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RINGS WITH GENERALIZED COMMUTATORS IN THE NUCLEI

Dedicated to my father on his 85th birthday

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Abstract. Let R be a prime weakly Novikov ring and $T_k = \underbrace{[[[\ldots[[R,R],R]\ldots,R],R],R]}_{kR's}$ where k is a positive integer. We prove that if $T_k \subseteq N_l \cap N_r$ or $T_k \subseteq N_m \cap N_r$ then R is associative or $T_k = 0$. Moreover, if T_k is contained in two of the three nuclei, and $k = 2$ or $k = 3$ then the same conclusions hold. We also consider such rings with derivations. Some similar results of weakly M-rings are obtained.

1. Introduction

Let R be a nonassociative ring. We shall denote the associator and commutator by $(x,y,z) = (xy)z - x(yz)$ and $[x,y] = xy - yx$ for all x,y,z in R respectively. In any ring R, one has the following nuclei:

$$\begin{aligned} N_l &= \{n \in R | (n,R,R) = 0\} \ - \ \text{left nucleus,} \\ N_m &= \{n \in R | (R,n,R) = 0\} \ - \ \text{middle nucleus,} \\ N_r &= \{n \in R | (R,R,n) = 0\} \ - \ \text{right nucleus.} \\ N &= N_l \cap N_m \cap N_r \ - \ \text{nucleus.} \end{aligned}$$

A ring R is called simple if R is the only nonzero ideal of R. Thus, $R^2 = R$.
A ring R is called semiprime if the only ideal of R which squares to zero is the zero ideal.
A ring R is called prime if the product of any two nonzero ideals of R is nonzero. Note that each associator and commutator are linear in each argument. Thus N_l, N_m and N_r are additive subgroups of $(R,+)$. If S is a nonempty subset of a ring R, then the ideal of R generated by S is $< S >$. A ring R is called weakly Novikov [4] if R satisfies the following identity.

$$(w,x,yz) = y(w,x,z) \quad \text{for all } w,x,y,z \text{ in } \ R. \tag{1}$$

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An additive mapping d on a ring R is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all x, y in R. For any ring R, let $T_k = \underbrace{[[[\ldots[[R,R],R]\ldots,R],R],R]}_{kR's}$ where k is a positive integer. Note that $T_2 = [R,R]$ and $T_3 = [[R,R],R]$. We also note that $[R,T_k] = [T_k,R] \subseteq T_k$, where k is a positive integer. Obviously, we have the following identities.

$$T_k + T_kR = T_k + RT_k \quad \text{for all positive integers } k. \tag{2}$$

$$d(R) + d(R)R = d(R) + Rd(R). \tag{3}$$

$$d((x,y,z)) = (d(x),y,z) + (x,d(y),z) + (x,y,d(z)) \quad \text{for all } x,y,z \text{ in } R. \tag{4}$$

$$S(x,y,z) = (x,y,z)+(y,z,x)+(z,x,y)=[xy,z]+[yz,x]+[zx,y]$$
$$\text{for all } x,y,z \text{ in } R. \tag{5}$$

We shall use the Teichmüller identity

$$(wx,y,z) - (w,xy,z) + (w,x,yz) = w(x,y,z) + (w,x,y)z \quad \text{for all } w,x,y,z \text{ in } R, \tag{6}$$

which is valid in every ring.
As a consequence of (6), we have that N_l, N_m and N_r are associative subrings of R. Suppose that $n \in N_l$ Then with $w = n$ in (6) we obtain

$$(nx,y,z) = n(x,y,z) \quad \text{for all } x,y,z \text{ in } R \text{ and } n \text{ in } N_l. \tag{7}$$

Suppose that $m \in N_r$. Then with $z = m$ in (6) we get

$$(w,x,ym) = (w,x,y)m \quad \text{for all } w,x,y \text{ in } R \text{ and } m \text{ in } N_r. \tag{8}$$

Suppose that $j \in N_l \cap N_m$. Then with $x = j$ in (6) we have

$$(wj,y,z) = (w,jy,z) \quad \text{for all } w,y,z \text{ in } R \text{ and } j \text{ in } N_l \cap N_m. \tag{9}$$

Definition 1. Let A be the associator ideal of a ring R.

