MEROMORPHIC FUNCTIONS WHOSE CERTAIN DIFFERENTIAL
POLYNOMIAL SHARE A POLYNOMIAL

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Abstract. In this paper, we use the idea of normal family to investigate the uniqueness problems of meromorphic functions when certain non-linear differential polynomial sharing a non-zero polynomial with certain degree. We obtain some results which will not only rectify the recent results of P. Sahoo and H. Karmakar [10] but also improve and generalize some recent results of L. Liu [7], H. Y. Xu, T. B. Cao and S. Liu [13] and P. Sahoo and H. Karmakar [10] in a large extent.

1. Introduction, Definitions and Results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let $f$ and $g$ be two non-constant meromorphic functions and let $a$ be a finite complex number. We say that $f$ and $g$ share a CM, provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. Similarly, we say that $f$ and $g$ share a IM, provided that $f - a$ and $g - a$ have the same zeros ignoring multiplicities. Let $k \in \mathbb{N} \cup \{0\} \cup \{\infty\}$. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all $a$-points of $f$, where an $a$-point of multiplicity $m$ is counted $m$ times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that $f$, $g$ share the value $a$ with weight $k$.

Let $m \in \mathbb{N} \cup \{\infty\}$ and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_m(a; f)$ the set of all $a$-points of $f$ with multiplicities not exceeding $m$, where an $a$-point is counted according to its multiplicity. Also we denote by $\overline{E}_m(a; f)$ the set of distinct $a$-points of $f(z)$ with multiplicities not greater than $m$. If $\overline{E}_m(a; f) = \overline{E}_m(a; g)$, we say that $a$ is a $m$-order pseudo common value of $f$ and $g$. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_\infty(a; f) = E_\infty(a; g)$ we say that $f$, $g$ share the value $a$ a CM (IM).

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We adopt the standard notations of value distribution theory (see [5]). For a non-constant meromorphic function \( f \), we denote by \( T(r, f) \) the Nevanlinna characteristic of \( f \) and by \( S(r, f) \) any quantity satisfying \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \) possibly outside a set of finite linear measure.

A meromorphic function \( a(z) \) is called a small function with respect to \( f \), if \( T(r, a) = S(r, f) \). We denote by \( T(r, f) \) the maximum of \( T(r, f) \) and \( T(r, g) \). The notation \( S(r) \) denotes any quantity satisfying \( S(r) = o(T(r)) \) as \( r \to \infty \), outside of a possible exceptional set of finite linear measure. Throughout this paper, we denote by \( \mu(f) \), \( \rho(f) \) and \( \lambda(f) \) the lower order of \( f \), the order of \( f \) and the exponent of convergence of zeros of \( f \) respectively (see [5, 15]).

Let \( f \) be a transcendental meromorphic function in the complex plane such that \( \rho(f) = \rho \leq \infty \). A complex number \( a \) is said to be a Borel exceptional value (see [15]) if 
\[
\limsup_{r \to \infty} \frac{\log^+ N(r, a; f)}{\log r} < \rho.
\]

For the sake of simplicity we also use the notations \( m^* := \chi_\mu m \), where \( \chi_\mu = 0 \) if \( \mu = 0 \), \( \chi_\mu = 1 \) if \( \mu \neq 0 \).

In 1959, W. K. Hayman (see [5], Corollary of Theorem 9) proved the following theorem.

**Theorem A.** Let \( f \) be a transcendental meromorphic function and \( n \in \mathbb{N} \setminus \{1, 2\} \). Then \( f^n f' = 1 \) has infinitely many solutions.

Theorem A was extended by Chen [3] in the following manner.

**Theorem B.** Let \( f \) be a transcendental entire function and \( n, k \in \mathbb{N} \) with \( n \geq k + 1 \). Then \( (f^n)^{(k)} - 1 \) has infinitely many zeros.

In 2002, Fang [4] proved the following result.

**Theorem C.** Let \( f, g \) be two non-constant entire functions and let \( n, k \in \mathbb{N} \) with \( n > 2k + 4 \). If \( (f^n)^{(k)} \) and \( (g^n)^{(k)} \) share 1 CM, then either \( f(z) = c_1 e^{cz} \) and \( g(z) = c_2 e^{-cz} \), where \( c, c_1, c_2 \in \mathbb{C} \) satisfying \((-1)^k (c_1 c_2)^n (nc)^{2k} = 1 \) or \( f \equiv t g \) for \( t \in \mathbb{C} \) such that \( t^n = 1 \).


**Theorem D.** Let \( f, g \) be two non-constant meromorphic functions and let \( n, m, k \in \mathbb{N} \) and \( \lambda, \mu \in \mathbb{C} \) such that \( |\lambda| + |\mu| \neq 0 \). If \( E_l(1, (f^n(\lambda f^m + \mu))^{(k)}) = E_l(1, (g^n(\lambda g^m + \mu))^{(k)}) \) and one of the following conditions holds:

1. \( l \geq 2 \) and \( n > 3m^* + 3k + 8 \);
2. \( l = 1 \) and \( n > 4m^* + 5k + 10 \);
3. \( l = 0 \) and \( n > 6m^* + 9k + 14 \).
Then

(i) when \(\lambda\mu \neq 0\), if \(m \geq 2\) and \(\delta(\infty; f) > \frac{3}{n+m}\), then \(f \equiv g\); if \(m = 1\) and \(\Theta(\infty; f) > \frac{3}{n+1}\), then \(f \equiv g\);

(ii) when \(\lambda\mu = 0\), if \(f, g \neq \infty\), then either \(f \equiv tg\), where \(t \in \mathbb{C}\) such that \(t^{n+m^*} = 1\) or \(f(z) = c_1e^{cz}\) and \(g(z) = c_2e^{-cz}\), where \(c, c_1, c_2 \in \mathbb{C}\) such that

\[(-1)^k \lambda^2(c_1c_2)^{n+m^*}((n+m^*)^2)^k = 1\] or \((-1)^k \mu^2(c_1c_2)^{n+m^*}((n+m^*)^2)^k = 1\).

Regarding Theorem D, following question arises.

**Question 1.** How two meromorphic functions \(f\) and \(g\) are related, if the condition

\[E_l(1,(f^n(\mu f^m + \lambda)))^{(k)} = E_l(1,(g^n(\mu g^m + \lambda)))^{(k)}\]

in Theorem D is replaced with the condition

\[E_l(1,(f^n(\mu f^m + \lambda)))^{(k)} = E_l(1,(g^n(\mu g^m + \lambda)))^{(k)}\]?

In 2012, Xu et al. [13] answer the above question by proving the following results which also improve Theorem D in some sense.

**Theorem E.** Let \(f, g\) be two non-constant meromorphic functions and let \(n, m, k \in \mathbb{N}\) with \(n > \frac{113}{3}k + \frac{26}{3}m^* + \frac{28}{3}\) and \(\lambda, \mu \in \mathbb{C}\) such that \(|\lambda| + |\mu| \neq 0\). If \(E_l(1,(f^n(\mu f^m + \lambda)))^{(k)} = E_l(1,(g^n(\mu g^m + \lambda)))^{(k)}\) and \(E_l(1,(f^n(\mu f^m + \lambda)))^{(k)} = E_l(1,(g^n(\mu g^m + \lambda)))^{(k)}\), then the conclusions of Theorem D still hold.

**Theorem F.** Let \(f, g\) be two non-constant meromorphic functions and let \(n, m, k \in \mathbb{N}\) with \(n > 3k + 3m^* + 6\) and \(\lambda, \mu \in \mathbb{C}\) such that \(|\lambda| + |\mu| \neq 0\). If \(E_l(1,(f^n(\mu f^m + \lambda)))^{(k)} = E_l(1,(g^n(\mu g^m + \lambda)))^{(k)}\) and \(E_l(1,(f^n(\mu f^m + \lambda)))^{(k)} = E_l(1,(g^n(\mu g^m + \lambda)))^{(k)}\), then the conclusions of Theorem D still hold.

Observing Theorems E and F, Sahoo and Karmakar [10] asked the following question.

**Question 2.** What can be said about the relationship between two meromorphic functions \(f\) and \(g\), if \((f^n P(f))^{(k)}\) and \((g^n P(g))^{(k)}\) share a non-zero polynomial, where \(P(z) = \sum_{i=0}^{m} a_i z^i\) is any non-zero polynomial, \(a_0, a_1, \ldots, a_m \in \mathbb{C}\)?

Let us define \(m^{**} = m\), if \(P(z) \neq a_0; m^{**} = 0\), if \(P(z) \equiv a_0\).

In the direction of the above question, Sahoo and Karmakar [10] obtained the following results.

