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ON SOME INEQUALITIES RELATED TO OPIAL-TYPE INEQUALITY IN TWO VARIABLES

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Abstract. In this paper, we generalized some inequalities related to Opial-Type inequality in two variables. The analysis used in the proofs is quite elementary.

1. Introduction

In 1982, G. S. Yang [1] proved the following Opial-Type inequality in two variables:

Theorem A. *If $f(s,t)$, $f_1(s,t)$, and $f_{12}(s,t)$ are continuous functions on $[a,b]\times[c,d]$, and if $f(a,t)=f(b,t)=f_1(s,c)=f_1(s,d)=0$, for $a\le s\le b$, $c\le t\le d$, then*

$$\int_a^b\int_c^d |f(s,t)||f_{12}(s,t)|dtds \le \frac{(b-a)(d-c)}{8}\int_a^b\int_c^d |f_{12}(s,t)|^2 dtds \tag{1}$$

In 1983, C. T. Lin and G. S. Yang [2] generalized (1) in the following form:

Theorem B. *If $f(s,t)$, $f_1(s,t)$, and $f_{12}(s,t)$ are continuous functions on $[a,b]\times[c,d]$, and if $f(a,t)=f(b,t)=f_1(s,c)=f_1(s,d)=0$, for $a\le s\le b$, $c\le t\le d$, then*

$$\int_a^b\int_c^d |f(s,t)|^m|f_{12}(s,t)|^n dtds \le \left(\frac{n}{m+n}\right)\left[\frac{(b-a)(d-c)}{4}\right]^m\int_a^b\int_c^d |f_{12}(s,t)|^{m+n} dtds \tag{2}$$

In 1984 and 1986, B. G. Pachpatte [3,4] generalized (1) in the following forms:

Theorem C. *Let $f(s,t)$, $f_1(s,t)$, $f_{12}(s,t)$ and $g(s,t)$, $g_1(s,t)$, $g_{12}(s,t)$ be continuous functions on $[a,b]\times[c,d]$, and let $f(a,t)=f(b,t)=f_1(s,c)=f_1(s,d)=0$, $g(a,t)=g(b,t)=g_1(s,c)=g_1(s,d)=0$, for $a\le s\le b$, $c\le t\le d$. Then*

$$\begin{aligned}&\int_a^b\int_c^d |f(s,t)g(s,t)|^m[|f(s,t)||g_{12}(s,t)|+|g(s,t)||f_{12}(s,t)|]dtds\\ &\le \frac{1}{2(m+1)}\left[\frac{(b-a)(d-c)}{4}\right]^{2m+1}\int_a^b\int_c^d [|f_{12}(s,t)|^{2(m+1)}+|g_{12}(s,t)|^{2(m+1)}]dtds\end{aligned} \tag{3}$$

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Theorem D. *Let* $f(s,t)$, $f_1(s,t)$, $f_{12}(s,t)$ *be real-valued continuous functions on* $[a,b]\times[c,d]$, *and* $f(a,t)=f(b,t)=f_1(s,c)=f_1(s,d)=0$, *for* $a\le s\le b$, $c\le t\le d$. *If* H *is a convex increasing function for* $x>0$ *and* $H(0)=0$, *then*

$$\int_a^b\int_c^d H'(|f(s,t)|)|f_{12}(s,t)|dtds \le H\left(\int_a^\alpha\int_c^\beta|f_{12}(s,t)|dtds\right)+H\left(\int_a^\alpha\int_\beta^d|f_{12}(s,t)|dtds\right) +H\left(\int_\alpha^b\int_c^\beta|f_{12}(s,t)|dtds\right)+H\left(\int_\alpha^b\int_\beta^d|f_{12}(s,t)|dtds\right) \tag{4}$$

for $a\le\alpha\le b$, $c\le\beta\le d$.

The aim of this paper is to establish some new integral inequalities which cover the inequalities (2), (3) and (4).

2. Main Results

Throughout, we assume that n is a real number such that $n\ge1$, and $k=\left[\frac{(b-a)(d-c)}{4}\right]^{n-1}$.