Ordinary by (6) A can by characterized as all finite sums of associators and left multiples of associators. In view of (1) it suffices to take all finite sums of associators if R is a weakly Novikov ring. Hence, in this case $A = (R,R,R)$. In the paper, we consider rings with generalized commutators in the nuclei. There had been other results concerning rings in which $[R,R] \subseteq N_l$. For example Thedy [5], Kleinfeld [1], Kleinfeld and Kleinfeld [2] as well as Kleinfeld and Smith [3].

Definition 2. For any ring R, let $V_k = T_k + RT_k$ for all positive integers k.

2. Results of Weakly Novikov Rings

Lemma 1. *If R is a weakly Novikov ring, then $RN_r \subseteq N_r$ and $A \cdot N_r = (R,R,R) \cdot N_r = 0$.*

Proof. Let $z \in N_r$ and $w,x,y \in R$. Then by (8) and (1), we have $(w,x,y)z = (w,x,yz) = y(w,x,z) = 0$. Thus, we get $A \cdot N_r = (R,R,R) \cdot N_r = 0$ and $RN_r \subseteq N_r$, as desired.

By (2) and the result of [8], we have the

Lemma 2. *If R is a ring such that T_k is contained in two of the three nuclei, then V_k is an ideal of R for every positive integer k.*
In the sequel, for the convenience we denote T_k and V_k by T and V respectively.

Theorem 1. *If R is a prime weakly Novikov ring such that $T \subseteq N_l \cap N_r$ or $T \subseteq N_m \cap N_r$, then R is associative or $T = 0$.*

Proof. Using $T \subseteq N_r$ and Lemma 1, we get

$$A \cdot V = A \cdot (T + RT) = 0. \tag{10}$$

By Lemma 2 and the primeness of R, (10) implies $A = 0$ or $V = 0$. Thus, R is associative or $T = 0$.

Lemma 3. *If R is a weakly Novikov ring such that $T \subseteq N_l \cap N_m$, then*

$$(R,R,T)R = 0 \tag{11}$$

Proof. Note that $[R,T] = [T,R] \subseteq T$. Using this, the hypotheses, (6),(1),(9) and (7), for all $y \in T$, and $w,x,z \in R$ we have $(w,x,y)z = w(x,y,z) + (w,x,y)z = (wx,y,z) - (w,xy,z) + (w,x,yz) = -(w,[x,y],z) - (w,yx,z) + y(w,x,z) = -(wy,x,z) + y(w,x,z) = -([w,y],x,z) - (yw,x,z) + y(w,x,z) = 0$, Hence, we get $(R,R,T)R = 0$,as desired.

Theorem 2. *Let R be a prime weakly Novikov ring such that $T \subseteq N_l \cap N_m$. If $S(x,y,z) \in N_m$ for all x,y,z in R, or $[T,(R,R,R)] = 0$, then R is associative or $T = 0$.*

Proof. Assume that $S(x,y,z) \in N_m$ for all x,y,z in R. Using this, (5) and the hypotheses, for all $x \in T$ and $y,z \in R$ we get $(y,z,x) = (x,y,z) + (y,z,x) + (z,x,y) = S(x,y,z) \in N_m$. Thus, $(R,R,T) \subseteq N_m$. Applying this, (1) and (11), we have $(R,R,RT)R = R(R,R,T) \cdot R = R \cdot (R,R,T)R = 0$.
Combining this with (11) results in

$$(R,R,V)R = 0. \tag{12}$$

Assume that $[T,(R,R,R)] = 0$. Using this, (1), (11) and (6), and noting that $[T,R] \subseteq T$, for all $w,x,y,t \in R$, and $z \in T$ we have $(w,x,y)z \cdot t = z(w,x,y) \cdot t = (w,x,zy)t =$

$(w,x,[z,y])t+(w,x,yz)t = w(x,y,z)\cdot t+(w,x,y)z\cdot t+(w,xy,z)t-(wx,y,z)t = w(x,y,z)\cdot t+(w,x,y)z\cdot t$ and so $(x,y,wz)t = w(x,y,z)\cdot t = 0$. Combining this with (11), we also obtain (12).

