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ON THE DEGREE OF APPROXIMATION OF THE CONJUGATE OF A FUNCTION BELONGING TO THE WEIGHTED $W(L^p, \xi(t))$ CLASS BY MATRIX MEANS OF THE CONJUGATE SERIES OF A FOURIER SERIES

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Abstract. In a recent paper Lal [1] obtained a theorem on the degree of approximation of the conjugate of a function belonging to the weighted $W(L^p, \xi(t))$ class using a triangular matrix transform of the conjugate series of the Fourier series representation of the function. The matrix involved was assumed to have monotone increasing rows. We establish the same result by removing the monotonicity conditon.

Let f be a 2π periodic Lebesgue integrable function, with Fourier series given by

$$f(x) \approx \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx). \tag{1}$$

The conjugate series of the series in (1) is given by

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx). \tag{2}$$

Let A be a lower triangular matrix with finite norm. The L^p norm of a function is defined by

$$||f||_p = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}, \qquad p \geq 1.$$

The degree of approximation is given by

$$E_n(f) = \min_{T_n} \|f - T_n\|_p,$$

where $T_n(x)$ denotes some nth degree trigonometric polynomial.

A function is said to be of class Lip α if

$$f(x+t) - f(x) = O(|t|^\alpha) \qquad \text{for} \qquad 0 < \alpha \leq 1,$$

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and $f \in \mathrm{Lip}(\alpha, p)$ if

$$\left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(t^\alpha), \qquad 0 < \alpha \leq 1,\, p \geq 1.$$

Given a positive increasing function $\xi(t)$ and a number $p > 1$,

$$f(x) \in \mathrm{Lip}(\xi(t), p)) \quad \text{if} \quad \left(\int_0^{2\pi} |f(x+t) - f(x)|^p dx\right)^{1/p} = O(\xi(t))$$

and that $f(x) \in W(L^p, \xi(t))$ if

$$\int_0^{2\pi} |f(x+t) - f(x)|^p sin^{\beta^p} dx^{1/p} = O(\xi(t)),\ (\beta \geq 0).$$

If $\beta = 0$, then the class $W(L^p, \xi(t))$ reduces to the class $\mathrm{Lip}(\xi(t), p)$.
We shall define

$$\Psi(t) = f(x+t) - f(x-t)$$

and

$$\bar{K}_n(t) = \frac{1}{2\pi} \sum_{k=0}^{n} a_{nk} \frac{\cos\left(k + \frac{1}{2}\right) t}{\sin\frac{t}{2}}.$$

Our main result is the following.

Theorem 1. *Let A be a lower triangular matrix with finite norm. Then the degee of aproximation of a function $\bar{f}(x)$, conjugate to a 2π periodic function f belonging to the weighted $W(L^p, \xi(t))$ class by the A means of its conjugate series, is given by*

$$||\bar{t}_n(x) - \bar{f}(x)||_p = O\left(\xi\left(\frac{1}{n}\right) n^{\beta + \frac{1}{p}}\right)$$

provided $\xi(t)$ satisfies the following conditions:

$$\left(\int_0^{1/n} \left(\frac{t|\psi(t)|}{\xi(t)}\right)^p \sin^{\beta p} t dt\right)^{1/p} = O\left(\frac{1}{n}\right) \tag{3}$$

$$\left(\int_{1/n}^{\pi} \left(\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right)^p dt\right)^{1/p} = O(n^\delta) \tag{4}$$

where δ is an arbitrary number such that $q(1-\delta) - 1 > 0$, conditions (3) *and* (4) *hold uniformly in x,*

$$\bar{t}_n(x) = \sum_{k=0}^{n} a_{nk} \bar{s}_k;$$

i.e., the matrix means of the conjugate series of the Fourier series (2), *q is the conjugate index of p, and*

$$\bar{f}(x) = -\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}tdt.$$

For the proof of our theorem we will require the following lemma.