Theorem 1. *For* $i=1,2$, *let* $f_i(s,t)$, $D_1f_i(s,t)$, $D_2D_1f_i(s,t)$ *be real-valued continuous functions on* $[a,b]\times[c,d]$, *and* $f_i(a,t)=f_i(b,t)=D_1f_i(s,c)=D_1f_i(s,d)=0$, *for* $a\le s\le b$, $c\le t\le d$. *Let* H_i *be convex, increasing functions on* $[0,\infty)$ *such that* $H_i(0)=0$. *Then*

$$\begin{aligned}\int_a^b\int_c^d&[H_1(|f_1(s,t)|^n)H_2{}'(|f_2(s,t)|^n)|D_2D_1f_2(s,t)|^n\\&+H_2(|f_2(s,t)|^n)H_1{}'(|f_1(s,t)|^n)|D_2D_1f_1(s,t)|^n]dtds\\\le\frac{1}{k_1}\prod_{i=1}^2 H_i&\left(k_1\int_a^\alpha\int_c^\beta|D_2D_1f_i(s,t)|^ndtds\right)+\frac{1}{k_2}\prod_{i=1}^2H_i\left(k_2\int_a^\alpha\int_\beta^d|D_2D_1f_i(s,t)|^ndtds\right)\\+\frac{1}{k_3}\prod_{i=1}^2H_i&\left(k_3\int_\alpha^b\int_c^\beta|D_2D_1f_i(s,t)|^ndtds\right)+\frac{1}{k_4}\prod_{i=1}^2H_i\left(k_4\int_\alpha^b\int_\beta^d|D_2D_1f_i(s,t)|^ndtds\right)\end{aligned}\tag{5}$$

where

$k_1=[(\alpha-a)(\beta-c)]^{n-1}$, $k_2=[(\alpha-a)(d-\beta)]^{n-1}$, $k_3=[(b-\alpha)(\beta-c)]^{n-1}$,
$k_4=[(b-\alpha)(d-\beta)]^{n-1}$, *and* $a\le\alpha\le b$, $c\le\beta\le d$.

Proof. For $a\le s\le\alpha\le b$, $c\le t\le\beta\le d$, and $i=1,2$; define $Z_i(s,t)=\int_a^s\int_c^t|D_2D_1f_i(u,v)|^ndvdu$. Then, $Z_i(s,t)\le Z_i(s,\beta)'$, $Z_i(a,t)=0$, and $D_1Z_i(s,t)=$

$\int_c^t |D_2D_1f_i(s,v)|^n dv$. Since $|f_i(s,t)|^n \le \left(\int_a^s |D_1f_i(u,t)|du\right)^n$ and $|D_1f_i(s,t)|^n \le \left(\int_c^t |D_2D_1f_i(s,v)|dv\right)^n$, it follows from Holder inequality with indices $\frac{n-1}{n}$ and $\frac{1}{n}$ that

$$|f_i(s,t)|^n \le (s-a)^{n-1}\left(\int_a^s |D_1f_i(u,t)|^n du\right)$$

and

$$|D_1f_i(s,t)|^n \le (t-c)^{n-1}\left(\int_c^t |D_2D_1f_i(s,v)|^n dv\right) = (t-c)^{n-1}D_1Z_i(s,t),$$

so that $|f_i(s,t)|^n \le k_1Z_i(s,\beta)$.
Since H_i, $H_i{}'$ are increasing on $[0,\infty)$, $D_2D_1Z_i(s,t) = |D_2D_1f_i(s,t)|^n$, $D_1Z_i(s,c) = 0$, and $H_1(0) = 0$, we have

