n-1}^{(1)}\right|^k$$

$$= O(1)\sum_{n=1}^{\infty}\frac{n^{k-1}}{P_n^k}|s_{n-1}^{(1)}|^k$$

$$= O(1)\sum_{n=1}^{\infty}\frac{|s_{n-1}^{(1)}|^k}{p_n^{k-1}P_n}$$

$$= O(1)\sum_{n=1}^{\infty}\frac{|s_{n-1}^{(1)}|^k}{p_{n-1}^{k-1}P_{n-1}} = O(1).$$

$$\sum_{n=1}^{\infty}n^{k-1}|T_5|^k = \sum_{n=1}^{\infty}n^{k-1}\left|\frac{p_0s_n^{(1)}}{P_n}\right|^k$$

$$= O(1)\sum_{n=1}^{\infty}\frac{n^{k-1}|s_n^{(1)}|^k}{P_n^k}$$

$$= O(1)\sum_{n=1}^{\infty}\frac{|s_n^{(1)}|^k}{p_n^{k-1}P_n} = O(1).$$

$$\sum_{n=1}^{\infty}n^{k-1}|T_6|^k = \sum_{n=1}^{\infty}n^{k-1}\left|\frac{p_0s_{n-1}^{(1)}}{P_n}\right|^k$$

$$= O(1)\sum_{n=1}^{\infty}\frac{n^{k-1}|s_{n-1}^{(1)}|^k}{P_n^k}$$
$$= O(1)\sum_{n=1}^{\infty}\frac{|s_{n-1}^{(1)}|^k}{p_n^{k-1}P_n}$$
$$= O(1)\sum_{n=1}^{\infty}\frac{|s_{n-1}^{(1)}|^k}{p_{n-1}^{k-1}P_{n-1}} = O(1).$$

Proof of Theorem 1.

$$\sum_{v=0}^{n}[s_v(x)-f(x)] = \frac{2}{\pi}\int_0^{\pi}\{t^{-1}\phi_1(t)+\phi_1'(t)\}\sum_{v=0}^{n}\sin vt dt + o(n)$$
$$= \frac{2}{\pi}\{I_1+I_2\}, \qquad \text{say.}$$

$$I_1 = \int_0^{1/n} + \int_{1/n}^{\pi} = I_{11}+I_{12}, \qquad \text{say.}$$
$$I_{11} = \int_0^{1/n} t^{-1}\phi_1(t)\sum_{v=0}^{n}\sin vt dt$$
$$= O(n^2)\int_0^{1/n} t^{-1}\phi_1(t)dt = O(nh(n)),$$

since, as shown in [3],

$$\Phi_1(t) = O\{h(t^{-1})\} \quad \text{implies that} \quad \int_0^t \phi_1(u)du = O\{th(t^{-1})\}.$$

$$I_{12} = O(n)\int_{1/n}^{\pi} t^{-1}\phi_1(t)dt = O(nh(n)).$$
$$I_2 = \int_0^{\pi}\phi_1'(t)\sum_{v=0}^{n}\sin vt dt$$
$$= -\left(\int_0^{1/n}+\int_{1/n}^{\pi}\right)\phi_1(t)\sum_{v=0}^{n} v\cos vt dt$$
$$= -(I_{21}+I_{22}), \qquad \text{say.}$$
$$I_{21} = O(n^2)\int_0^{1/n}\phi_1(t)dt = O(nh(n)).$$
$$I_{22} = O(1)\int_{1/n}^{\pi}\phi_1(t)\left|\sum_{v=0}^{n} v\cos vt\right| dt$$

$$= O(n)\int_{1/n}^{\pi}\phi_1(t)\max_{0\le r\le 1}\left|\sum_{v=0}^{n}\cos vt\right|dt$$
$$= O(n)\int_{1/n}^{\pi}t^{-1}\phi_1(t)dt = O(nh(n)).$$

Therefore

$$\sum_{v=0}^{n}[s_v(x)-f(x)] = O\{nh(n)\}.$$

$$\left|\sum_{v=0}^{n}s_v(x)\right| \le \left|\sum_{v=0}^{n}[s_v(x)-f(x)]\right| + \left|\sum_{v=0}^{n}f(x)\right|$$
$$= O\{nh(n)\}.$$

The proof of Theorem 2 is identical with that of Theorem 2 in [3], and so will be omitted.

Corollary 1. *If*

$$\sum_{n=0}^{\infty}\frac{(nh(n))^k}{n} = O(1)$$

and

$$\Phi_1(t) \ := \int_t^{\delta}u^{-1}\phi_1(u)du = O\{h(t^{-1})\},$$

then $\sum A_n(x)$ *is summable* $|C,1|_k, k\ge 1$.

Proof. with each $p_n = 1$, all of the conditions of Theorem 1 are satisfied.

Corollary 2. *If* (1.6) *and* (1.4) *hold, then the conjugate series is summable* $|C,1|_k$, $k\ge 1$.

References

[1] G. S. Pandey, *Multipliers for $|C,1|$ summability of Fourier seris,* Pacific J. Math. **54**(1978), 177-182.

[2] B. E. Rhoades, *Inclusion theorems for absolute matrix summability,* J. Math. Anal. Appl. **238**(1999), 82-90.

[3] W. T. Sulaiman, *On absolute Nörlund summability of Fourier series and the derived Fourier series,* Tamkang J. Math. **21**(1990), 79-87.

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