TAMKANG JOURNAL OF MATHEMATICS
Volume 33, Number 2, Summer 2002

ON A SUBCLASS OF BAZILEVIČ FUNCTIONS

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Abstract. For $\alpha > 0$, $0 \leq \beta < 1$, we denote $B_1(\alpha, \beta)$ to be the class of normalised analytic functions satisfying the condition $Re\left(\frac{f(z)}{z}\right)^{\alpha-1} f'(z) > \beta$ for z in the unit disc $D = \{z : |z| < 1\}$. Sharp estimates for $Re\left(\frac{f(z)}{z}\right)^{\alpha}$ is established. In fact a more generalised result concerning iterated integrals is obtained.

1. Introduction

For $\alpha > 0$, $0 \leq \beta < 1$, let $B_1(\alpha, \beta)$ be the class of Bazilevič functions defined in the unit disc $D = \{z : |z| < 1\}$ normalized such that $f(0) = 0, f'(0) = 1$ and satisfying.

$$Re \frac{z^{1-\alpha} f'(z)}{f(z)^{1-\alpha}} > \beta, z \in D.$$

We note that $B_1(1, 0) \equiv R$, the class of functions whose derivative has positive real part. Macgregor [6] studied this class and Hallenbeck [4] showed the sharp result below

$$Re\Big(\frac{f(z)}{z}\Big) \geq -1 + \frac{2}{r} \log(1 + r) > -1 + 2 \log 2$$

for $z = re^{i\theta} \in D$.

The class $B_1(\alpha) \equiv B_1(\alpha, 0)$ was first looked at by Singh [9] and later followed by other authors including [5] and [10]. The sharp estimate for the lower bound of $Re\left(\frac{f(z)}{z}\right)^{\alpha}$ was established for $B_1(\alpha)$ and further extended to include estimates for the real part of some iterated integral operators in [3]. The author in [1] and [2] give some results concerning the class $B_1(\alpha, \beta)$. In [7] the following result was attained by Owa and Obradovič:

$$Re\Big(\frac{f(z)}{z}\Big)^{\alpha} > (1 + 2\alpha\beta)/(1 + 2\alpha) \qquad \text{for } z \in D.$$

We now give a sharp result for this and generalise further to the real part of some iterated integral operators.

Received March 1, 2001.
2000 *Mathematics Subject Classification*. Primary 30C45.
Key words and phrases. Bazilevič functions and integral operators.

2. Results

First, for $z \in D, a > -1$ and $n = 1, 2, 3, \ldots$, define

$$I_n(z) = \frac{a+1}{z^{a+1}} \int_0^z t^a I_{n-1}(t)dt, \tag{1}$$

where $I_0(z) = (f(z)/z)^\alpha$.

Theorem 2.1. *Let $f \in B_1(\alpha, \beta)$ and $z = re^{i\theta} \in D$. Then for $n \geq 0$,*

$$Re I_n(z) \geq \gamma_n(r) > \gamma_n(1) \tag{2}$$

and

$$\gamma_n(r) < 1,$$

where for $n = 1, 2, 3, \ldots$

$$\gamma_n(r) = \frac{a+1}{r^{a+1}} \int_0^r \rho^a \gamma_{n-1}(\rho)d\rho$$

and

$$\gamma_0(r) = \frac{\alpha(1-\beta)}{r^\alpha} \int_0^r \rho^{\alpha-1}\Big(\frac{1-\rho}{1+\rho}\Big)d\rho + \beta.$$

Equality is attained for the function f_0 defined by

$$f_0(z) = \Big(\alpha \int_0^z t^{\alpha-1}\Big(\frac{(1-t)(1-\beta)}{(1+t)} + \beta\Big)dt\Big)^{1/\alpha}.$$

We note that when $n = 0$, the theorem gives the sharp lower bound for $Re\Big(\frac{f(z)}{z}\Big)^\alpha$ i.e.

$$Re\Big(\frac{f(z)}{z}\Big)^\alpha \geq \gamma_0(r) = 1 + 2\alpha(1-\beta)\sum_{j=1}^{\infty} \frac{(-r)^j}{(j+\alpha)}.$$

This improves Owa and Obradovićs result [loc. cit]. Furthermore, in the cae $n = 1$, the sharp result also improves that of Owa et al. [8] for the class $B_1(1, \beta)$.