$$\begin{aligned}
&\int_a^\alpha \int_c^\beta [H_1(|f_1(s,t)|^n)H_2{}'(|f_2(s,t)|^n)|D_2D_1f_2(s,t)|^n \\
&\qquad\qquad +H_2(|f_2(s,t)|^n)H_1{}'(|f_1(s,t)|^n)|D_2D_1f_1(s,t)|^n]dtds \\
&\le \int_a^\alpha \int_c^\beta [H_1(k_1Z_1(s,\beta))H_2{}'(k_1Z_2(s,\beta))D_2D_1Z_2(s,t) \\
&\qquad\qquad +H_2(k_1Z_2(s,\beta))H_1{}'(k_1Z_1(s,\beta))D_2D_1Z_1(s,t)]dtds \\
&= \int_a^\alpha [H_1(k_1Z_1(s,\beta))H_2{}'(k_1Z_2(s,\beta))D_1Z_2(s,\beta) \\
&\qquad\qquad +H_2(k_1Z_2(s,\beta))H_1{}'(k_1Z_1(s,\beta))D_1Z_1(s,\beta)]ds \\
&= \frac{1}{k_1}\int_a^\alpha \frac{d}{ds}[H_1(k_1Z_1(s,\beta))H_2(k_1Z_2(s,\beta))]ds \\
&= \frac{1}{k_1}\prod_{i=1}^2 H_i\left(k_1\int_a^\alpha \int_c^\beta |D_2D_1f_i(u,v)|^n dvdu\right). \qquad (6)
\end{aligned}$$

For $a \le s \le \alpha \le b,\ c \le \beta \le t \le d,\ i = 1,2,$ define $Z_i(s,t) = \int_a^s \int_t^d |D_2D_1f_i(u,v)|^n dvdu$ Then, $Z_i(s,t) \le Z_i(\alpha,t)$, $Z_i(a,t) = 0$ and $D_2Z_i(s,t) = -\int_a^s |D_2D_1f_i(u,t)|^n du$ Since $|f_i(s,t)|^n \le \left(\int_a^s |D_1f_i(u,t)|du\right)^n$ and $|D_1f_i(s,t)|^n \le \left(\int_t^d |D_2D_1f_i(s,v)|dv\right)^n$, it follows from Holder inequality with indices $\dfrac{n-1}{n}$ and $\dfrac{1}{n}$ that

$$|f_i(s,t)|^n \le (s-a)^{n-1}\left(\int_a^s |D_1f_i(u,t)|^n du\right)$$
$$|D_1f_i(s,t)|^n \le (d-t)^{n-1}\left(\int_t^d |D_2D_1f_i(s,v)|^n dv\right) = (d-t)^{n-1}D_1Z_i(s,t),$$

so that

$$\begin{aligned}|f_i(s,t)|^n &\le (s-a)^{n-1}(d-t)^{n-1}\left(\int_a^s D_1 Z_i(u,t)du\right)\\ &= (s-a)^{n-1}(d-t)^{n-1} Z_i(s,t)\\ &\le k_2 Z_i(\alpha,t)\end{aligned}$$

Since H_i, ${H_i}'$ are increasing on $[0,\infty)$, $D_2D_1Z_i(s,t) = -|D_2D_1f_i(s,v)|^n$, $D_2Z_i(a,t)=0$, and $H_1(0)=0$, we have

$$\begin{aligned}&\int_a^\alpha\int_\beta^d [H_1(|f_1(s,t)|^n){H_2}'(|f_2(s,t)|^n)|D_2D_1f_2(s,t)|^n\\ &\qquad\qquad +H_2(|f_2(s,t)|^n){H_1}'(|f_1(s,t)|^n)|D_2D_1f_1(s,t)|^n]dtds\\ &\le -\int_a^\alpha\int_\beta^d [H_1(k_2Z_1(\alpha,t)){H_2}'(k_2Z_2(\alpha,t))D_2D_1Z_2(s,t)\\ &\qquad\qquad +H_2(k_2Z_2(\alpha,t)){H_1}'(k_2Z_1(\alpha,t))D_2D_1Z_1(s,t)]dtds\\ &= -\int_\beta^d [H_1(k_2Z_1(\alpha,t)){H_2}'(k_2Z_2(\alpha,t))D_2Z_2(\alpha,t)\\ &\qquad\qquad +H_2(k_2Z_2(\alpha,t)){H_1}'(k_2Z_1(\alpha,t))D_2Z_1(\alpha,t)]dt\\ &= \frac{1}{k_2}\prod_{i=1}^2 H_i\left(k_2\int_a^\alpha\int_\beta^d |D_2D_1f_i(u,v)|^n dvdu\right)\end{aligned} \tag{7}$$