Using (1) and (12), we see that $<(R,R,T)> = (R,R,V)$. By the semiprimeness of R, (12), implies $(R,R,V) = 0$. Hence, $V \subseteq N_r$
Consequently, $T \subseteq N$. By Theorem 1, R is associative or $T = 0$.

In [3], Kleinfeld and Smith had proved that if R is a prime left alternative ring with $[R,R] \subseteq N_l$ and characteristic$\neq$ 2,3 then R is associative. A linearization of the left alternative identity shows that $N_l = N_m$. We have the similar result for the weakly Novikov ring case.

Theorem 3. *If R is a prime weakly Novikov ring such that $[R,R]$ is contained in two of the three nuclei, then R is associative or commutative.*
In the latter case, $N_r = 0$ or R is associative.

Proof. In view of Theorem 1, we may assume $[R,R] \subseteq N_l \cap N_m$. Let $B = [R,R] + R[R,R]$. By Lemma 2, B is an ideal of R. Using Lemma 3,
we get

$$(R,R,[R,R])R = 0. \tag{13}$$

Applying (5) and $[R,R] \subseteq N_l \cap N_m$, for all $x,y,z \in R$ we have $S(x,y,z) = (x,y,z) + (y,z,x) + (z,x,y) \in N_l \cap N_m$. Let $x \in [R,R]$. Then we get $(y,z,x) \in N_l \cap N_m$. Thus, we obtain $(R,R,[R,R]) \subseteq N_l \cap N_m$. Using this and (13), we have $R(R,R[R,R])\cdot R = R\cdot(R,R,[R,R])R = 0$.
Hence, applying this, (1) and (13), and noting that B is an ideal of R, we obtain that $(R,R,B)\cdot R = 0$ and $<(R,R,[R,R])> = (R,R,B)$. Thus, by the semiprimeness of R we get $(R,R,B) = 0$ and so $[R,R] \subseteq N_r$. By Theorem 1, R is associative or commutative.

Assume that R is commutative. Thus we have $N_rR = RN_r \subseteq N_r$ and $A\cdot N_r = 0$ by Lemma 1. Hence N_r is an ideal of R. By the primeness of R,$A\cdot N_r = 0$ implies $A = 0$ or $N_r = 0$.

By Theorem 3, we obtain the

Corollary 1. *If R is a prime weakly Novikov ring such that $[R,R]$ is contained in two of the three nuclei with $N_r \neq 0$ or $[R,R] \neq 0$, then R is associative, that is $N_r = R$.*

In the sequel, for the convenience we denote V_3 by D.

Lemma 4. *If R is a weakly Novikov ring such that $[[R,R],R] \subseteq N_l \cap N_m$ then $<(R,R,D)>\cdot(R,R,R) = 0$, where $<(R,R,D)> = (R,R,D) + (R,R,D)R + R\cdot(R,R,D)R$.*

Proof. Let $D = [[R,R],R] + R[[R,R],R]$ and $[[R,R],R] \subseteq N_l \cap N_m$. By Lemma 3, we obtain

$$(R,R,[[R,R],R])R = 0. \tag{14}$$

Thus (14) implies

$$(R,R,[[R,R],R]) \subseteq N_l. \tag{15}$$

Assume that $y \in [[R,R],R]$ and $w,x,z,u,v,t \in R$. Using (14), the hypotheses and (5) we have $z(w,x,y) = [z,(w,x,y)] = [z,S(w,x,y)] \in [[R,R],R] \subseteq N_l \cap N_m$ and so by (1) twice we get $(w,x,[z,y]) + y(w,x,z) = (w,x,[z,y]) + (w,x,yz) = (w,x,zy) = z(w,x,y) \in N_l \cap N_m$. Applying these, (1) and (15) we obtain the following two inclusions.

