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BOUNDS FOR UNIFORM RESOLVENT CONDITIONS

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Abstract. For a bounded linear operator on a Banach space, the uniform resolvent condition implies the absolute summability of the powers of the operator. In this paper, we study the bounds for the absolute sum of the powers of an operator that satisfies the uniform resolvent condition. Some known bounds on general Banach spaces as well as on finite-dimensional Banach spaces are improved.

1. Introduction

Let X be a Banach space and T be a bounded linear operator on X. T is said to satisfy the uniform resolvent condition if there is a constant U such that

$$\|(\lambda - T)^{-1}\| \le U \tag{1.1}$$

for all $|\lambda| \ge 1$. Clearly, if T satisfies uniform resolvent condition, then its spectrum $\sigma(T)$ is a subset of the open unit disk. In fact, $\sigma(T)$ is a subset of smaller disk of radius $1-1/U$ by (3.2) in Section 3.

In the stability study of operators on Banach spaces, one often needs to estimate the growth rate of the powers of the operators. We say that the powers of T are absolutely summable if there is a constant B such that

$$\sum_{n=0}^{\infty} \|T^n\| \le B. \tag{1.2}$$

If the powers of T are absolutely summable (1.2), then using the power expansion of the resolvent, we can easily see that T satisfies the uniform resolvent condition (1.1) with $U \le B$.

On the other side, Lubich and Nevanlinna [2] proved that (1.1) implies (1.2) too. They even obtained a bound for B, that is, $B \le eU^2(1+\ln(2U))$. Since then, several papers have been published in searching the best possible growth rate of B. The following is a summary of the progress on this direction.

$$\begin{aligned} \text{Lubich and Nevalinna [2] (1991)} &: eU^2(1+\ln(2U)) \\ \text{Nevalinna [3] (1997)} &: 6U^2-6U+1 \\ \text{Chen [1] (2000) and Nevalinna [4] (2001)} &: 4U^2-4U+1 \end{aligned}$$

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In this paper, we will improve the bound of B to $3U^2+1$. We will also improve the corresponding bounds for operators on the finite-dimensional spaces.

2. Some lemmas

Lemma 2.1. *If T satisfies the uniform resolvent condition (1.1) with $U \geq 1$, then for each real number q with $0<q<1$, T satisfies*

$$\|(\lambda-T)^{-1}\| \leq \frac{U}{1-q} \tag{2.1}$$

for all λ such that

$$|\lambda| \geq 1-\frac{q}{U}.$$

Proof. Suppose T satisfies (1.1). Let q be any positive real number less than 1 and let p be any complex number such that $|p|=q$. If $|\lambda| \geq 1$, then

$$\left\|\frac{p}{U}(\lambda-T)^{-1}\right\| \leq q,$$

which implies that $1-\frac{p}{U}(\lambda-T)^{-1}$ is invertible and

$$\left\|\left(1-\frac{p}{U}(\lambda-T)^{-1}\right)^{-1}\right\| \leq \frac{1}{1-q}.$$

Now

$$1-\frac{p}{U}(\lambda-T)^{-1}=(\lambda-T)^{-1}\left(\lambda-\frac{p}{U}-T\right)$$

thus $\lambda-\frac{p}{U}-T$ is invertible, and

$$\left\|\left(\lambda-\frac{p}{U}-T\right)^{-1}\right\| \leq \left\|\left(1-\frac{p}{U}(\lambda-T)^{-1}\right)^{-1}\right\| \|(\lambda-T)^{-1}\| \leq \frac{U}{1-q}.$$

Let $|\mu| \geq 1-\frac{q}{U}$. Set $\lambda=\mu+\frac{\mu q}{|\mu| U}$, and $p=\frac{\mu q}{|\mu|}$. Then $|\lambda|=|\mu|+\frac{q}{U} \geq 1$ and $\lambda-\frac{p}{U}=\mu$. Thus we have

$$\|(\mu-T)^{-1}\| \leq \frac{U}{1-q}.$$ □

The following is a collection of some easy results of calculus which will be used in the next sections.