Similarly, we have

$$\begin{aligned}&\int_\alpha^b\int_c^\beta [H_1(|f_1(s,t)|^n){H_2}'(|f_2(s,t)|^n|D_2D_1f_2(s,t)|^n\\ &\qquad\qquad +H_2(|f_2(s,t)|^n){H_1}'(|f_1(s,t)|^n)|D_2D_1f_1(s,t)|^n]dtds\\ &\le \frac{1}{k_3}\prod_{i=1}^2 H_i\left(k_3\int_\alpha^b\int_c^\beta |D_2D_1f_i(u,v)|^n dvdu\right),\end{aligned} \tag{8}$$

and

$$\begin{aligned}&\int_\alpha^b\int_\beta^d [H_1(|f_1(s,t)|^n){H_2}'(|f_2(s,t)|^n)|D_2D_1f_2(s,t)|^n\\ &\qquad\qquad +H_2(|f_2(s,t)|^n){H_1}'(|f_1(s,t)|^n)|D_2D_1f_1(s,t)|^n]dtds\\ &\le \frac{1}{k_4}\prod_{i=1}^2 H_i\left(k_4\int_\alpha^b\int_\beta^d |D_2D_1f_i(u,v)|^n dvdu\right).\end{aligned} \tag{9}$$

The desired inequality then follows from (6), (7), (8), and (9).

Remark 1. Let $n=1$, $H_1=H$, $H_2=1$, $f_1=f$ in Theorem 1. Then, it follows from the inequality (5) that

$$\int_a^b\int_c^d H'(|f(s,t)|)|f_{12}(s,t)|dtds \le H\left(\int_a^\alpha\int_c^\beta |f_{12}(s,t)|dtds\right)+H\left(\int_a^\alpha\int_\beta^d |f_{12}(s,t)|dtds\right) +H\left(\int_\alpha^b\int_c^\beta |f_{12}(s,t)|dtds\right)+H\left(\int_\alpha^b\int_\beta^d |f_{12}(s,t)|dtds\right)$$

which is the inequality (4) [see[4], Theorem 1]

Theorem 2. *Let $f_i(s,t)$, $D_1f_i(s,t)$, $D_2D_1f_i(s,t)$, H_i be as in Theorem 1. Then*

$$\begin{aligned}&\int_a^b\int_c^d [H_1(|f_1(s,t)|)^n H_2{}'(|f_2(s,t)|^n)|D_2D_1f_2(s,t)|^n \\ &\qquad +H_2(|f_2(s,t)|^n)H_1{}'(|f_1(s,t)|^n)|D_2D_1f_1(s,t)|^n]dtds \\ &\le \frac{1}{k}\Bigg[\prod_{i=1}^2 H_i\left(k\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|D_2D_1f_i(s,t)|^n dtds\right)+\prod_{i=1}^2 H_i\left(k\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d|D_2D_1f_i(s,t)|^n dtds\right) \\ &\quad +\prod_{i=1}^2 H_i\left(k\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}|D_2D_1f_i(s,t)|^n dtds\right)+\prod_{i=1}^2 H_i\left(k\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d|D_2D_1f_i(s,t)|^n dtds\right)\Bigg]\end{aligned} \tag{10}$$

Proof. By taking $\alpha=\dfrac{a+b}{2}$, $\beta=\dfrac{c+d}{2}$ in Theorem 1, we have $k_1=k_2=k_3=k_4=k$. Hence, the inequality (10) follows from (5).