$$(R,R,R[[R,R],R]) = R(R,R,[[R,R],R]) \subseteq N_l \cap N_m. \tag{16}$$

$$[[R,R],R]A = [[R,R],R](R,R,R) \subseteq N_l. \tag{17}$$

Then (17) implies

$$[[R,R],R]A \cdot R = [[R,R],R] \cdot AR \subseteq [[R,R],R]A \subseteq N_l. \tag{18}$$

Combined (15) with (16) results in

$$(R,R,D) \subseteq N_l. \tag{19}$$

Using (1), (17), (7) and (18), we have $(w,x,yz)(u,v,t) = y(w,x,z) \cdot (u,v,t) = (y(w,x,z) \cdot u,v,t) = 0$. Hence applying this, (2) and (14) we obtain

$$(R,R,D)A = (R,R,D)(R,R,R) = 0. \tag{20}$$

Then by (20), (19), (7) and (1) we get $0 = (R,R,D)(R,R,R) = ((R,R,D)R,R,R)$ and $0 = (R,R,D)(R,R,R) = (R,R,(R,R,D)R)$. Thus, by these, (14) and (1) we have

$$R(R,R,[[R,R],R]) \cdot R = (R,R,D)R \subseteq N_l \cap N_r. \tag{21}$$

Let $x \in R(R,R,[[R,R],R])$ and $w,y,z \in R$. Then by (16) and (1) we get $x \in N_l \cap N_m$ and $wx \in (R,R,D)$. Hence by (9) and (19), we obtain $(w,xy,z) = (wx,y,z) = 0$. Combined this, (1), (14) and (21) results in

$$(R,R,D)R \subseteq N. \tag{22}$$

Using (1), (19) and (22) we see that $<(R,R,D)> = (R,R,D) + (R,R,D)R + R \cdot (R,R,D)R$.
Combined (19) with (20) results in

$$(R,R,D)R \cdot A = (R,R,D) \cdot RA \subseteq (R,R,D)A = 0. \tag{23}$$

Apping (22) and (23), we get $\{R \cdot (R,R,D)R\} \cdot A = R \cdot \{(R,R,D)R \cdot A\} = 0$. Thus using this, (20) and (23), we have $<(R,R,D)> \cdot A = 0$, as desired.

Theorem 4. *If R is a prime weakly Novikov ring such that $[[R,R]R]$ is contained in two of the three nuclei, then R is associative of $[[R,R],R] = 0$.*

Proof. In view of Theorem 1, we may assume $[[R,R],R] \subseteq N_l \cap N_m$.
Let $D = [[R,R],R] + R[[R,R],R]$. Then by Lemma 4 we obtain $<(R,R,D)> \cdot A = 0$, where $<(R,R,D)> = (R,R,D) + (R,R,D)R + R \cdot (R,R,D)R$. By the semiprimeness

of R, this implies $<(R,R,D)>=0$. Hence $[[R,R],R]\subseteq N_r$. Thus by Theorem 1, R is associative or $[[R,R],R]=0$.

By Theorem 4, we have the

Corollary 2. *If R is a prime weakly Novikov ring such that $[[R,R],R]$ is contained in two of the three nuclei with $[[R,R],R]\neq 0$, then R is associative.*

The following is very easy.

Remark 1. If R is a simple weakly Novikov ring such that $T\subseteq N_r$, then R is associative or $T=0$.

Proof. Assume that $R=A=(R,R,R)$. By Lemma 1, we have $RT=AT=0$. Thus, we get $TR=[T,R]\subseteq T$. Hence, we see that $<T>=T$. By the simplicity of R, we obtain $T=0$, as desired.

Remark 2. If R is a semiprime weakly Novikov ring such that $(R,R,R)\subseteq N_l$ or $(R,R,R)\subseteq N_r$ then R is associative.