Lemma 2.2. *If $0<x<1$, then*

$$\sum_{n=N}^{\infty}(n+1)x^n = x^N(1-x)^{-2}(N+1-Nx), \tag{2.2}$$

and the functions

$$g(t)=x^t(t+1-tx)$$

and

$$h(t)=\ln t+ex^t$$

are decreasing and increasing respectively for $t>0$.

Proof. We know for $0 < x < 1$,

$$\sum_{n=N}^{\infty}(n+1)x^n = \frac{d}{dx}\sum_{n=N}^{\infty}x^{n+1} = \frac{d}{dx}\left(\frac{x^{N+1}}{1-x}\right) = x^N(1-x)^{-2}(N+1-Nx),$$

which proves (2.2).

For $0 < x < 1$, we know $\ln x < 0$, $\ln x + 1 - x < 0$, thus for $t > 0$,

$$g'(t) = x^t(t+1-tx)\ln x + x^t(1-x) = x^t\left[t(1-x)\ln x + \ln x + 1 - x\right] < 0,$$

which implies $g(t)$ is decreasing on $(0,\infty)$.

Set $h_1(t) = 1 + tx^t e\ln x$. Then $h_1'(t) = ex^t\ln x(1+t\ln x)$. It follows that for $0 < x < 1$, $h_1(t)$ has absolute minimum 0 over $(0,\infty)$ at $t = -\frac{1}{\ln x}$. Now

$$h'(t) = \frac{1}{t} + ex^t\ln x = \frac{h_1(t)}{t} \geq 0.$$

Hence $h(t)$ is increasing on $(0,\infty)$. □

3. Some improved bounds

In this section, we will give some better bounds for B. First we have an estimate for the powers of T.

Theorem 3.1. *If T satisfies the uniform resolvent condition (1.1) with a constant $U \geq 1$, then for all positive integer n and any positive real number q that is less than 1,*

$$\|T^n\| \leq \frac{U}{1-q}\left(1-\frac{q}{U}\right)^{n+1}. \tag{3.1}$$

Proof. Suppose T satisfies (1.1), then by Lemma 2.1, we know that T satisfies (2.1) for all q and λ such that $0 < q < 1$ and $|\lambda| \geq 1 - \frac{q}{U}$. Thus for any positive integer n

$$T^n = \frac{1}{2\pi i}\int_{|\lambda|=1-\frac{q}{U}}\lambda^n(\lambda-T)^{-1}d\lambda.$$

It follows that

$$\|T^n\| \leq \left(1-\frac{q}{U}\right)^{n+1}\frac{U}{1-q},$$

which completes the proof. □

From Lemma 2.1, we see that

$$\sigma(T) \subseteq \left\{|z| : |z| \leq 1 - \frac{1}{U}\right\}. \tag{3.2}$$

Thus if $U = 1$, then $\sigma(T) = \{0\}$, and T is quasinilpotent. In fact, by Theorem 3.1, we know that if $U = 1$, then $\|T\| \leq 1-q$. Letting $q \to 1^-$, we have $T = 0$. Hence

Corollary 3.2. *If T satisfies (1.1) with $U = 1$, then $T = 0$.*

Now the bound $(2U-1)^2$ in [1, 4] is just an easy consequence of Theorem 3.1.

Corollary 3.3. *If T satisfies (1.1) with a constant $U \geq 1$, then T satisfies (1.2) with $B \leq (2U-1)^2$.*

Proof. By Theorem 3.1, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \|T^n\| &\leq \sum_{n=1}^{\infty} \left(1-\frac{q}{U}\right)^{n+1} \frac{U}{1-q} \\ &= \frac{(U-q)^2}{q(1-q)}.\end{aligned}$$

Taking $q = \frac{U}{2U-1}$, we get

$$\sum_{n=1}^{\infty} \|T^n\| \leq 4U(U-1).$$

or

$$\sum_{n=0}^{\infty} \|T^n\| \leq (2U-1)^2. \qquad \square$$

When U is large, the bound in the above corollary is equivalent to $4U^2$. If we minimize each term in Theorem 1, we can reduce the bound to $3U^2$.

