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A SOLUTION OF ONE PROBLEM OF COMPLEX INTEGRATION

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Abstract. In this paper the following identity

$$(\sqrt{\pi}+\int_0^i e^{-1/t^2}dt)e^{-1}=i\Big(1-\frac{2^1}{1}+\frac{2^2}{1\cdot 3}-\frac{2^3}{1\cdot 3\cdot 5}\cdots\Big)$$

is proved, where the integration is done over a curve with tangent vector at 0 toward the positive part of x-axis.

1. Formulation of the problem.

K. Trenčevski has set the following numerical expansion

$$(\sqrt{\pi}+\int_0^i e^{-1/t^2}dt)e^{-1}=i\Big(1-\frac{2^1}{1}+\frac{2^2}{1\cdot 3}-\frac{2^3}{1\cdot 3\cdot 5}\cdots\Big)$$

where the integration is done over a curve with tangent vector at 0 toward the positive part of x-axis.

This problem has not appeared until now in the literature and it is a subject of our consideration. However, applying some known results obtained by D. S. Mitrinović, J. D. Kečkić [1] and M. R. Spiegel [2], this identity finally is proved by the author Ž. Tomovski.

2. Solution of the problem.

Let $F(z)=\int_z^\infty \frac{e^{-t}}{t^p}dt$, where $p>0$ and $Rez>0$. In [2], p.288-289, was proved the following identity

$$F(z)=e^{-z}\Big[\frac{1}{z^p}-\frac{p}{z^{p+1}}+\frac{p(p+1)}{z^{p+2}}-\ldots+(-1)^n\frac{p(p+1)(p+2)\cdots(p+n-1)}{z^{p+n}}\Big]$$
$$+(-1)^{n+1}p(p+1)(p+2)\cdots(p+n)\int_z^\infty\frac{e^{-t}}{t^{p+n+1}}dt. \qquad (2.1)$$

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We have

$$A=(\sqrt{\pi}+\int_0^i e^{-1/t^2}dt)e^{-1}=(\sqrt{\pi}+\int_0^1 e^{-1/t^2}dt+\int_1^i e^{-1/t^2}dt)e^{-1}.$$

Let us evaluate the integral $I=\int_0^1 e^{-1/t^2}dt$. By putting $u=1/t$ and applying the identity (2.1), we obtain

$$\begin{aligned} I=\frac{1}{2}\Big[e^{-1}\Big(1-\frac{3}{2}+\frac{3\cdot 5}{2^2}-\ldots+(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^n}\Big)\\ +(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n+3)}{2^{n+1}}\int_1^\infty \frac{e^{-u}}{u^{n+5/2}}du\Big] \end{aligned} \tag{2.2}$$

On the other hand,

$$\sqrt{\pi}=2\int_0^\infty e^{-u^2}du=2\int_0^1 e^{-u^2}du+2\int_1^\infty e^{-u^2}du.$$

It is obvious that $\int_0^1 e^{-t^2}dt=\sum_{n=0}^\infty \frac{(-1)^n}{n!}\frac{1}{2n+1}$. From (2.1), we get

$$\begin{aligned} \int_1^\infty e^{-u^2}du&=\frac{1}{2}\int_1^\infty e^{-t}t^{-1/2}dt\\ &=\frac{1}{2}\Big[e^{-1}\Big(1-\frac{1}{2}+\frac{1\cdot 3}{2^2}-\ldots+(-1)^n\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\Big)\\ &\quad+(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}}\int_1^\infty \frac{e^{-u}}{u^{n+3/2}}du\Big]. \end{aligned}$$

Integrating by parts, we obtain

$$\int_1^\infty \frac{e^{-u}}{u^{n+3/2}}du=e^{-1}-\frac{2n+3}{2}\int_1^\infty \frac{e^{-u}}{u^{n+5/2}}du.$$