Lemma 1. *Let A be a lower triangular matrix with finite norm. Then*

$$\left|\sum_{k=0}^{n} a_{nk}\cos\left(k+\frac{1}{2}\right)t\right| = O\left\{\frac{1}{t}\left[\sum_{k=0}^{n-1}|\Delta_k a_{nk}| + |a_{nn}|\right]\right\}.$$

Proof. Using summation by parts we may write

$$\sum_{k=0}^{n} a_{nk}\cos\left(k+\frac{1}{2}\right)t = a_{nn}\sum_{k=0}^{n}\cos\left(k+\frac{1}{2}\right)t + \sum_{k=0}^{n-1}\Delta_k a_{nk}\sum_{i=0}^{k}\cos\left(i+\frac{1}{2}\right)t.$$

Using the well known identity

$$\sum_{k=0}^{n}\cos\left(k+\frac{1}{2}\right)t = \frac{\sin(n+1)t}{2\sin\frac{t}{2}},$$

and the fact that

$$\frac{\sin x}{x} \geq \frac{2\sqrt{2}}{\pi} \qquad \text{for} \qquad 0 < x \leq \frac{\pi}{4},$$

$$\left|\sum_{k=0}^{n} a_{nk}\cos\left(k+\frac{1}{2}\right)t\right| \leq \frac{\pi}{2\sqrt{2}}\left\{|a_{nn}| + \sum_{k=0}^{n-1}|\Delta_k a_{nk}|\right\}.$$

Proof of Theorem 1. With $\bar{S}_n(x)$ denoting the nth partial sum of the series (2),

$$\bar{S}_n(x) - \left(-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}tdt\right) = \frac{1}{2\pi}\int_0^\pi \psi(t)\frac{\cos(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt.$$

$$\sum_{k=0}^{n} a_{nk}\left[\bar{S}_k - \Big(-\frac{1}{2\pi}\int_0^\pi \psi(t)\cot\frac{1}{2}tdt\Big)\right] = \frac{1}{2\pi}\int_0^\pi \psi(t)\sum_{k=0}^{n} a_{nk}\frac{\cos\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}dt$$

or

$$\begin{aligned}\bar{t}_n - \bar{f}(x) &= \frac{1}{2\pi}\int_0^\pi \psi(t)\sum_{k=0}^{n} a_{nk}\frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}dt \\ &= \int_0^\pi \psi(t)\bar{K}_n(t)dt \\ &= \int_0^{1/n} \psi(t)\bar{K}_n(t)dt + \int_{1/n}^{\pi} \psi(t)\bar{K}_n(t)dt \\ &= I_1 + I_2, \qquad \text{say.} \end{aligned} \tag{5}$$

Applying Hölder's inequality and the fact that $\psi(t) \in W(L^p, \xi(t))$, we get

$$\begin{aligned}|I_1| &\le \int_0^{1/n} |\psi(t)||\bar{K}_n(t)|dt \\ &\le \int_0^{1/n} |\psi(t)|\frac{1}{t}\sum_{k=0}^{n}|a_{nk}|dt \\ &= O(1)\left[\int_0^{1/n}\left(\frac{t\psi(t)}{\xi(t)}\sin^{\beta t}\right)^p dt\right]^{1/p}\left[\int_0^{1/n}\left(\frac{\xi(t)}{t^2\sin^{\beta t}}\right)^q dt\right]^{1/q} \\ &= O\left(\frac{1}{n}\right)\left[\int_0^{1/n}\left(\frac{\xi(t)}{t^{2+\beta}}\right)^q dt\right]^{1/q} \\ &= O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{1}{n}\right)\right)\left[\int_0^{1/n}\frac{dt}{t^{q(\beta+2)}}\right]^{1/q} \\ &= O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{1}{n}\right)\right)\left[\frac{t^{-q(\beta+2)+1}}{-q(\beta+2)+1}\Big|_0^{1/n}\right]^{1/q} \\ &= O\left(\frac{1}{n}\right)O\left(\xi\left(\frac{1}{n}\right)\right)O((n^{q(\beta+2)-1})^{1/q}) \\ &= O\left(\frac{\xi(1/n)}{n}n^{\beta+2-1/q}\right) \\ &= O(\xi(1/n)n^{\beta+1-1/q}) = O(\xi(1/n)n^{\beta+1/p}). \end{aligned} \tag{6}$$