Theorem 3.4. *If T satisfies the uniform resolvent condition* (1.1) *with a constant $U > 1$, then T satisfies* (1.2) *with $B < 3U^2 + 1$.*

Proof. For each positive integer n, set

$$f_n(q) = \left(1-\frac{q}{U}\right)^{n+1} \frac{U}{1-q}.$$

By (3.1), we know $\|T^n\| \leq f_n(q)$ for $0 < q < 1$. In particular, letting $q \to 0^+$, we see that

$$\|T^n\| \leq f_n(0) = U.$$

Let $N = \lfloor U \rfloor + 1$. Since

$$f_n{}'(q) = \left(1-\frac{q}{U}\right)^{n} \frac{1}{(1-q)^2}(nq-n-1+U),$$

we see that if $n \geq N > U$, then we can choose $q = 1 - \frac{U-1}{n}$ which is positive and less than 1, and hence by (3.1)

$$\|T^n\| \leq f_n\left(1-\frac{U-1}{n}\right) \leq (n+1)\left(1+\frac{1}{n}\right)^n \left(1-\frac{1}{U}\right)^n < e(n+1)\left(1-\frac{1}{U}\right)^n,$$

where the last inequality holds since the sequence $\left(1+\frac{1}{n}\right)^n$ is increasing to e. Hence

$$\sum_{n=N}^{\infty}\|T^n\| \leq \sum_{n=N}^{\infty}(n+1)e\left(1-\frac{1}{U}\right)^n = eU^2\left(1-\frac{1}{U}\right)^N\left[N+1-N\left(1-\frac{1}{U}\right)\right].$$

Applying Lemma 2.2 on the right side of the above inequality with $N > U$, we know

$$\sum_{n=N}^{\infty}\|T^n\| < eU^2\left(1-\frac{1}{U}\right)^U\left[U+1-U\left(1-\frac{1}{U}\right)\right] = 2eU^2\left(1-\frac{1}{U}\right)^U.$$

It is easy to verify that $\left(1-\frac{1}{x}\right)^x$ is increasing. It follows $\left(1-\frac{1}{U}\right)^U < \frac{1}{e}$ and hence we have

$$\sum_{n=N}^{\infty}\|T^n\| < 2U^2.$$

Therefore

$$\begin{aligned}\sum_{n=0}^{\infty}\|T^n\| &= 1+\sum_{n=1}^{\lfloor U\rfloor}\|T^n\| + \sum_{n=\lfloor U\rfloor+1}^{\infty}\|T^n\| \\ &< 1+U\lfloor U\rfloor+2U^2 \\ &\leq 3U^2+1.\end{aligned}$$

□

4. On finite dimensional spaces

When the dimension of X is finite, Lubich and Nevanlinna found a better bound with $B \leq 2edU(1+\ln(2U))$. This bound can also be improved.

Theorem 4.1. *On a Banach space of dimension d, the uniform resolvent condition implies*

$$\sum_{n=0}^{\infty}\|T^n\| \leq U(\sqrt{\ln U}+1)^2 d. \tag{4.1}$$

Proof. Set $\Gamma = \{|z| = 1-\frac{q}{U}\}$. By using partial integration, we can get

$$T^n = \frac{1}{2\pi\mathrm{i}(n+1)}\int_\Gamma \lambda^{n+1}(\lambda-T)^{-2}d\lambda.$$

Let u, v be any two unit vectors respectively in X and X'. Set

$$q(\lambda) = \langle v, (\lambda-T)^{-1}u\rangle.$$

Then following the same idea as in the proof of Theorem 2.3 of [2] (also see the proof on page 137 of [5]), we have

$$|\langle v, T^n u\rangle| \leq \frac{1}{2\pi(n+1)}\left(1-\frac{q}{U}\right)^{n+1}\int_\Gamma |q'(\lambda)||d\lambda|.$$

But

$$\int_\Gamma |q'(\lambda)||d\lambda| \le 2\pi d \cdot \max_\Gamma |q(\lambda)| \qquad \text{(by Spijker's Lemma)}$$
$$\le 2\pi d \frac{U}{1-q}.$$