Thus,

$$\begin{aligned} \int_1^\infty e^{-u^2}du=\frac{1}{2}\Big[e^{-1}\Big(1-\frac{1}{2}+\frac{1\cdot 3}{2^2}-\ldots+(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}}\Big)\\ +(-1)^{n+2}\frac{1\cdot 3\cdot 5\cdots(2n+3)}{2^{n+2}}\int_1^\infty \frac{e^{-u}}{u^{n+5/2}}du\Big]. \end{aligned}$$

Hence,

$$\begin{aligned} \sqrt{\pi}=2\sum_{s=0}^\infty \frac{(-1)^s}{s!}\frac{1}{2s+1}+e^{-1}\Big(1-\frac{1}{2}+\frac{1\cdot 3}{2^2}-\ldots+(-1)^{n+1}\frac{1\cdot 3\cdot 5\cdots(2n+1)}{2^{n+1}}\Big)\\ +(-1)^{n+2}\frac{1\cdot 3\cdot 5\cdots(2n+3)}{2^{n+2}}\int_1^\infty \frac{e^{-u}}{u^{n+5/2}}du. \end{aligned} \tag{2.3}$$

According to (2.2) and (2.3), we obtain

$$\sqrt{\pi}+\int_0^1 e^{-1/t^2}=2\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\frac{1}{2s+1}+e^{-1}. \tag{2.4}$$

But,

$$\begin{aligned}\int_1^i e^{-1/t^2}dt&=\int_1^i\Big(\sum_{s=0}^{\infty}\frac{1}{s!}(-1)^s\frac{1}{t^{2s}}\Big)dt\\&=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(2s-1)}-i\sum_{s=0}^{\infty}\frac{1}{s!(2s-1)}.\end{aligned} \tag{2.5}$$

Then, according to (2.4) and (2.5), we obtain

$$\begin{aligned}A&=\Big[2\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\frac{1}{2s+1}+\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(2s-1)}+e^{-1}-i\sum_{s=0}^{\infty}\frac{1}{s!(2s-1)}\Big]e^{-1}\\&=\Big[2\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\frac{1}{2s+1}-1+\sum_{s=0}^{\infty}\frac{(-1)^{s+1}}{(s+1)!(2s+1)}+e^{-1}-i\sum_{s=0}^{\infty}\frac{1}{s!(2s-1)}\Big]e^{-1}\\&=\Big[\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(2s+1)}\Big(2-\frac{1}{s+1}\Big)-1+e^{-1}-i\sum_{s=0}^{\infty}\frac{1}{s!(2s-1)}\Big]e^{-1}\\&=\Big[\sum_{s=0}^{\infty}\frac{(-1)^s}{(s+1)!}-1+e^{-1}-i\sum_{s=0}^{\infty}\frac{1}{s!(2s-1)}\Big]e^{-1}\\&=-i\sum_{s=0}^{\infty}\frac{1}{s!(2s-1)}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}.\end{aligned}$$

Let $a_s=\frac{(-1)^s}{s!}$,$b_s=\frac{1}{s!(2s-1)}$. Then,

$$\Big(\sum_{s=0}^{\infty}a_s\Big)\Big(\sum_{s=0}^{\infty}b_s\Big)=\sum_{s=0}^{\infty}c_s,$$

where

$$\begin{aligned}c_s&=\sum_{k=0}^{s}a_{s-k}b_k=\sum_{k=0}^{s}\frac{(-1)^{s-k}}{(s-k)!}\cdot\frac{1}{k!(2k-1)}\\&=\frac{(-1)^s}{s!}\sum_{k=0}^{s}\frac{(-1)^k}{2k-1}\frac{s!}{k!(s-k)!}\\&=\frac{(-1)^s}{s!}\sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{1}{2k-1}.\end{aligned}$$

Then we shall apply the following identity, obtained by D. S. Mitrinović and J. D. Kečkić in [1], p.146.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{ak+b}=\frac{n!a^n}{b(a+b)(2a+b)\cdots(na+b)}.$$