Remark 2. Let $n=1$, $m\ge 0$, $f_1=f$, $f_2=g$, $H_1=H_2=x^{m+1}$ in Theorem 2. Then, it follows from the inequality (10) that

$$\begin{aligned}&(m+1)\int_a^b\int_c^d |f(s,t)g(s,t)|^m[|f(s,t)||g_{12}(s,t)|+|g(s,t)||f_{12}(s,t)|]dtds \\ &\le \left[\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|f_{12}(s,t)|dtds\right)\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|g_{12}(s,t)|dtds\right)\right]^{m+1} \\ &\quad +\left[\left(\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d|f_{12}(s,t)|dtds\right)\left(\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^d|g_{12}(s,t)|dtds\right)\right]^{m+1} \\ &\quad +\left[\left(\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}|f_{12}(s,t)|dtds\right)\left(\int_{\frac{a+b}{2}}^b\int_c^{\frac{c+d}{2}}|g_{12}(s,t)|dtds\right)\right]^{m+1} \\ &\quad +\left[\left(\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d|f_{12}(s,t)|dtds\right)\left(\int_{\frac{a+b}{2}}^b\int_{\frac{c+d}{2}}^d|g_{12}(s,t)|dtds\right)\right]^{m+1}\end{aligned}$$

Using Holder inequality twice with indices $\dfrac{2m+1}{2(m+1)}$ and $\dfrac{1}{2(m+1)}$, we have

$$\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|f_{12}(s,t)|dtds\right]^{m+1}$$
$$\leq\left[\int_a^{\frac{a+b}{2}}\left(\frac{d-c}{2}\right)^{\frac{2m+1}{2(m+1)}}\left(\int_c^{\frac{c+d}{2}}|f_{12}(s,t)|^{2(m+1)}dt\right)^{\frac{1}{2(m+1)}}ds\right]^{m+1}$$
$$\leq\left(\frac{d-c}{2}\right)^{\frac{2m+1}{2}}\left(\frac{b-a}{2}\right)^{\frac{2m+1}{2}}\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|f_{12}(s,t)|^{2(m+1)}dtds\right)^{\frac{1}{2}}$$

Also,

$$\left[\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|g_{12}(s,t)|dtds\right]^{m+1}$$
$$\leq\left(\frac{d-c}{2}\right)^{\frac{2m+1}{2}}\left(\frac{b-a}{2}\right)^{\frac{2m+1}{2}}\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|g_{12}(s,t)|^{2(m+1)}dtds\right)^{\frac{1}{2}}$$

Since $2xy\leq x^2+y^2$ for all x and y, we have

$$\left[\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|f_{12}(s,t)|dtds\right)\left(\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}|g_{12}(s,t)|dtds\right)\right]^{m+1}$$
$$\leq\frac{1}{2}\left[\frac{(b-a)(d-c)}{4}\right]^{2m+1}\int_a^{\frac{a+b}{2}}\int_c^{\frac{c+d}{2}}(|f_{12}(s,t)|^{2(m+1)}+|g_{12}(s,t)|^{2(m+1)})dtds.$$

Similarly,

$$\left[\left(\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}|f_{12}(s,t)|dtds\right)\left(\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}|g_{12}(s,t)|dtds\right)\right]^{m+1}$$
$$\leq\frac{1}{2}\left[\frac{(b-a)(d-c)}{4}\right]^{2m+1}\int_a^{\frac{a+b}{2}}\int_{\frac{c+d}{2}}^{d}(|f_{12}(s,t)|^{2(m+1)}+|g_{12}(s,t)|^{2(m+1)})dtds,$$
$$\left[\left(\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}|f_{12}(s,t)|dtds\right)(\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}|g_{12}(s,t)|dtds)\right]^{m+1}$$
$$\leq\frac{1}{2}\left[\frac{(b-a)(d-c)}{4}\right]^{2m+1}\int_{\frac{a+b}{2}}^{b}\int_c^{\frac{c+d}{2}}(|f_{12}(s,t)|^{2(m+1)}+|g_{12}(s,t)|^{2(m+1)})dtds,$$

and

$$\left[\left(\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}|f_{12}(s,t)|dtds\right)\left(\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}|g_{12}(s,t)|dtds\right)\right]^{m+1}$$
$$\leq\frac{1}{2}\left[\frac{(b-a)(d-c)}{4}\right]^{2m+1}\int_{\frac{a+b}{2}}^{b}\int_{\frac{c+d}{2}}^{d}(|f_{12}(s,t)|^{2(m+1)}+|g_{12}(s,t)|^{2(m+1)})dtds.$$