**Theorem G.** Let \(f, g\) be two non-constant meromorphic functions, \(p\) be a non-zero polynomial of degree \(q\) and \(n, k \in \mathbb{N}\), \(m \in \mathbb{N} \cup \{0\}\) with \(n > \max\{\frac{113}{3}k + \frac{26}{3}m^* + \frac{28}{3}, k+2q\}\). Suppose that either \(k, q\) are co-prime or \(k > q\), when \(q \geq 2\). Let \(E_l(1,(p f^q P(f)))^{(k)} = E_l(1,(p g^q P(g)))^{(k)}\) and \(E_l(1,(p f^q P(f)))^{(k)} = E_l(1,(p g^q P(g)))^{(k)}\), where \(P(z) = \sum_{i=0}^{m} a_i z^i\) is any non-zero polynomial and \(l \in \mathbb{N} \setminus \{1, 2\}\). Then the following conclusions hold.
(i) If \( P(z) = \sum_{i=0}^{m} a_i z^i \) is not a monomial, then either \( f \equiv t g, t \in \mathbb{C} \) such that \( t^d = 1 \), where 
\[ d = (n + m) - i \leq n \] 
\( a_{m-i} \neq 0 \) for some \( i = 0, 1, 2, \ldots, m \) or \( f \) and \( g \) satisfy the 

algebraic equation \( R(f, g) = 0 \), where \( R(f, g) \) is given by 

\[
R(w_1, w_2) = w_1^n (a_m w_1^m + \ldots + a_1 w_1 + a_0) - w_2^n (a_m w_2^m + \ldots + a_1 w_2 + a_0).
\]

(ii) When \( P(z) = a_0 \) or \( P(z) = a_m z^m \), then either \( f \equiv t g, t \in \mathbb{C} \) such that \( t^{n+m} = 1 \) or \( f(z) = b_1 e^{bQ(z)} \) and \( g(z) = b_2 e^{-bQ(z)} \), where \( Q \) is a polynomial without constant such that \( Q' = p; b, b_1, b_2 \in \mathbb{C} \) such that 

\[
a_0^2 (nb)^2 (b_1 b_2)^n = 1 \text{ or } a_m^2 ((n + m)b)^2 (b_1 b_2)^{n+m} = -1.
\]

**Theorem H.** Let \( f, g \) be two transcendental meromorphic functions, \( p \) be a non-zero polynomial of degree \( q \) and \( n, k \in \mathbb{N} \), \( m \in \mathbb{N} \cup \{0\} \) with \( n > \max\{3k + 3m^2 + 6, k + 2q\} \). Suppose that 
either \( k, q \) are co-prime or \( k > q \), when \( q \geq 2 \). \( \ell \in \mathbb{N} \setminus \{1, 2, 3\} \). Then the conclusions of Theorem G still hold.

**Remark 1.** In the proof of Theorem 1 [10], one can easily point out a number of gaps.

Firstly the authors [10] declare that Lemma 10 [10] can be proved in the line of the proof of Lemma 9 [21]. But this is not possible here. Actually in Lemma 9 [21], \( f, g \) share \( \infty \) IM. But in Lemma 10 [10], authors did not consider the condition “\( f, g \) share \( \infty \) IM”. Therefore existence of Lemma 10 [10] is questionable here.

Secondly in the proof of Lemma 11 [10] there is a big gap. From the relation 

\[
(a_m f^{n+m})'(a_m g^{n+m})' \equiv p^2
\]

authors conclude that \( f = e^x \) and \( g = e^y \). Again from the relation 

\[
(a_m f^{n+m})^{(k)}(a_m g^{n+m})^{(k)} \equiv p^2
\]  

(1.1)

authors conclude that 

\[
N(r, \infty; a_m f^{n+m}) + N(r, 0; a_m f^{n+m}) = O(\log r).
\]

(1.2)

The calculations are not true. A question arises: When zeros of \( f(g) \) are neutralized by the poles of \( g(f) \)? Actually the authors did not consider this case. As for example we consider the case. Suppose \( k = 4, m = 1, q = 1 \) and \( n = 7 \). Let \( z_0 \) be a zero of \( f \) of multiplicity 2. One can easily think that \( z_0 \) is a simple pole of \( g \). It is clear that \( z_0 \) is a zero of \( (a_m f^{n+m})^{(k)} \) of multiplicity 12 and a pole of \( (a_m g^{n+m})^{(k)} \) of multiplicity 12. This shows that zeros of \( f(g) \) can be neutralized by the poles of \( g(f) \). Also poles of \( f \) can be neutralized by the zeros of \( (a_m g^{n+m})^{(k)} \), but not the zeros of \( g \). As a result from (1.1) we can not easily arrive at (1.2). Therefore existence of Lemma 11 [10] is questionable here.
Finally, since Lemmas 9 and 10 [10] play an important role in proving Theorems 1 and 2 [10], so existence of Theorem 1 [10] as well as Theorem 2 [10] are questionable here.

The above discussion is sufficient enough to make oneself inquisitive to investigate the accurate form of Theorems G and H. In this paper we study Theorem 1 [10] as well as Theorem 2 [10] again in more general form without help of Lemma 10 [10] as well as Lemma 11 [10].

Also it is quite natural to ask the following questions.

**Question 3.** Can the lower bound of \( n \) be further reduced in Theorems G and H?

**Question 4.** Can one remove the condition “Suppose that either \( k, q \) are co-prime or \( k > q \), when \( q \geq 2 \)” in Theorems G and H?

**Question 5.** Can one remove the condition \( f \neq \infty, g \neq \infty \) keeping all the conclusions intact when \( \lambda \mu = 0 \) in Theorems D, E, F?

2. **Main results**

In this paper, we always use \( P(z) \) denoting an arbitrary polynomial of degree \( n \) as follows:

\[
P(z) = \sum_{i=0}^{n} a_i z^i = a_n \prod_{i=1}^{s} (z - c_i)^{l_i}, \tag{2.1}
\]

where \( a_0, a_1, \ldots, a_n (\neq 0) \in \mathbb{C} \) and \( c_{l_i} \in \mathbb{C} \) are distinct and \( l_1, l_2, \ldots, l_s, s, n, k \in \mathbb{N} \) such that \( \sum_{i=1}^{s} l_i = n \). Also we let \( l = \max(l_1, l_2, \ldots, l_s) \) and \( e \) be the zero of \( P(z) \) of multiplicity \( l \).

From (2.1) we have \( P(z) = (z - e)^l P_*(z) \), where \( P_*(z) \) is a polynomial in degree \( n - l = m(\geq 0) \), say. We also use \( P_1(z_1) \) as an arbitrary non-zero polynomial defined by

\[
P_1(z_1) = a_n \prod_{i=1}^{s} (z_1 + e - c_i)^{l_i} = \sum_{i=0}^{m} b_i z_1^i, \tag{2.2}
\]

where \( z_1 = z - e \) and \( \deg(P_1(z_1)) = m \geq 0 \). Obviously \( P(z) = z_1^l P_1(z_1) \).

Taking the possible answers of the above questions into backdrop we obtain the following results which are not only rectify Theorems G, H, but also improve and generalize Theorems D-H.

**Theorem 1.** Let \( f, g \) be two transcendental meromorphic functions and \( p \) be a non-zero polynomial with \( \deg(p) \leq l - 1 \), where \( m \in \mathbb{N} \cup \{0\}, k, l \in \mathbb{N} \) such that \( l > \frac{13}{3} k + \frac{11}{3} m + \frac{28}{3} \). Suppose \( \overline{E}_l(p, (P(f))^k) = \overline{E}_l(p, (P(g))^k) \) and \( E_{1l}(p, (P(f))^k) = E_{1l}(p, (P(g))^k) \), where \( P(z) \) is defined as in (2.1) and \( l \in \mathbb{N} \setminus \{1, 2\} \). Now

(I) when \( P_1(z_1) \) is not a monomial, then one of the following three cases holds
(II) \( f(z) - e \equiv t(g(z) - e) \) for \( t \in \mathbb{C} \) such that \( t^{d_0} = 1 \), where 
\[ d_0 = \text{GCD}(l + m, \ldots, l + m - i, \ldots, l), \quad b_{m-i} \neq 0 \text{ for some } i = 0, 1, \ldots, m; \]

(II) \( f_1 = f - e \) and \( g_1 = g - e \) satisfy the algebraic equation \( R(f_1, g_1) = 0 \), where 
\[ R(a_1, a_2) = a_1^l(b_m b_m^{m-1} + \cdots + b_0) - a_2^l(b_m b_m^{m-1} + \cdots + b_0); \]

(II) when \( P_1(z_1) \) is a monomial, say \( P_1(z_1) = b_1z_1^i \neq 0 \), where \( i \in \{0, 1, \ldots, m\} \), then one of the following two cases holds

(II1) \( f - e \equiv t(g - e) \) for \( t \in \mathbb{C} \) such that \( t^{l+i} = 1 \),

(II2) if \( p \notin \mathbb{C} \), then \( f(z) = c_1 e^{Q(z)} + e \) and \( g(z) = c_2 e^{-Q(z)} + e \), where \( Q(z) = \int_0^z p(t) \, dt \), \( c, c_1, c_2 \in \mathbb{C} \) such that \( b_1^2(c_1 c_2)^{l+i}(l+i+1) = -1 \); if \( p(z) = b \in \mathbb{C} \setminus \{0\} \), then \( f(z) = c_3 e^{cz} + e \) and \( g(z) = c_4 e^{-cz} + e \), where \( c, c_3, c_4 \in \mathbb{C} \) such that \((-1)^k b_1^2(c_3 c_4)^{l+i}(l+i+1) = b^2\).