In order to prove the above Theorem, we first require the following Lemma.

Lemma 2.1. *Let $\gamma_n(r)$ be defined as in Theorem 2.1. Then for fixed r and $n \geq 1$,*

$$\gamma_{n-1}(r) < \gamma_n(r).$$

Proof of Lemma. We use induction. However, first we prove the following inequality

$$\frac{r^a}{(1+r)^2}[(1+a)(1-r^2) - 2r] < \alpha(1+a)\gamma^{a-\alpha} \int_0^\gamma \rho^{\alpha-1}\Big(\frac{1-\rho}{1+\rho}\Big)d\rho. \tag{3}$$

Observe that for $0 < r < 1$, $\alpha > 0$ and $a > -1$, the inequality below is true.

$$0 < \frac{2r(1-r)}{(1+a)(1+r)} + \frac{2\alpha r}{1+a} + 2r.$$

With the introduction of the term $\alpha(1-r^2)$ to both sides of the inequality, this implies

$$\alpha(1-r^2) - 2r < \frac{2r}{(1+a)(1+r)}\{(1-r) + \alpha(1+r)\} + \alpha(1-r)(1+r)$$

and, on multiplying throughout by $\dfrac{r^{\alpha-1}}{(1+r)^2}$, we have

$$\frac{r^{\alpha-1}}{(1+r)^2}(\alpha - \alpha r^2 - 2r) < \frac{2r^\alpha}{(1+a)(1+r)^3}\{(1+\alpha)(1+r) - 2r\} + \frac{\alpha(1-r)r^{\alpha-1}}{(1+r)}.$$

Integrating both sides from 0 to r and using the Comparison Theorem, we obtain

$$\begin{aligned}&\int_0^r \frac{\rho^{\alpha-1}}{(1+\rho)^2}(\alpha - 2\rho - \alpha\rho^2)d\rho \\ &< \alpha\int_0^r \frac{\rho^{\alpha-1}(1-\rho)}{(1+\rho)}d\rho + \frac{2}{1+a}\int_0^r \frac{\rho^\alpha}{(1+\rho)^3}((1+\alpha)(1+\rho) - 2\rho)d\rho\end{aligned}$$

which implies

$$\frac{r^\alpha(1-r)}{1+r} < \alpha\int_0^r \frac{\rho^{\alpha-1}(1-\rho)}{(1+\rho)}d\rho + \frac{2}{1+a}\Big(\frac{r^{\alpha+1}}{(1+r)^2}\Big).$$

Next, multiplying both sides by the term $r^{a-\alpha}(1+a)$ gives inequality (3).
And, after integrating both sides of (3), we have

$$r^{a+1}\Big(\frac{1-r}{1+r}\Big) < \alpha(1+a)\int_0^r \rho^{a-\alpha}\int_0^\rho \xi^{\alpha-1}\Big(\frac{1-\xi}{1+\xi}\Big)d\xi d\rho,$$

which implies that $\gamma_0 < \gamma_1$.
Next, note that

$$\begin{aligned}\gamma_k - \gamma_{k+1} &= \frac{a+1}{r^{a+1}}\int_0^r \rho^a[\gamma_{k-1}(\rho) - \gamma_k(\rho)]d\rho \\ &< 0,\end{aligned}$$

and so the Lemma follows by induction.

Proof of Theorem 2.1. We again use induction. Suppose first that $n = 0$. Then since $f \in B_1(\alpha, \beta)\exists$ a function p with $p(0) = 1$ and $Re\ p(z) > 0$ such that

$$Re\Big(\frac{f(z)}{z}\Big)^\alpha = \alpha Re\frac{1}{z^\alpha}\int_0^z t^{\alpha-1}[p(t)(1-\beta) + \beta]dt.$$