The left-hand side of this identity is in fact, equal to $(1/b)F(-n,b/a;1+b/a;1)$ and so it is merely a specialized version of the Chu-Vandermonde summation theorem for the finite Gauss hypergeometric series: $F(-n,b;c;1)$ with b replaced by b/a and $c=1+b/a$. If $a=2$ and $b=-1$, we obtain

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{1}{2k-1}=\frac{s!2^s}{(-1)\cdot(2-1)\cdot(2\cdot2-1)\cdots(2s-1)}=-\frac{s!2^s}{(2s-1)!!},$$

i.e.

$$c_s=(-1)^{s+1}\frac{2^s}{(2s-1)!!}.$$

Finally, $A=i\sum_{s=0}^{\infty}(-1)^s\frac{2^s}{(2s-1)!!}$,where $(2s-1)!!|_{s=0}=1$. In the last step we used that

$$(2s-1)!!=1\cdot3\cdot5\cdots(2s-1)=\frac{1\cdot2\cdot3\cdot4\cdots(2s-1)(2s)}{2\cdot4\cdot6\cdots(2s)}=\frac{(2s)!}{2^s s!}$$

and hence

$$(2s-1)!!|_{s=0}=\frac{(2s)!}{2^s s!}|_{s=0}=\frac{0!}{2^0 0!}=1.$$

That finishes the proof.

Remark. In [3] was proved the following numerical identity

$$1+\sum_{s=1}^{\infty}\frac{2^s}{(2s-1)!!}=e(\sqrt{\pi}+\int_0^1 e^{-1/t^2}dt).$$

Applying this identity, the proof of our problem can be refined. Indeed,

$$\begin{aligned}A&=(\sqrt{\pi}+\int_0^1 e^{-1/t^2}dt)e^{-1}+e^{-1}\int_1^i e^{-1/t^2}dt\\&=e^{-1}\Big[\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\sum_{s=0}^{\infty}\frac{2^s}{(2s-1)!!}+\sum_{s=0}^{\infty}\frac{(-1)^s}{s!(2s-1)}-i\sum_{s=0}^{\infty}\frac{1}{s!(2s-1)}\Big].\end{aligned}$$

But,

$$\begin{aligned}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\sum_{s=0}^{\infty}\frac{2^s}{(2s-1)!!}&=\sum_{s=0}^{\infty}\Big(\sum_{k=0}^{s}\frac{(-1)^{s-k}}{(s-k)!}\frac{2^k}{(2k-1)!!}\Big)\\&=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\sum_{k=0}^{s}(-1)^k\frac{s!}{(s-k)!k!}\frac{k!2^k}{(2k-1)!!}\\&=\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{(2k)!!}{(2k-1)!!}.\end{aligned}$$

Then, we shall apply the well-known formulae (see [1], p145)

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}f(k)=(-1)^n\Delta^n f(0),$$

where $f(k)=\frac{(2k)!!}{(2k-1)!!}$ and $\Delta^n f$ is the finite difference of f of order n. Applying the mathematical induction of n, we get

$$\Delta^n f(k)=(-1)^{n-1}\frac{(2k)!!}{(2n+2k-1)!!}(2n-3)!!, n\geq 2 \qquad \text{and } \Delta^1 f(k)=\frac{(2k)!!}{(2k+1)!!}.$$

Hence, $\Delta^n f(0)=(-1)^{n-1}\frac{(2n-3)!!}{(2n-1)!!}=\frac{(-1)^{n-1}}{2n-1}$,i.e.

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k}\frac{(2k)!!}{(2k-1)!!}=-\frac{1}{2s-1}.$$

Then

$$A=e^{-1}\Big(-\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s-1)s!}+\sum_{s=0}^{\infty}\frac{(-1)^s}{(2s-1)s!}-i\sum_{s=0}^{\infty}\frac{1}{s!(2s-1)}\Big)$$
$$=-ie^{-1}\sum_{s=0}^{\infty}\frac{1}{s!(2s-1)}=i\sum_{s=0}^{\infty}(-1)^s\frac{2^s}{(2s-1)!!}.$$

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