In particular when \( \rho(f) > 2 \) and \( p \in \mathbb{C} \setminus \{0\} \), then (II) does not hold.

**Theorem 2.** Let \( f, g \) be two transcendental meromorphic functions and \( p \) be a non-zero polynomial with \( \deg(p) \leq l - 1 \), where \( m \in \mathbb{N} \cup \{0\} \), \( k, l \in \mathbb{N} \) such that \( l > 3k + 3m + 6 \). Suppose \( E_1(p, (P(f))^{(k)}) = E_1(p, (P(g))^{(k)}) \) and \( E_2(p, (P(f))^{(k)}) = E_2(p, (P(g))^{(k)}) \), where \( P(z) \) is defined as in (2.1) and \( l \in \mathbb{N} \setminus \{1, 2, 3\} \). Then the conclusion of Theorem 1 holds.

With the help of Theorem 1.5 [8] and Theorem 1 we get the following corollary immediately.

**Corollary 1.** Let \( f, g \) be two transcendental meromorphic functions and \( p \) be a non-zero polynomial with \( \deg(p) \leq l - 1 \), where \( m \in \mathbb{N} \cup \{0\} \), \( k, l \in \mathbb{N} \) such that \( l > 3k + m + 8 \). Suppose \( (P(f))^{(k)} - p \) and \( (P(g))^{(k)} - p \) share \( (0,2) \). Then the conclusion of Theorem 1 holds.

We now explain some definitions and notations which are used in the paper.

**Definition 1 ([6]).** For \( a \in \mathbb{C} \cup \{\infty\} \) we denote by \( N(r, a; f \geq 1) \) the counting function of simple \( a \) points of \( f \). For \( m \in \mathbb{N} \) we denote by \( N(r, a; f \leq m) \) \( N(r, a; f \geq m) \) the counting function of those \( a \) points of \( f \) whose multiplicities are not greater (less) than \( m \) where each \( a \) point is counted according to its multiplicity. \( \overline{N}(r, a; f \leq m) \) \( \overline{N}(r, a; f \geq m) \) are defined similarly, where in counting the \( a \)-points of \( f \) we ignore the multiplicities. Also \( N(r, a; f \leq m) \) \( N(r, a; f \geq m) \) \( \overline{N}(r, a; f \leq m) \) \( \overline{N}(r, a; f \geq m) \) are defined analogously.

**Definition 2 ([17]).** For \( a \in \mathbb{C} \cup \{\infty\} \) and \( p \in \mathbb{N} \) we let \( N_p(r, a; f) = \sum_{i=1}^p \overline{N}(r, a; f \geq i). \)
**Definition 3.** Let \(a, b \in \mathbb{C} \cup \{\infty\}\) and \(p \in \mathbb{N}\). We denote by \(N(r, a; f | \geq p | g = b) = \overline{N}(r, a; f | \geq p | g \neq b)\) the reduced counting function of those \(a\)-points of \(f\) with multiplicities \(\geq p\), which are the \(b\)-points (not the \(b\)-points) of \(g\).

**Definition 4 ([5]).** Let \(a \in \mathbb{C} \cup \{\infty\}\) and \(k \in \mathbb{N}\). We define

\[
\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)} \quad \text{and} \quad \delta_k(a; f) = 1 - \limsup_{r \to \infty} \frac{N_k(r, a; f)}{T(r, f)}.
\]

### 3. Lemmas

Let \(h\) be a meromorphic function in \(\mathbb{C}\). Then \(h\) is called a normal function if there exists a positive real number \(M\) such that \(h^\#(z) \leq M\), for all \(z \in \mathbb{C}\), where

\[
h^\#(z) = \frac{|h'(z)|}{1 + |h(z)|^2}
\]

denotes the spherical derivative of \(h\). Let \(\mathcal{F}\) be a family of meromorphic functions in a domain \(D \subset \mathbb{C}\). We say that \(\mathcal{F}\) is normal in \(D\) if every sequence \(\{f_n\}_n \subseteq \mathcal{F}\) contains a subsequence which converges spherically and uniformly on the compact subsets of \(D\) (see [11]).

Let \(F\) and \(G\) be two non-constant meromorphic functions defined in \(\mathbb{C}\). We denote by \(H\) the function as follows:

\[
H = \left(\frac{F''}{F'} - \frac{2G'}{F - 1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G - 1}\right). \tag{3.1}
\]

**Lemma 1 ([14]).** Let \(f\) be a non-constant meromorphic function and let \(a_n(z)(\neq 0)\), \(a_{n-1}(z)\), \ldots, \(a_0(z)\) be meromorphic functions such that \(T(r, a_i(z)) = S(r, f)\) for \(i = 0, 1, 2, \ldots, n\). Then

\[
T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).
\]

**Lemma 2 ([20]).** Let \(f\) be a non-constant meromorphic function and \(p, k \in \mathbb{N}\). Then

\[
N_p \left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f),
\]
\[
N_p \left(r, 0; f^{(k)}\right) \leq kN(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).
\]

**Lemma 3 ([12]).** Let \(f, g\) be two non-constant meromorphic functions and \(k, n \in \mathbb{N}\) such that \(n > 2k + 1\). If \((f^n)^{(k)} \equiv (g^n)^{(k)}\), then \(f \equiv tg\) for \(t \in \mathbb{C}\) such that \(t^n = 1\).

**Lemma 4.** Let \(f, g\) be two non-constant meromorphic functions. Let \(l, k \in \mathbb{N}\) and \(m \in \mathbb{N} \cup \{0\}\) such that \(l > m + 3k\). Suppose \((P(f))^{(k)} \equiv (P(g))^{(k)},\) where \(P(z)\) be defined as in (2.1). Now

(I) when \(P_1(z_1)\) is not a monomial, then one of the following two cases holds:
(ii) \( f(z) - e \equiv t(g(z) - e) \) for \( t \in \mathbb{C} \) such that \( t^{d_0} = 1 \), where 
\[ d_0 = \text{GCD}(l + m, \ldots, l + m - i, \ldots, l), \] 
where \( b_{m-i} \neq 0 \) for some \( i = 0, 1, \ldots, m \);

(iii) \( f_1 = f - e \) and \( g_1 = g - e \) satisfy the equation \( R(f_1, g_1) = 0 \), where
\[ R(\omega_1, \omega_2) = \omega_1^n - \omega_2^n = (b_m \omega_1^m + b_{m-1} \omega_1^{m-1} + \ldots + b_0) - (b_m \omega_2^m + b_{m-1} \omega_2^{m-1} + \ldots + b_0). \]

(II) when \( P_1(z_1) \) is a monomial, say \( P_1(z_1) = b_1 z_1^i \not= 0 \), where \( i \in \{0, 1, \ldots, m\} \), then 
\( f - e \equiv t(g - e) \) for \( t \in \mathbb{C} \) such that \( t^{l+i} = 1 \).

**Proof.** We have \( (P(f))^{(k)} \equiv (P(g))^{(k)} \). Integrating we get \( (P(f))^{(k-1)} \equiv (P(g))^{(k-1)} + c_{k-1} \). If possible suppose \( c_{k-1} \neq 0 \). Now in view of Lemma 2 for \( p = 1 \) and using the second fundamental theorem we get

\[
n T(r, f) = T(r, P(f)) + O(1)
\leq T(r, (P(f))^{(k-1)}) - N(r, 0; (P(f))^{(k-1)}) + N_k(r, 0; P(f)) + S(r, f)
\leq \frac{N(r, 0; (P(f))^{(k-1)}) + N(r, \infty; f) + \frac{1}{N(r, c_{k-1}; (P(f))^{(k-1)})} - N(r, 0; (P(f))^{(k-1)})}{N_k(r, 0; P(f)) + S(r, f)}
\leq \frac{N(r, \infty; f) + (k-1)N(r, \infty; g) + N_k(r, 0; P(g)) + kN(r, 0; P(f)) + S(r, f)}{N(r, \infty; f) + (k-1)N(r, \infty; g) + kN(r, \infty; g) + N(r, 0; P(g) \mid g \neq e) + kN(r, \infty; g)}
\leq \frac{N(r, 0; P(f) \mid f \not= e) + S(r, f)}{(n-l+k+1) T(r, f) + (n-l+2k-1) T(r, g) + S(r, f) + S(r, g)}
\leq (2n-2l+3k) T(r) + S(r).
\]

Similarly we get \( n T(r, g) \leq (2n-2l+3k) T(r) + S(r) \). Combining these we get \( (2l-n-3k) T(r) \leq S(r) \), which is a contradiction since \( l > m+3k \). Therefore \( c_{k-1} = 0 \). So \( (P(f))^{(k-1)} \equiv (P(g))^{(k-1)} \). Proceeding in this way we get \( (P(f))' \equiv (P(g))' \). Integrating we get \( P(f) \equiv P(g) + c_0 \). If possible let \( c_0 \neq 0 \). Using the second fundamental theorem we get

\[
n T(r, f) = T(r, P(f)) + O(1)
\leq \frac{N(r, 0; P(f)) + N(r, \infty; P(f)) + N(r, c_0; P(f))}{N(r, 0; P(f)) + N(r, \infty; P(f))}
\leq \frac{(n-l+2) T(r, f) + (n-l+1) T(r, g) + S(r, f)}{(n-l+2) T(r, f) + (n-l+1) T(r, g) + S(r, f)}
\leq (2n-2l+3) T(r) + S(r).
\]

Similarly we get \( n T(r, g) \leq (2n-2l+3) T(r) + S(r) \). Combining these we get \( (2l-n-3) T(r) \leq S(r) \), which is a contradiction since \( l > m+3 \). Therefore \( c_0 = 0 \) and so \( P(f) \equiv P(g) \), i.e.,

\[
f_1^r (b_m f_1^m + b_{m-1} f_1^{m-1} + \ldots + b_0) \equiv g_1^r (b_1 g_1^m + b_{m-1} g_1^{m-1} + \ldots + b_0), \quad (3.2)
\]
where \( f_1 = f - e \) and \( g_1 = g - e \). Suppose \( P_1(z_1) \) is not a monomial.