Write $t = \rho e^{i\theta}$ so that

$$
\begin{aligned}
Re\Big(\frac{f(z)}{z}\Big)^{\alpha} &= \frac{\alpha}{r^{\alpha}}\int_0^r \rho^{\alpha-1}[1-\beta Rep(\rho e^{i\theta})+\beta]d\rho \\
&\geq \frac{\alpha}{r^{\alpha}}\int_0^r \rho^{\alpha-1}\Big[\frac{(1-\beta)(1-\rho)}{1+\rho}+\beta\Big]d\rho \\
&= \gamma_o(r),
\end{aligned}
$$

since $Re\ p(z) \geq (1-r)/(1+r)$ for $|z| = r < 1$ (see [6]). Elementary calculus now shows that for $0 < \rho < r < 1, \gamma_0(1) < \gamma_0(r) < 1$.
Next from (1)

$$
\begin{aligned}
ReI_{n+1}(z) &= Re\frac{a+1}{z^{a+1}}\int_0^z t^a I_n(t)dt \\
&= \frac{a+1}{r^{a+1}}\int_0^r \rho^a ReI_n(\rho e^{i\theta})d\rho \\
&\geq \gamma_{n+1}(r),
\end{aligned}
$$

where the inequality follows by induction. This proves the first inquality in (2).
Now

$$
\gamma_n(r) = 1 + 2\alpha(1+a)^n(1-\beta)\sum_{j=1}^{\infty}\frac{(-r)^j}{(j+\alpha)(j+a+1)^n}
$$

and so for $n \geq 1$, this series is absolutely convergent. Suitably rearranging pairs of terms in $\gamma_n(r)$ as

$$
\begin{aligned}
\gamma(n) = 1 &- 2\alpha(1+a)^n(1-\beta)\Big(\frac{r}{(1+\alpha)(2+a)^n} - \frac{r^2}{(2+\alpha)(3+a)^n}\Big) \\
&-2\alpha(1+a)^n(1-\beta)\Big(\frac{r^3}{(3+\alpha)(4+a)^n} - \frac{r^4}{(4+\alpha)(5+a)^n}\Big) + \cdots \\
&= 1 - 2\alpha(1+a)^n(1-\beta)\sum_{k=1}^{\infty}\Big(\frac{r^{2k-1}}{(2k-1+\alpha)(2k+a)^n} - \frac{r^{2k}}{(2k+\alpha)(2k+1+a)^n}\Big),
\end{aligned}
$$

shows that $\gamma_n(r) < 1$.

Finally, we show that $\gamma_n(r) > \gamma_n(1)$(the second inequality in (2)). Using Lemma 2.1 in the following

$$
r{\gamma'}_n(r) + (a+1)\gamma_n(r) = (a+1)\gamma_{n-1}(r)
$$

shows that for a fixed $n \geq 1, r\gamma_n'(r) < 0$. Hence $\gamma_n(r)$ decreases with r as $r \to 1$.
This completes the proof of the Theorem.

To obtain the sharp lower bound for $Re\Big(\frac{f(z)}{z}\Big)$ requires the following Lemma, which is elementary and as such we have omitted the proof.

Lemma 2.2. *Let $z \in \mathbb{C}$ with $Re z \geq \sigma > 0$ where σ is a constant. Then for fixed m, with $0 < m < 1$,*

$$Re z^m \geq \sigma^m.$$

With the above Lemma, one can easily deduce the following theorems.

Theorem 2.2. *Let $f \in B_1(\alpha, \beta)$. Then for $z = re^{i\theta} \in D$ and $\alpha \geq 1$,*

$$Re\Big(\frac{f(z)}{z}\Big) > \gamma_0(1)^{1/\alpha},$$

where γ_0 is defined as in Theorem 2.1.

Theorem 2.3. *If $f \in B_1(\alpha, \beta)$ then for $0 \leq \beta < 1$,*

$$Re\sqrt{\frac{f(z)}{z}} > \Big(1 + 2(1-\beta)\ln\Big(\frac{2}{e}\Big)\Big)^{1/2}.$$

The function $f(z) = -z(1-2\beta) - 2(1-\beta)\log(1-z)$ illustrates that the result is sharp.

Proof. Let $\alpha = 1$ in Theorem 2.2. Then Lemma 2.2 with $m = 1/2$ gives us the result.

Remark. The above result improves that of Owa, Fukui and Altintas [8].

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