It follows that

$$\|T^n\| \le \frac{1}{n+1}\left(1-\frac{q}{U}\right)^{n+1} \frac{U}{1-q} d. \tag{4.2}$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{n} r^n = -\ln(1-r) \quad \text{for } 0<r<1,$$

we have,

$$\sum_{n=0}^{\infty} \|T^n\| \le \sum_{n=0}^{\infty} \frac{U}{1-q} d \frac{1}{n+1}\left(1-\frac{q}{U}\right)^{n+1}$$
$$= \frac{U}{1-q} d\left(-\ln\frac{q}{U}\right).$$

Taking $q = \frac{1}{1+\sqrt{\ln U}}$, we have

$$\sum_{n=0}^{\infty} \|T^n\| \le Ud \frac{\ln U - \ln q}{1-q}$$
$$= Ud\left(\frac{\ln U}{1-q} + \ln\left(1+\frac{1-q}{q}\right)\frac{1}{1-q}\right)$$
$$\le Ud\left(\frac{\ln U}{1-q} + \frac{1}{q}\right)$$
$$= Ud\left(\sqrt{\ln U}+1\right)^2,$$

which is (4.1). ∎

Note that the bound $Ud\left(\sqrt{\ln U}+1\right)^2$ is better than $2edU(1+\ln(2U))$. In [2], it was proved that under Euclidean norm, $B \le 2Ud$. (4.1) is even better than this bound when U is close to 1. When U is large, we can improve (4.1) further.

Theorem 4.2. *On a Banach space of dimension d, the uniform resolvent condition implies*

$$\sum_{n=0}^{\infty} \|T^n\| \le Ud(\ln U + 1) + 1.$$

Proof. We see in the proof of Theorem 3.4, we eventually show that the function $\left(1-\frac{q}{U}\right)^{n+1}(1-q)^{-1}$ has minimum at $q = 1-\frac{U-1}{n}$ when $n > U-1$. Set $N = \lfloor U \rfloor$. Then by (4.2), if $n \ge N$, then

$$\|T^n\| \le d\left(1-\frac{1}{U}\right)^n\left(1+\frac{1}{n}\right)^n \le ed\left(1-\frac{1}{U}\right)^n.$$

So

$$\sum_{n=N}^{\infty} \|T^n\| \le \sum_{n=N}^{\infty} ed\left(1-\frac{1}{U}\right)^n = edU\left(1-\frac{1}{U}\right)^N$$

We see that for any n, if we take $q \to 0^+$ in (4.2), then

$$\|T^n\| \le \frac{Ud}{n+1}.$$

Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} \|T^n\| &= 1 + \sum_{n=1}^{N-1} \|T^n\| + \sum_{n=N}^{\infty} \|T^n\| \\
&\le 1 + \sum_{n=1}^{N-1} \frac{Ud}{n+1} + edU\left(1-\frac{1}{U}\right)^N \\
&\le 1 + Ud\left[\ln N + e\left(1-\frac{1}{U}\right)^N\right] \\
&\le 1 + Ud\left[\ln U + e\left(1-\frac{1}{U}\right)^U\right] \\
&\le 1 + Ud(\ln U + 1),
\end{aligned}$$

where the second inequality is from the estimate of harmonic series $\sum_{n=1}^{k} \frac{1}{n} < \ln k+1$, the third comes from Lemma 2.2, and the fourth is from the fact the function $\left(1-\frac{1}{U}\right)^U$ is increasing to $\frac{1}{e}$. □

References

[1] Y. CHEN, *Sur les multifonctions analytiques et la théorie spectrale*, doctoral dissertation (2000), Laval University.

[2] C. Lubich and O. Nevanlinna, *On resolvent conditions and stability estimates*, BIT **31** (1991), 293–313.

[3] O. Nevanlinna, *On the growth of the resolvent operators for power bounded operators*, Banach Center Publications, **38** (1997), 247–264.

[4] O. Nevanlinna, *Resolvent conditions and powers of operators*, Studia Mathematica, **145** (2001), 113-134.

[5] E. Wegert and L. N. Trefethen, *From the Buffon Needle Problem to the Kreiss Matrix Theorem*, The American Mathematical Monthly, **101** (1994), 132–139

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