Let \( h = \frac{f}{g_1} \). If \( h \) is a constant, then substituting \( f_1 = g_1 h \) into (3.2) we deduce that

\[
b_{m}g_{1}^{l+m}(h^{l+m}-1) + b_{m-1}g_{1}^{b+m-1}(h^{l+m-1}-1) + \ldots + c_{0}g_{1}^{l}(h^{l}-1) \equiv 0,
\]

which implies \( h^{d_0} = 1 \), where \( d_0 = GCD(l + m, \ldots, l + m - i, \ldots, l) \), \( b_{m-i} \neq 0 \) for some \( i = 0, 1, \ldots, m \). Thus \( f_1 \equiv tg_1 \), i.e., \( f(z) - e \equiv t(g(z) - e) \) for \( t \in \mathbb{C} \) such that \( t^{d_0} = 1 \), where \( d_0 = GCD(l + m, \ldots, l + m - i, \ldots, l) \), \( b_{m-i} \neq 0 \) for some \( i = 0, 1, \ldots, m \). If \( h \) is not a constant, then we know by (3.2) that \( f_1 \) and \( g_1 \) satisfying the equation \( R(f_1, g_1) = 0 \), where \( R(w_1, w_2) = \omega_1(b_m \omega_1^{m} + b_{m-1} \omega_1^{m-1} + \ldots + b_0) - \omega_2(b_m \omega_2^{m} + b_{m-1} \omega_2^{m-1} + \ldots + b_0) \).

Suppose \( P_1(z_1) \) is a monomial, say \( P_1(z_1) = b_i z_1^i \neq 0 \), where \( i \in \{0, 1, \ldots, m\} \). Then by Lemma 3 we have \( f \equiv t(g - e) \) for \( t \in \mathbb{C} \) such that \( t^{l+1} = 1 \). This proves the proof.

\[\square\]

**Lemma 5.** Let \( f, g \) be two non-constant meromorphic functions and let \( F = (P(f))^{(k)}/\alpha, G = (P(g))^{(k)}/\alpha \), where \( P(z) \) be defined as in (2.1), \( \alpha \) be a small function with respect to \( f, g \) and \( m \in \mathbb{N} \cup \{0\}, k, l \in \mathbb{N} \) such that \( l > m + 3k + 3 \). Suppose \( H \equiv 0 \). Then either \((P(f))^{(k)}(P(g))^{\alpha} \equiv \alpha^2\), where \((P(f))^{(k)} - \alpha \) and \((P(g))^{(k)} - \alpha \) share 0 CM or \((P(f))^{(k)}(P(g))^{\alpha} \equiv (P(g))^{(k)}\).

**Proof.** We have \( H \equiv 0 \). By integration, we get \( \frac{F'}{(F-1)^2} \equiv d \frac{G'}{(G-1)^2} \), where \( d \in \mathbb{C} \setminus \{0\} \), i.e.,

\[
\left(\frac{F_1-a}{a}\right)' \equiv d \left(\frac{G_1-a}{a}\right)',
\]

where \( F_1 = (P(f))^{(k)} \) and \( G_1 = (P(g))^{(k)} \). This shows that \( \frac{F_1-a}{a} \) and \( \frac{G_1-a}{a} \) share 0 CM and so \( F_1 - \alpha \) and \( G_1 - \alpha \) share 0 CM. Finally, by integration we get

\[
\frac{1}{F-1} \equiv \frac{bG + a - b}{G-1}, \tag{3.3}
\]

where \( a(\neq 0), b \in \mathbb{C} \). We now consider the following cases.

**Case 1.** Let \( b \neq 0 \) and \( a \neq b \). If \( b = -1 \), from (3.3) we have \( F \equiv \frac{-a}{G-a-1} \). Therefore \( \overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + S(r, f) \). So in view of Lemma 2 and the second fundamental theorem we get

\[
n T(r, g) = T(r, P(f)) + O(1)
\]

\[
\leq T(r, G) + N_{k+1}(r, 0; P(g)) - \overline{N}(r, 0; G)
\]

\[
\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, a+1; G) + N_{k+1}(r, 0; P(g)) - \overline{N}(r, 0; G) + S(r, g)
\]

\[
\leq \overline{N}(r, \infty; g) + (k+1)\overline{N}(r, e; g) + N(r, 0; P(g) \mid g \neq e) + \overline{N}(r, \infty; f) + S(r, g)
\]

\[
\leq T(r, f) + (n - l + k + 2) T(r, g) + S(r, f) + S(r, g).
\]
Suppose that there exists a set $I$ with infinite measure such that $T(r, f) \leq T(r, g)$ for $r \in I$. So for $r \in I$ we have $(l - k - 3) T(r, g) \leq S(r, g)$, which is a contradiction since $l > k + 3$.

If $b \neq -1$, from (3.3) we obtain that $F - (1 + \frac{1}{b}) \equiv \frac{-a}{b^r(G+a-b)}$. So $\overline{N}(r, \frac{b-a}{b}; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + S(r, f)$. Using Lemma 2 and the same argument as used in the case when $b = -1$ we can get a contradiction.

**Case 2.** Let $b \neq 0$ and $a = b$. If $b = -1$, then from (3.3) we have $FG \equiv a^2$, i.e., $(P(f))^{(k)}(P(g))^{(k)} \equiv a^2$, where $(P(f))^k - a$ and $(P(g))^k - a$ share 0 CM.

If $b \neq -1$, from (3.3) we have $\frac{1}{F} \equiv \frac{bG}{(1+bG-1)}$. Therefore $\overline{N}(r, \frac{1}{1+b}; G) = \overline{N}(r, 0; F)$. So in view of Lemma 2 and the second fundamental theorem we get

$$
n T(r, g) \leq T(r, G) + N_{k+1}(r, 0; P(g)) - \overline{N}(r, 0; G) + S(r, g)
$$

$$
\leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, \frac{1}{1+b}; G) + N_{k+1}(r, 0; P(g)) - \overline{N}(r, 0; G) + S(r, g)
$$

$$
\leq \overline{N}(r, \infty; G) + N_{k+1}(r, 0; P(g)) + N_{k+1}(r, 0; P(f)) + k\overline{N}(r, \infty; f) + S(r, f) + S(r, g)
$$

$$
\leq (n - l + k + 2) T(r, g) + (n - l + 2k + 1) T(r, f) + S(r, f) + S(r, g).
$$

So for $r \in I$ we have $(2l - n - 3k - 3) T(r, g) \leq S(r, g)$, which is a contradiction since $l > m+3k+3$.

**Case 3.** Let $b = 0$. From (3.3) we obtain $F \equiv \frac{G+a-1}{a}$. If $a \neq 1$ then we obtain $\overline{N}(r, 1 - a; G) = \overline{N}(r, 0; F)$. We can similarly deduce a contradiction as in Case 2. Therefore $a = 1$ and so $F \equiv G$, i.e., $(P(f))^{(k)} \equiv (P(g))^{(k)}$. This completes the proof.

**Lemma 6** ([15], Theorem 1.24). Suppose that $f$ is a non-constant meromorphic function in the complex plane and $k \in \mathbb{N}$. Then

$$
N(r, 0; f^{(k)}) \leq N(r, 0; f) + k\overline{N}(r, \infty; f) + O(\log T(r, f) + \log r),
$$

as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

**Lemma 7** ([5], Lemma 3.5). Suppose that $F$ is meromorphic in a domain $D$ and set $f = \frac{F}{T}$. Then for $n \in \mathbb{N}$,

$$
\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2} f' + a_n f^{n-3} f'' + b_n f^{n-4} (f')^2 + P_{n-3}(f),
$$

where $a_n = \frac{1}{n(n-1)(n-2)}$, $b_n = \frac{1}{n(n-1)(n-2)(n-3)}$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

**Lemma 8** ([2], Lemma 1). Let $f$ be a meromorphic function on $\mathbb{C}$. If $f$ has bounded spherical derivative on $\mathbb{C}$, then $\rho(f) \leq 2$. If in addition $f$ is entire, then $\rho(f) \leq 1$. 
Lemma 9 ([15], Theorem 2.11). Let $f$ be a transcendental meromorphic function in the complex plane such that $\rho(f) > 0$. If $f$ has two distinct Borel exceptional values in the extended complex plane, then $\mu(f) = \rho(f)$ and $\rho(f) \in \mathbb{N} \cup \{\infty\}$.