Using Hölder's inequality and Lemma 1 we have

$$\begin{aligned}|I_2| &\le \int_{1/n}^{\pi} |\psi(t)||\bar{K}_n(t)|dt \\ &O\ (1)\int_{1/n}^{\pi}|\psi(t)|\left|a_{nn}\frac{\sin(n+1)t}{2\sin\frac{t}{2}} + \sum_{k=0}^{n-1}\Delta_k a_{nk}\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right|dt \end{aligned}$$

$$O\ (1)\int_{1/n}^{\pi}|\psi(t)|\frac{1}{t}\left[|a_{nn}|+\sum_{k=0}^{n-1}|\Delta_k a_{nk}|\right]dt$$
$$=O(1)\left(\int_{1/n}^{\pi}\left(\frac{t^{-\delta}\sin^{\beta}t\psi(t)}{\xi(t)}\right)^{p}dt\right)^{1/p}\left(\int_{1/n}^{\pi}\left(\frac{\xi(t)}{t^{1-\delta}\sin^{\beta}t}\right)^{q}dt\right)^{1/q}$$
$$=O(n^{\delta})\left(\int_{n}^{1/\pi}\left(\frac{\xi(1/y)}{(1/y)^{\beta+1-\delta}}\right)^{q}\left(-\frac{dy}{y^2}\right)\right)^{1/q}$$
$$=O(n^{\delta})\left(\int_{1/\pi}^{n}(\xi(1/y))^{q}y^{q(\beta+1-\delta)-2}dy\right)^{1/q}$$
$$O\ (n^{\delta}\xi(1/n))\left(\int_{1/\pi}^{n}y^{q(\beta+1-\delta)-2}dy\right)^{1/q}$$
$$=O(n^{\delta}\xi(1/n))\left(\frac{y^{q(\beta+1-\delta)-1}}{q(\beta+1-\delta)-1}\Big|_{1/\pi}^{n}\right)^{1/q}$$
$$O\ (n^{\delta}\xi(1/n))(n^{q(\beta+1-\delta)-1})^{1/q})$$
$$=O(\xi(1/n))(n^{\beta+1-1/q})=O(\xi(1/n))(n^{\beta+1/p}). \qquad (7)$$

Combining (6) and (7) finishes the proof.

Corollary 2. *If $\beta=0$ and $\xi(t)=t^{\alpha}, 0<\alpha\leq 1$, then the degree of approximation of a function $\bar{f}(x)$, conjugate to a 2π periodic function f belonging to the class $Lip(\alpha,p)$ is given by*

$$|\bar{t}_n(x)-\bar{f}(x)|=O\left(\frac{1}{n^{\alpha-1/p}}\right).$$

Proof.

$$\|\bar{t}_n(x)-\bar{f}(x)\|_p=\left(\int_0^{2\pi}|\bar{t}_n(x)-\bar{f}(x)|^p dx\right)^{1/p}$$

or,

$$O\left(\xi\left(\frac{1}{n}\right)n^{\beta+1/p}\right)=\left(\int_0^{2\pi}|\bar{t}_n(x)-\bar{f}(x)|^p dx\right)^{1/p}$$

or,

$$O(1)=\left(\int_0^{2\pi}|\bar{t}_n(x)-\bar{f}(x)|^p dx\right)^{1/p}O\left(\frac{1}{\xi(1/n)n^{\beta+1/p}}\right),$$

since otherwise the right hand side of the above equation will not be $O(1)$.

Hence

$$|\bar{t}_n(x)-\bar{f}(x)|=O\left(\left(\frac{1}{n}\right)^{\alpha}n^{1/p}\right)=O\left(\frac{1}{n^{\alpha-1/p}}\right).$$

Corollary 3. *If $p \to \infty$ in Corollary 1, then, for $0 < \alpha < 1$,*

$$||\bar{t}_n(x) - \bar{f}(x)|| = O\left(\frac{1}{n^\alpha}\right).$$

The results of [1] are all special cases of the results of this paper.

References

[1] Shyam Lal, *On the degreee of approximation of conjugate of a function belonging to weighted $W(L^p, \xi(t))$ class by matrix summability means of conjugate series of a Fourier series*, Tamkang J. Math. **31**(2000), 279-288.

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