Proof. We see that the associator ideal A of R is all finite sums of associators. Assume that $(R,R,R)\subseteq N_l$. Then by this and (7), for all $w\in(R,R,R)$ and $x,y,z\in R$ we get $w(x,y,z)=(wx,y,z)\in(A,R,R)=0$.
Thus, we have $(R,R,R)(R,R,R)=0$ and so $A^2=0$.
Assume that $(R,R,R)\subseteq N_r$. Then by Lemma 1, we obtain

$$(R,R,R)(R,R,R)=(R,R,R(R,R,R))=(R,R,A)=0$$

In either case, we have $A^2=0$. By the semiprimeness of R, this implies $A=0$. Thus, R is associative.

In view of Theorem 1 of [6], we have the

Remark 3. If R is a semiprime weakly Novikov ring with a derivation d such that $d(R)\subseteq N_r$, then $d(A)=0$. Moreover, if R is prime such that $d(R)\subseteq N_l\cap N_r$ or $d(R)\subseteq N_m\cap N_r$, then R is associative or $d=0$.

Proof. By the definition of d, $d(R)\subseteq N_r$, (8), (1) and $A=(R,R,R)$, for all $w,x,y,z,t\in R$ we get $(w,x,y)d(z)=(w,x,yd(z))=y(w,x,d(z))=0$, $(w,x,y)\cdot td(z)=(w,x,y)t\cdot d(z)=0$ and so $d(y)(w,x,z)=(w,x,d(y)z)=(w,x,d(y)z)+(w,x,yd(z))=(w,x,d(yz))=0$.

Let $E=d(R)+Rd(R)$. Then the above three equalities imply

$$A\cdot E=0 \text{ and } d(R)\cdot A=0. \tag{24}$$

Using (24), we have that $d(A)R\subseteq d(A)$ and $Rd(A)\subseteq d(A)$. Hence $<d(A)>=d(A)$. Applying (4), we see that $d(A)\subseteq A$. Thus by (24), $d(A)\cdot A=0$ and so by the semiprimeness of R, this implies $d(A)=0$.

Assume that R is prime such that $d(R) \subseteq N_l \cap N_r$ or $d(R) \subseteq N_m \cap N_r$. Then by (3) and the result of [8], E is an ideal of R. By the primeness of R, (24) implies $A = 0$ or $E = 0$. Hence, R is associative or $d = 0$.

In Remark 3, if R is a semiprime weakly Novikov ring with a derivation d such that $d(R) \subseteq N_r$, then $d(A) = 0$. Hence, the results of [7] can be applied.

3. Results of weakly M-rings

In the sequel, we denote T_k and V_k by T and V respectively.

A ring R is called a weakly M-ring if R satisfies the following identity.

$$(w, xy, z) = x(w, y, z) \text{ for all } w, x, y, z \text{ in } R. \tag{25}$$

Note that if R is a weakly M-ring then by (6) and (25) we obtain $A = (R, R, R)$.

Theorem 5. If R is a prime weakly M-ring such that $T \subseteq N_l \cap N_m$ or $T \subseteq N_m \cap N_r$, then R is associative or $T = 0$.

Proof. Note that $[T, R] \subseteq T$. Using this, $T \subseteq N_m$ and (25), for all $x \in T$ and $w, y, z, t \in R$ we have $x(w, y, z) = x(w, y, z) - y(w, x, z) = (w, xy, z) - (w, yx, z) = (w, [x, y], z) = 0$, and so $tx \cdot (w, y, z) = t \cdot x(w, y, z) = 0$. These two identities yield

$$V \cdot A = 0 \tag{26}$$

Since V is an ideal of R, by the primeness of R, (26) implies $A = 0$ or $V = 0$. Hence, R is associative or $T = 0$.

The following three remarks are similar to those in section 2. The proofs are also similar, so we omit it.

Remark 4. If R is a simple weakly M-ring such that $T \subseteq N_m$, then R is associative or $T = 0$.

Remark 5. If R is a semiprime weakly M-ring such that $(R, R, R) \subseteq N_m$, then R is associative.

Remark 6. If R is a prime weakly M-ring with a derivation d such that $d(R) \subseteq N_l \cap N_m$ or $d(R) \subseteq N_m \cap N_r$, then R is associative or $d = 0$.

Finally, we ask if the theorem or the remark is valid for the other cases.

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