Lemma 10 ([18]). Let $F$ be a family of meromorphic functions in the unit disc $\Delta$ such that all zeros of functions in $F$ have multiplicity greater than or equal to $l$ and all poles of functions in $F$ have multiplicity greater than or equal to $j$ and $\alpha$ be a real number satisfying $-l < \alpha < j$. Then $F$ is not normal in any neighborhood of $z_0 \in \Delta$, if and only if there exist

(i) points $z_n \in \Delta$, $z_n \to z_0$,

(ii) positive numbers $\rho_n$, $\rho_n \to 0^+$ and

(iii) functions $f_n \in F$,

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where $g$ is a non-constant meromorphic function. The function $g$ may be taken to satisfy the normalisation $g^0(\zeta) = g^0(0) = 1(\zeta \in \mathbb{C})$.

Lemma 11. Let $f, g$ be two transcendental entire functions and $p$ be a non-zero polynomial with $\deg(p) \leq n - 1$, where $n, k \in \mathbb{N}$ such that $n > \max\{2k, k + 2\}$. Suppose $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$, where $(f^n)^{(k)} - p$ and $(g^n)^{(k)} - p$ share CM. Now (i) if $p \not\equiv 0 \in \mathbb{C}$, then $f(z) = c_1 e^{cQ(z)}$ and $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t) dt$, $c, c_1, c_2 \in \mathbb{C}$ such that $(nc)^2(c_1 c_2)^n = -1$; (ii) if $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = c_3 e^{dz}$ and $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$.

Proof. The proof of lemma follows from the proof of Lemma 11 [1].

Lemma 12. Let $f, g$ be two transcendental meromorphic functions and $p$ be a non-zero polynomial with $\deg(p) \leq n - 1$, where $n, k \in \mathbb{N}$ such that $n > \max\{2k, k + 2\}$. Suppose $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$, where $(f^n)^{(k)} - p$ and $(g^n)^{(k)} - p$ share 0 CM. Now (i) if $p \not\equiv 0 \in \mathbb{C}$, then $f(z) = c_1 e^{cQ(z)}$ and $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t) dt$, $c, c_1, c_2 \in \mathbb{C}$ such that $(nc)^2(c_1 c_2)^n = -1$; (ii) if $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = c_3 e^{dz}$ and $g(z) = c_4 e^{-dz}$, where $c_3, c_4, d \in \mathbb{C}$ such that $(-1)^k (c_3 c_4)^n (nd)^{2k} = b^2$.

Proof. Suppose

$$(f^n)^{(k)}(g^n)^{(k)} \equiv p^2. \tag{3.4}$$

We consider the following cases.

Case 1. Suppose $\infty$ is a Picard exceptional value of both $f$ and $g$. Then $f$ and $g$ are transcendental entire functions. Remaining part follows from Lemma 11.

Case 2. Suppose $\infty$ is not a Picard exceptional value of either $f$ or $g$, or both of $f$ and $g$.

First we suppose $\infty$ is a Picard exceptional value of $g$. Let $z_0$ be a zero of $f$ with multiplicity $q_0$. Clearly $z_0$ is a zero of $(f^n)^{(k)}$ with multiplicity $nq_0 - k$. Now from (3.4) we observe that
$z_0$ must be a zero of $p$ with multiplicity $nq_0 - k$. Since $\deg(p) \leq n - 1$ and $n > 2k$, from (3.4) we conclude that either $f$ has no zeros or $f$ has finitely many zeros. Therefore in either cases we have $N(r, 0; f) = O(\log r)$ as $r \to \infty$.

Next we suppose $\infty$ is not a Picard exceptional value of $g$. Let $z_1 (p(z_1) \neq 0)$ be a zero of $f$ with multiplicity $q_1$. Clearly $z_1$ is a zero of $(f^n)^{(k)}$ with multiplicity $nq_1 - k$. From (3.4) we observe that $z_1$ must be a pole of $g$ with multiplicity $r_1$, say. Note that $z_1$ is a pole of $(g^n)^{(k)}$ with multiplicity $nr_1 + k$. Therefore $nq_1 - k = nr_1 + k$ and so $q_1 > r_1$. Now $nq_1 - k = nr_1 + k$ implies that $n(q_1 - r_1) = 2k$. Since $n > 2k$, we arrive at a contradiction. This shows that $z_1$ is a zero of $p$ and so we have $N(r, 0; f) = O(\log r)$ as $r \to \infty$. Similarly we can prove that $N(r, 0; g) = O(\log r)$ as $r \to \infty$.

Let $H = f^n, \tilde{H} = g^n, F = \frac{H}{p}$ and $G = \frac{\tilde{H}}{p}$. Let $\mathcal{F} = \{F_\omega\}$ and $\mathcal{G} = \{G_\omega\}$, where $F_\omega(z) = F(z + \omega) = \frac{H(z + \omega)}{p(z + \omega)}$ and $G_\omega(z) = G(z + \omega) = \frac{\tilde{H}(z + \omega)}{p(z + \omega)}$, $z \in \mathbb{C}$. Clearly $\mathcal{F}$ and $\mathcal{G}$ are two families of meromorphic functions defined on $\mathbb{C}$. We now consider following two sub-cases.

Sub-case 2.1. Suppose one of the families $\mathcal{F}$ and $\mathcal{G}$, say $\mathcal{F}$, is normal on $\mathbb{C}$. Then by Marty’s theorem $F^#(\omega) = F_\omega^#(0) \leq M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence by Lemma 8 we have $\rho(F) \leq 2$. Now from (3.4) we have

$$\rho(f) = \rho\left(\frac{f^n}{p}\right) = \rho(f^n) = \rho((f^n)^{(k)}) = \rho((g^n)^{(k)}) = \rho(g^n) = \rho\left(\frac{g^n}{p}\right) = \rho(g) \leq 2. \quad (3.5)$$

From (3.4), (3.5) and Lemma 6 we have

$$(n + k) \overline{N}(r, \infty; f) \leq N(r, \infty; (f^n)^{(k)}) = N(r, \infty; \frac{p^2}{(g^n)^{(k)}})$$

$$\leq N(r, 0; (g^n)^{(k)}) + O(\log r)$$

$$\leq N(r, 0; g^n) + k \overline{N}(r, \infty; g^n) + O(\log r) \leq k \overline{N}(r, \infty; g) + O(\log r),$$

as $r \to \infty$. Similarly $(n + k) \overline{N}(r, \infty; g) \leq k \overline{N}(r, \infty; f) + O(\log r)$, as $r \to \infty$. Therefore we have $\overline{N}(r, \infty; f) + \overline{N}(r, \infty; g) \leq O(\log r)$, as $r \to \infty$. This shows that $f$ and $g$ have at most finitely many poles. On the other hand $f$ and $g$ have at most finitely many zeros. Let

$$f = \frac{P_1}{Q_1} e^{\alpha} \quad \text{and} \quad g = \frac{P_2}{Q_2} e^{\beta}, \quad (3.6)$$

where $P_1, P_2, Q_1$ and $Q_2$ are non-zero polynomials and $\alpha$ and $\beta$ are non-constant polynomials. Since every zeros of $f(z)$ are the poles of $g(z)$ as well as the zeros of $p(z)$, it follows that $\deg(P_1) < \deg(Q_2)$. Similarly $\deg(P_2) < \deg(Q_1)$. Let $R_1 = \frac{P_1}{Q_1}$ and $R_2 = \frac{P_2}{Q_2}$. Since $f$ and $g$ are transcendental meromorphic functions, from (3.6) we have $\rho(f) > 0$ and $\rho(g) > 0$. We observe from (3.5) and Lemma 9 that $\mu(f) = \rho(f) = 1$ or $\mu(f) = \rho(f) = 2$ and so $\deg(\alpha) \leq 2$. Similarly we have $\deg(\beta) \leq 2$. Now from (3.6) and Lemma 7 we have