Hence,

$$\int_a^b \int_c^d |f(s,t)g(s,t)|^m[|f(s,t)||g_{12}(s,t)| + |g(s,t)||f_{12}(s,t)|]dtds$$
$$\leq \frac{1}{2(m+1)}\left[\frac{(b-a)(d-c)}{4}\right]^{2m+1} \int_a^b \int_c^d [|f_{12}(s,t)|^{2(m+1)} + |g_{12}(s,t)|^{2(m+1)}]dtds$$

which is the inequality (3) [See [3], Theorem 1]

Remark 3. Let $H_1 = x^{\frac{m+n}{n}}$, $H_2 = 1$, $f_1 = f_2 = f$ in Theorem 2. Then for $m \geq 0$, $n \geq 1$, it follows from the inequality (10) that

$$\int_a^b \int_c^d \left(\frac{m+n}{n}\right)|f(s,t)|^m|f_{12}(s,t)|^n dtds$$
$$\leq \frac{1}{k}\Bigg[\left(k\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} |f_{12}(s,t)|^n dtds\right)^{\frac{m+n}{n}} + \left(k\int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d |f_{12}(s,t)|^n dtds\right)^{\frac{m+n}{n}}$$
$$+\left(k\int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} |f_{12}(s,t)|^n dtds\right)^{\frac{m+n}{n}} + \left(k\int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d |f_{12}(s,t)|^n dtds\right)^{\frac{m+n}{n}}\Bigg]$$

Using Holder inequality twice with indices $\dfrac{m}{m+n}$ and $\dfrac{n}{m+n}$ to each term on the right hand side, we have

$$\left(\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} |f_{12}(s,t)|^n dtds\right)^{\frac{m+n}{n}} \leq \left[\frac{(b-a)(d-c)}{4}\right]^{\frac{m}{n}} \left(\int_a^{\frac{a+b}{2}} \int_c^{\frac{c+d}{2}} |f_{12}(s,t)|^{m+n} dtds\right),$$
$$\left(\int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d |f_{12}(s,t)|^n dtds\right)^{\frac{m+n}{n}} \leq \left[\frac{(b-a)(d-c)}{4}\right]^{\frac{m}{n}} \left(\int_a^{\frac{a+b}{2}} \int_{\frac{c+d}{2}}^d |f_{12}(s,t)|^{m+n} dtds\right),$$
$$\left(\int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} |f_{12}(s,t)|^n dtds\right)^{\frac{m+n}{n}} \leq \left[\frac{(b-a)(d-c)}{4}\right]^{\frac{m}{n}} \left(\int_{\frac{a+b}{2}}^b \int_c^{\frac{c+d}{2}} |f_{12}(s,t)|^{m+n} dtds\right),$$

and

$$\left(\int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d |f_{12}(s,t)|^n dtds\right)^{\frac{m+n}{n}} \leq \left[\frac{(b-a)(d-c)}{4}\right]^{\frac{m}{n}} \left(\int_{\frac{a+b}{2}}^b \int_{\frac{c+d}{2}}^d |f_{12}(s,t)|^{m+n} dtds\right).$$

Therefore,

$$\int_a^b \int_c^d |f(s,t)|^m|f_{12}(s,t)|^n dtds \leq (\frac{n}{m+n})\left[\frac{(b-a)(d-c)}{4}\right]^m \int_a^b \int_c^d |f_{12}(s,t)|^{m+n} dtds$$

which is the inequality (2) [see [2], Theorem 5]

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