$$(f^n)^{(k)} = \left(\left(\alpha' + \frac{R_1'}{R_1}\right)^k + P_{k-1}^* \left(\alpha' + \frac{R_1'}{R_1}\right)\right) A R_1^n e^{n\alpha}, \quad (3.7)$$
there exist \( \alpha \) and \( \beta \), there exists at least one \( z \) such that
\[
\beta + R_2 \bigg( \beta + \frac{R_2}{R_1} \bigg) e^{n \alpha + \beta} = \frac{R_1^k}{(R_1 R_2)^n},
\]
i.e.,
\[
AB \bigg( \alpha + \frac{R_1}{R_1} \bigg)^k + P_{k-1}^* \bigg( \alpha + \frac{R_1}{R_1} \bigg) \bigg( \beta + \frac{R_2}{R_2} \bigg)^k + P_{k-1}^* \bigg( \beta + \frac{R_2}{R_2} \bigg) e^{n (\alpha + \beta)} = (P^* Q^*)^n p_*^2,
\]
where \( P^*, Q^* \) are non-constant polynomials and \( p_* \) is a non-zero polynomial. Since \( P^*, Q^*, \alpha \) and \( \beta \) are non-constant polynomials, from (3.9) we have \( \alpha + \beta = d_1 \), where \( d_1 \in \mathbb{C} \). Therefore \( \alpha' + \beta' = 0 \). From (3.9) we have
\[
AB \bigg( \alpha' + \frac{R_1}{R_1} \bigg)^k + P_{k-1}^* \bigg( \alpha' + \frac{R_1}{R_1} \bigg) \bigg( - \alpha' + \frac{R_1}{R_1} \bigg)^k + P_{k-1}^* \bigg( - \alpha' + \frac{R_1}{R_1} \bigg) e^{n d_1} = (P^* Q^*)^n p_*^2. \tag{3.10}
\]
Letting \( |z| \to \infty \), we see that \( 2k \deg(\alpha') = n \deg(P^* Q^*) + 2 \deg(p_*) \). Since \( \deg(\alpha') \leq 1 \) and \( n > 2k \), we arrive at a contradiction.

**Sub-case 2.2.** Suppose that one of the families \( \mathcal{F} \) and \( \mathcal{G} \), say \( \mathcal{G} \) is not normal on \( \mathbb{C} \). Then there exists at least one \( z_0 \in \Delta \) such that \( \mathcal{G} \) is not normal \( z_0 \), we assume that \( z_0 = 0 \). Now by Marty’s theorem there exists a sequence of meromorphic functions \( \{ F(z + \omega_j) \} \subset \mathcal{F} \), where \( z \in \{ z : |z| < 1 \} \) and \( \{ \omega_j \} \subset \mathbb{C} \) is some sequence of complex numbers such that \( F^{\omega_j}(\omega_j) \to \infty \), as \( |\omega_j| \to \infty \). Note that \( p \) has only finitely many zeros. So there exists a \( r > 0 \) such that \( p(z) \neq 0 \) in \( D = \{ z : |z| \geq r \} \). Since \( p(z) \) is a polynomial, for all \( z \in \mathbb{C} \) satisfying \( |z| \geq r \), we have
\[
0 \leq \frac{|p'(z)|}{|p(z)|} \leq \frac{M_1}{|z|} < 1, \quad p(z) \neq 0. \tag{3.11}
\]
Also since \( \omega_j \to \infty \) as \( j \to \infty \), without loss of generality we may assume that \( |\omega_j| \geq r + 1 \) for all \( j \). Let \( D_1 = \{ z : |z| < 1 \} \) and
\[
F(w_j + z) = \frac{H(w_j + z)}{p(w_j + z)}.
\]
Since \( |w_j + z| \geq |w_j| - |z| \), it follows that \( w_j + z \in D \) for all \( z \in D_1 \). Also since \( p(z) \neq 0 \) in \( D \), it follows that \( p(\omega_j + z) \neq 0 \) in \( D_1 \) for all \( j \). Then by Lemma 10 there exist
(i) points \( z_j, |z_j| < 1 \),
(ii) positive numbers \( \rho_j, \rho_j \to 0^* \),
(iii) a subsequence \( \{ F(\omega_j + z_j + \rho_j \xi) \} \) of \( \{ F(\omega_j + z) \} \)
such that 
\[ h_j(\zeta) = \rho_j^{-k}f(\omega_j + z_j + \rho_j \zeta) \to h(\zeta), \]
i.e.,
\[ h_j(\zeta) = \rho_j^{-k} \frac{H(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to h(\zeta) \]  \hspace{2cm} (3.12)
spherically locally uniformly in \( \mathbb{C} \), where \( h(\zeta) \) is some non-constant meromorphic function such that \( h^{\#}(\zeta) \leq h^{\#}(0) = 1 \). Now from Lemma 8 we see that \( \rho(\zeta) \leq 2 \). In the proof of Zalcman’s lemma (see [9, 19]) we see that
\[ \rho_j = \frac{1}{F^\#(b_j)} \text{ and } F^\#(b_j) \geq F^\#(\omega_j), \] where \( b_j = \omega_j + z_j \). Note that
\[ \frac{p'(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to 0, \] as \( j \to \infty \). By Hurwitz’s theorem we can see that the multiplicity of every zero and pole of \( h(\zeta) \) is a multiple of \( n \). Therefore we can deduce that \( h = \tilde{h}^n \), where \( \tilde{h} \) is some non-constant meromorphic function in the complex plane. We now prove that
\[ (h_j(\zeta))^{(k)} = \frac{H^{(k)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to h^{(k)}(\zeta). \] Note that from (3.12)
\[ \rho_j^{-k+1} \frac{H'(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} = h_j'(\zeta) + \rho_j^{-k+1} \frac{p'(\omega_j + z_j + \rho_j \zeta)}{p^2(\omega_j + z_j + \rho_j \zeta)} H(\omega_j + z_j + \rho_j \zeta) \]
\[ = h_j'(\zeta) + \rho_j \frac{p'(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} h_j(\zeta). \] (3.16)
Now from (3.12), (3.14) and (3.16) we observe that \( \rho_j^{-k+1} \frac{H^{(l)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to h'(\zeta) \). Suppose
\[ \rho_j^{-k+l} \frac{H^{(l)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to h^{(l)}(\zeta). \]
Let
\[ G_j(\zeta) = \rho_j^{-k+l} \frac{H^{(l)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)}. \]
Then \( G_j(\zeta) \to h^{(l)}(\zeta) \). Note that
\[ \rho_j^{-k+l+1} \frac{H^{(l+1)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} = G_j'(\zeta) + \rho_j^{-k+l+1} \frac{p'(\omega_j + z_j + \rho_j \zeta)}{p^2(\omega_j + z_j + \rho_j \zeta)} H^{(l)}(\omega_j + z_j + \rho_j \zeta) \]
\[= G_j'(\zeta) + \rho_j \frac{p'(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} G_j(\zeta) . \] (3.17)

So from (3.14) and (3.17) we see that
\[\rho_j^{-k+1+1} \frac{H^{(l+1)}(\omega_j + z_n + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to G_j' \big| (3.17), \text{i.e.,} \]
\[\rho_j^{-k+1+1} \frac{H^{(l+1)}(\omega_j + z_n + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to h_j^{(l+1)}(\zeta) . \]

Then by mathematical induction we get desired result (3.15). Let
\[(\hat{h}_j(\zeta))(k) = \frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} . \] (3.18)

From (3.4) we have
\[\frac{H^{(k)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} = 1 \]
and so from (3.15) and (3.18) we get
\[(h_j(\zeta))(k) (\hat{h}_j(\zeta))(k) = 1 . \] (3.19)

From (3.15), (3.19) and formula of higher derivatives we can deduce that \(\hat{h}_j(\zeta) \to \hat{h}_1(\zeta)\), i.e.,
\[\frac{\hat{H}(\omega_j + z_j + \rho_j \zeta)}{p(\omega_j + z_j + \rho_j \zeta)} \to \hat{h}_1(\zeta) , \] (3.20)
spherically locally uniformly in \(\mathbb{C}\), where \(\hat{h}_1(\zeta)\) is some non-constant meromorphic function in the complex plane. By Hurwitz's theorem we can see that the multiplicity of every zero and pole of \(\hat{h}_1(\zeta)\) is a multiple of \(n\). Therefore we can deduce that \(\hat{h}_1 = \hat{h}^n\), where \(\hat{h}\) is some non-constant meromorphic function in the complex plane. Therefore (3.20) can be rewritten as \(\hat{h}_j(\zeta) \to \hat{h}^n(\zeta)\), spherically locally uniformly in \(\mathbb{C}\) and so
\[(\hat{h}_j(\zeta))(k) \to (\hat{h}^n(\zeta))(k) \] (3.21)
spherically locally uniformly in \(\mathbb{C}\). From (3.15), (3.19) and (3.21) we get
\[(\hat{h}^n(\zeta))(k)(\hat{h}^n(\zeta))(k) \equiv 1 . \] (3.22)

Since \(\rho(h) \leq 2\), from (3.22) we see that
\[\rho(h) = \rho(\hat{h}^n) = \rho((\hat{h}^n)(k)) = \rho((\hat{h}^n)(k)) = \rho(\hat{h}^n) = \rho(h) \leq 2 . \] (3.23)

Let \(\zeta_0\) be a zero of \(\hat{h}\) with multiplicity \(q_0\). Note that \(\zeta_0\) is a zero of \((\hat{h}^n)(k)\) with multiplicity \(nq_0 - k\). From (3.22) we see that \(\zeta_0\) must be a pole of \(\hat{h}\) with multiplicity \(r_0\), say. Note that
\( \zeta_0 \) is a pole of \((\hat{h}^n)^{(k)}\) with multiplicity \(nr_0 + k\). Therefore \(nq_0 - k = nr_0 + k\) and so \(q_0 > r_0\). Now \(nq_0 - k = nr_0 + k\) implies that \(n(q_0 - r_0) = 2k\). Since \(n > 2k\), we arrive at a contradiction. Therefore \(\bar{h} \neq 0\). Similarly we can prove that \(\hat{h} \neq 0\). From (3.22), (3.23) and Lemma 6 we have

\[
(n + k) \overline{N}(r, \infty; \bar{h}) \leq N(r, \infty; (\hat{h}^n)^{(k)}) = N\left(r, \infty; \frac{1}{(\hat{h}^n)^{(k)}} \right) \\
\leq N(r, 0; (\hat{h}^n)^{(k)}) + O(\log r) \\
\leq N(r, 0; \hat{h}^n) + k \overline{N}(r, \infty; \hat{h}^n) + O(\log r) \leq k N(r, \infty; \bar{h}) + O(\log r),
\]

as \(r \to \infty\). Similarly \((n + k) \overline{N}(r, \infty; \bar{h}) \leq k \overline{N}(r, \infty; \hat{h}) + O(\log r)\), as \(r \to \infty\). Therefore we have \(\overline{N}(r, \infty; \bar{h}) + \overline{N}(r, \infty; \hat{h}) \leq O(\log r)\), as \(r \to \infty\). This shows that \(\bar{h}\) and \(\hat{h}\) have at most finitely many poles. Let

\[
\tilde{h} = \frac{1}{P_3} e^{\alpha_2} \text{ and } \hat{h} = \frac{1}{Q_3} e^{\beta_2},
\]

(3.24)

where \(P_3, Q_3\) are non-zero polynomials and \(\alpha_2, \beta_2\) are non-constant polynomials. Since \(\bar{h}\) and \(\hat{h}\) are transcendental meromorphic functions, from (3.24) we have \(\rho(\bar{h}) > 0\) and \(\rho(\hat{h}) > 0\). We observe from (3.23) and Lemma 9 that \(\mu(\bar{h}) = \rho(\hat{h}) = 1\) or \(\mu(\hat{h}) = \rho(\bar{h}) = 2\) and so \(\deg(\hat{h}) \leq 2\). Similarly we have \(\deg(\bar{h}) \leq 2\). Next in the same manner as in Sub-case 2.1, we get \(2k \deg(a_2') = n \deg(P_3Q_3)\). Since \(\deg(a_2') \leq 1\) and \(n > 2k\), we can deduce that \(P_3, Q_3 \in \mathbb{C}\). This shows that \(\infty\) is a Picard exceptional value of both \(\bar{h}\) and \(\hat{h}\). Combining this with Theorem 1 in Fang [4] and the assumption \(n > 2k\), we get

\[
\tilde{h}(z) = c_1 e^{cz} \text{ and } \hat{h}(z) = c_2 e^{-cz},
\]

(3.25)

where \(c, c_1, c_2 \in \mathbb{C}\) such that \((-1)^k(c_1 c_2)^n(nc)^{2k} = 1\). Since \(h = \bar{h}^n\), from (3.12) and (3.25) we have

\[
\frac{h_j'(\zeta)}{h_j'(\zeta)} = \rho_j \frac{F'(w_j + \zeta + \rho_j \zeta)}{F'(w_j + \zeta + \rho_j \zeta)} \frac{h'(\zeta)}{h'(\zeta)} = nc,
\]

(3.26)

spherically locally uniformly in \(\mathbb{C}\). From (3.13) and (3.26) we get

\[
\rho_j \left| \frac{F'(\omega_j + z_j)}{F'(\omega_j + z_j)} \right| = 1 + \left| F'(\omega_j + z_j) \right|^2 \left| \frac{F'(\omega_j + z_j)}{F'(\omega_j + z_j)} \right| = 1 + \left| F'(\omega_j + z_j) \right|^2 \left| F'(\omega_j + z_j) \right| \to \left| \frac{h'(0)}{h(0)} \right| = n|c|,
\]

which implies that

\[
\lim_{j \to \infty} F(\omega_j + z_j) \neq 0, \infty.
\]

(3.27)

From (3.12) and (3.27) we see that

\[
h_j(0) = \rho_j^{-k} F(\omega_j + z_j) \to \infty.
\]

(3.28)
Again from (3.12) and (3.25) we have
\[ h_j(0) \to h(0) = \hat{c}_1^n. \] (3.29)

Now from (3.28) and (3.29) we arrive at a contradiction. This completes the proof. \( \square \)

From Lemma 12 we have the following lemma.

**Lemma 13.** Let \( f, g \) be two transcendental meromorphic functions and \( p \) be a non-zero polynomial with \( \deg(p) \leq n - 1 \), where \( n, k \in \mathbb{N} \) with \( n > \max\{2k, k+2\} \). Let \(((f - a)^n)_{(k)} - p, ((g - a)^n)_{(k)} - p \) share 0 CM and \(((f - a)^n)_{(k)} ((g - a)^n)_{(k)} \equiv p^2 \). Now (i) if \( p \not\in \mathbb{C} \), then \( f(z) = c_1 e^{Q(z)} + a \) and \( g(z) = c_2 e^{-cQ(z)} + a \), where \( Q(z) = \int_0^z p(t) dt \), \( c, c_1, c_2 \in \mathbb{C} \) such that \( (nc)^2(c_1 c_2)^n = -1 \); (ii) if \( p(z) = b \in \mathbb{C} \setminus \{0\} \), then \( f(z) = c_3 e^{dz} + a \) and \( g(z) = c_4 e^{-dz} + a \), where \( d, c_3, c_4 \in \mathbb{C} \) such that \((-1)^k(c_3 c_4)^n(n)^2 = b^2 \).

**Lemma 14.** Let \( f, g \) be two transcendental meromorphic functions and let \( m, n, k \in \mathbb{N} \) such that \( n > 2k \). Let \( P(z) = \sum_{i=0}^m a_i z^i \) be a non-zero polynomial such that \( P(z) \) is not a monomial. If \((f^n P(f)) \equiv (g^n P(g)) \equiv 1 \), then \( f \) is of order at most 2.

**Proof.** We have
\[ (f^n P(f)) \equiv (g^n P(g)) \equiv 1. \] (3.30)

Without loss of generality we may assume that \( a_m, a_0 \neq 0 \). Let \( \mathcal{F} = \{f_\omega\} \) and \( \mathcal{G} = \{g_\omega\} \), where \( f_\omega(z) = f(z + \omega) \) and \( g_\omega(z) = g(z + \omega) \), \( z \in \mathbb{C} \). Clearly \( \mathcal{F} \) and \( \mathcal{G} \) are two families of meromorphic functions defined on \( \mathbb{C} \). We now consider following two sub-cases.

**Sub-case 2.1.** Suppose one of the families \( \mathcal{F} \) and \( \mathcal{G} \), say \( \mathcal{F} \), is normal on \( \mathbb{C} \). Then by Marty’s theorem \( f^\#(\omega) = f^\#_0(0) \leq M \) for some \( M > 0 \) and for all \( \omega \in \mathbb{C} \). By Lemma 8, \( \rho(f) \leq 2 \).

**Sub-case 2.2.** Suppose one of the families \( \mathcal{F} \) and \( \mathcal{G} \), say \( \mathcal{F} \), is not normal on \( \mathbb{C} \). Then there exists at least one \( z_0 \in \Delta \) such that \( \mathcal{F} \) is not normal \( z_0 \), we assume that \( z_0 = 0 \). Now by Marty’s theorem there exists a sequence of meromorphic functions \( \{f(z + \omega_j)\} \subset \mathcal{F} \), where \( z \in \{z : |z| < 1\} \) and \( \{\omega_j\} \subset \mathbb{C} \) is some sequence of complex numbers such that \( f^\#(\omega_j) \to \infty \), as \( |\omega_j| \to \infty \). Then by Lemma 10 there exist

(i) points \( z_j, |z_j| < 1 \),
(ii) positive numbers \( \rho_j, \rho_j \to 0^+ \),
(iii) a subsequence \( \{f(\omega_j + z_j + \rho_j \zeta) = f_j(z_j + \rho_j \zeta)\} \) of \( \{f(\omega_j + z)\} \)
such that
\[ h_j(\zeta) = \rho_j^{-\frac{k}{n}} f_j(z_j + \rho_j \zeta) \to h(\zeta) \] (3.31)
spherically locally uniformly in $\mathbb{C}$, where $h(\zeta)$ is some non-constant meromorphic function such that $h^\#(\zeta) \leq h^\#(0) = 1$. Now from Lemma 8 we see that $\rho(h) \leq 2$. In the proof of Zalcman’s lemma (see [9, 19]) we see that

$$\rho_j = \frac{1}{f^\#(b_j)} \quad \text{and} \quad f^\#(b_j) \geq f^\#(\omega_j),$$

(3.32)

where $b_j = \omega_j + z_j$. Now (3.31) yields

$$a_m \rho_j \frac{k_m}{j} h_j^{n+m}(\zeta) + \cdots + a_1 \rho_j \frac{k}{j} h_j^{n+1}(\zeta) + a_0 h_j^n(\zeta) \overset{(k)}{=} \left( a_m f_j^{n+m}(z_j + \rho_j \zeta) + \cdots + a_1 f_j^{n+1}(z_j + \rho_j \zeta) + a_0 f_j^n(z_j + \rho_j \zeta) \right)^{(k)} \rightarrow a_0 (h^n(\zeta))^{(k)},$$

(3.33)

spherically locally uniformly in $\mathbb{C}$. Next we suppose $\hat{h}_j(\zeta) = \rho_j \frac{k_m}{j} g_j(z_j + \rho_j \zeta)$. Therefore

$$a_m \rho_j \frac{k_m}{j} \hat{h}_j^{n+m}(\zeta) + \cdots + a_1 \rho_j \frac{k}{j} \hat{h}_j^{n+1}(\zeta) + a_0 \hat{h}_j^n(\zeta) \overset{(k)}{=} \left( a_m g_j^{n+m}(z_j + \rho_j \zeta) + \cdots + a_1 g_j^{n+1}(z_j + \rho_j \zeta) + a_0 g_j^n(z_j + \rho_j \zeta) \right)^{(k)}.$$

(3.34)

Now from (3.30), (3.33) and (3.34) we have

$$a_m \rho_j \frac{k_m}{j} h_j^{n+m}(\zeta) + \cdots + a_0 h_j^n(\zeta) \overset{(k)}{=} a_m \rho_j \frac{k_m}{j} \hat{h}_j^{n+m}(\zeta) + \cdots + a_0 \hat{h}_j^n(\zeta) \overset{(k)}{=} 1.$$

(3.35)

Letting $j \rightarrow \infty$, from (3.33), (3.35) and the formula of higher derivatives we can deduce that

$$a_m \rho_j \frac{k_m}{j} \hat{h}_j^{n+m}(\zeta) + \cdots + a_0 \hat{h}_j^n(\zeta) \rightarrow a_0 \hat{h}_1(\zeta),$$

(3.36)

spherically locally uniformly in $\mathbb{C}$, where $\hat{h}_1(\zeta)$ is some non-constant meromorphic function in the complex plane. Now from (3.33), (3.35) and (3.36) we observe that

$$a_0^2 (h^n(\zeta))^{(k)}(\hat{h}_1(\zeta))^{(k)} \equiv 1.$$

(3.37)

By Hurwitz’s theorem we can see that the multiplicity of every zero and pole of $\hat{h}_1(\zeta)$ is a multiple of $n$. Therefore we can deduce that $\hat{h}_1 = \hat{h}_1$, where $\hat{h}$ is some non-constant meromorphic function in the complex plane. Therefore (3.36) can be rewritten as

$$a_m \rho_j \frac{k_m}{j} h_j^{n+m}(\zeta) + \cdots + a_0 \hat{h}_j^n(\zeta) \rightarrow a_0 \hat{h}_1^n(\zeta),$$

spherically locally uniformly in $\mathbb{C}$ and so $\left( a_m \rho_j \frac{k_m}{j} h_j^{n+m}(\zeta) + \cdots + a_0 \hat{h}_j^n(\zeta) \right)^{(k)} \rightarrow a_0 (\hat{h}_1^n(\zeta))^{(k)}$ spherically locally uniformly in $\mathbb{C}$. From (3.37) we see that $a_0^2 (h^n(\zeta))^{(k)}(\hat{h}_1(\zeta))^{(k)} \equiv 1$. Now by applying Sub-case 2.2 of Lemma 12, we arrive at a contradiction.

This completes the proof. □
Lemma 15. Let \( f \) and \( g \) be two transcendental meromorphic functions such that
\[
E_l(p; (P(f))^{(k)}) = E_l(p; (P(g))^{(k)}) \quad \text{and} \quad E_1(p; (P(f))^{(k)}) = E_1(p; (P(g))^{(k)}),
\]
where \( l \in \mathbb{N} \setminus \{1, 2\} \), \( P(z) \) be defined as in (2.1) and \( p(z) \neq 0 \) is a polynomial. If
\[
\Delta_{1l} = \left( \frac{8}{3} + \frac{2}{3} k \right) \Theta(\infty, P(f)) + (k + 2) \Theta(\infty, P(g)) + \Theta(0, P(f)) + \Theta(0, P(g)) \\
+ \frac{5}{3} \delta_{k+1}(0, P(f)) + \delta_{k+1}(0, P(g)) > \frac{5}{3}(k + 5), \quad (3.38)
\]
then \( H \equiv 0 \).

Proof. Proof of lemma follows from Lemma 8 [10].

Lemma 16. Let \( f \) and \( g \) be two transcendental meromorphic functions such that
\[
E_l(p; (P(f))^{(k)}) = E_l(p; (P(g))^{(k)}) \quad \text{and} \quad E_2(p; (P(f))^{(k)}) = E_2(p; (P(g))^{(k)}),
\]
where \( l \in \mathbb{N} \setminus \{1, 2, 3\} \), \( P(z) \) be defined as in (2.1) and \( p(z) \neq 0 \) is a polynomial. If
\[
\Delta_{2l} = \left( \frac{1}{2} k + 2 \right) [\Theta(\infty, f) + \Theta(\infty, g)] + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) \\
> (k + 7), \quad (3.39)
\]
then \( H \equiv 0 \).

Proof. Proof of lemma follows from Lemma 9 [10].

4. Proof of the Theorem

Proof of Theorem 1. Let \( F = P(f) \) and \( G = P(g) \). Then we see that \( E_l(p; F^{(k)}) = E_l(p; G^{(k)}) \)
and \( E_1(p; F^{(k)}) = E_1(p; G^{(k)}) \), where \( l \in \mathbb{N} \setminus \{1, 2\} \). Note that
\[
\Theta(\infty; P(f)) = 1 - \limsup_{r \to \infty} \frac{N(r, \infty; P(f))}{T(r, P(f))} \\n= 1 - \limsup_{r \to \infty} \frac{N(r, \infty; f)}{n T(r, f)} \geq 1 - \limsup_{r \to \infty} \frac{T(r, f)}{n T(r, f)} \geq 1 - \frac{1}{n} = \frac{n - 1}{n}, \quad (4.1)
\]
\[
\delta_{k+1}(0; P(f)) = 1 - \limsup_{r \to \infty} \frac{N_{k+1}(r, 0; P(f))}{T(r, P(f))} \\
\geq 1 - \limsup_{r \to \infty} \frac{\sum_{j=1}^{s} N_{k+1}(r, 0; (f - c_l)^{l_j}) + N_{k+1}(r, 0; (f - c_l)^{l_j})}{n T(r, f)} \\
\geq 1 - \limsup_{r \to \infty} \frac{(s - 1) T(r, f) + (k + 1) T(r, f) + S(r, f)}{n T(r, f)} \\
\geq 1 - \frac{s + k}{n} \geq \frac{l - k - 1}{n}, \quad (4.2)
\]
\[
\Theta(0; P(f)) = 1 - \limsup_{r \to \infty} \frac{N(r, 0; P(f))}{T(r, P(f))} \\
\geq 1 - \limsup_{r \to \infty} \frac{\sum_{j=1}^{s} N(r, 0; (f - c_{j})^{l}) + N(r, 0; (f - c_{j})^{l})}{nT(r, f)} \\
\geq 1 - \frac{1}{n} \frac{s - 1}{n} + \frac{1 - l}{n}. \tag{4.3}
\]

Similarly we have
\[
\Theta(0; P(g)) \geq \frac{n - 1}{n}, \quad \delta_{k+1}(0; P(g)) \geq \frac{l - k - 1}{n} \quad \text{and} \quad \Theta(0; P(g)) \geq \frac{l - 1}{n}. \tag{4.4}
\]

Now in view (3.38) and (4.1)-(4.4) we obtain
\[
\Delta_{1l} \geq \left(\frac{14}{3} + \frac{5}{3}k\right) \frac{n - 1}{n} + 2 \frac{l - 1}{n} + \frac{8}{3} \left(\frac{l - k - 1}{n}\right) \\
= \left(\frac{14}{3} + \frac{5}{3}k\right) \frac{l + r - 1}{l + r} + 2 \frac{l - 1}{l + r} + \frac{8}{3} \left(\frac{l - k - 1}{l + r}\right).
\]

Since \(l > \frac{13}{3}k + \frac{11}{3}r + \frac{28}{3}\), we get \(\Delta_{1l} > \frac{5}{3}(k + 5)\). Therefore by Lemma 15 we have \(H \equiv 0\).

Then theorem follows from Lemmas 4, 5, 13 and 14.

\[\square\]

**Proof of Theorem 2.** Using Lemma 16, theorem can be proved in the line of the proof of Theorem 1. So we omit the details.

\[\square\]

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