TAMKANG JOURNAL OF MATHEMATICS
Volume 50, Number 2, 165-185, June 2019
doi:10.5556/j.tkjm.50.2019.2896

This paper is available online at http://journals.math.tku.edu.tw/index.php/TKJM/pages/view/onlinefirst

MEROMORPHIC FUNCTIONS WHOSE CERTAIN DIFFERENTIAL POLYNOMIAL SHARE A POLYNOMIAL

SUJOY MAJUMDER AND RAJIB MANDAL

Abstract. In this paper, we use the idea of normal family to investigate the uniqueness problems of meromorphic functions when certain non-linear differential polynomial sharing a non-zero polynomial with certain degree. We obtain some results which will not only rectify the recent results of P. Sahoo and H. Karmakar [10] but also improve and generalize some recent results of L. Liu [7], H. Y. Xu, T. B. Cao and S. Liu [13] and P. Sahoo and H. Karmakar [10] in a large extent.

1. Introduction, Definitions and Results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have the same zeros ignoring multiplicities. Let $k \in \mathbb{N}\cup\{0\}\cup\{\infty\}$. For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$, we say that f, g share the value a with weight k.

Let $m \in \mathbb{N}\cup\{\infty\}$ and $a \in \mathbb{C}\cup\{\infty\}$. We denote by $E_{m)}(a;f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. Also we denote by $\overline{E}_{m)}(a;f)$ the set of distinct a-points of $f(z)$ with multiplicities not greater than m. If $\overline{E}_{m)}(a;f) = \overline{E}_{m)}(a;g)$, we say that a is a m-order pseudo common value of f and g. If for some $a \in \mathbb{C}\cup\{\infty\}$, $E_{\infty)}(a;f) = E_{\infty)}(a;g)$ $(\overline{E}_{\infty)}(a;f) = \overline{E}_{\infty)}(a;g))$ we say that f, g share the value a CM (IM).

Received June 16, 2018, accepted July 23, 2018.
2010 *Mathematics Subject Classification.* Primary 30D35, Secondary 30D30.
Key words and phrases. Uniqueness, meromorphic function, non-linear differential polynomials, normal families.
Corresponding author: Sujoy Majumder.

We adopt the standard notations of value distribution theory (see [5]). For a non-constant meromorphic function f, we denote by $T(r,f)$ the Nevanlinna characteristic of f and by $S(r,f)$ any quantity satisfying $S(r,f)=o\{T(r,f)\}$ as $r\to\infty$ possibly outside a set of finite linear measure.

A meromorphic function $a(z)$ is called a small function with respect to f, if $T(r,a)=S(r,f)$. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\longrightarrow\infty$, outside of a possible exceptional set of finite linear measure. Throughout this paper, we denote by $\mu(f)$, $\rho(f)$ and $\lambda(f)$ the lower order of f, the order of f and the exponent of convergence of zeros of f respectively (see [5, 15]).

Let f be a transcendental meromorphic function in the complex plane such that $\rho(f)=\rho\leq\infty$. A complex number a is said to be a Borel exceptional value (see [15]) if

$$\limsup_{r\longrightarrow\infty}\frac{\log^{+}N(r,a;f)}{\log r}<\rho.$$

For the sake of simplicity we also use the notations $m^{*}:=\chi_{\mu}m$, where $\chi_{\mu}=0$ if $\mu=0$, $\chi_{\mu}=1$ if $\mu\neq0$.

In 1959, W. K. Hayman (see [5], Corollary of Theorem 9) proved the following theorem.

Theorem A. *Let f be a transcendental meromorphic function and $n\in\mathbb{N}\setminus\{1,2\}$. Then $f^{n}f'=1$ has infinitely many solutions.*

Theorem A was extended by Chen [3] in the following manner.

Theorem B. *Let f be a transcendental entire function and $n,\ k\in\mathbb{N}$ with $n\geq k+1$. Then $(f^{n})^{(k)}-1$ has infinitely many zeros.*

In 2002, Fang [4] proved the following result.

Theorem C. *Let $f,\ g$ be two non-constant entire functions and let $n,\ k\in\mathbb{N}$ with $n>2k+4$. If $(f^{n})^{(k)}$ and $(g^{n})^{(k)}$ share 1 CM, then either $f(z)=c_{1}e^{cz}$ and $g(z)=c_{2}e^{-cz}$, where $c,\ c_{1},\ c_{2}\in\mathbb{C}$ satisfying $(-1)^{k}(c_{1}c_{2})^{n}(nc)^{2k}=1$ or $f\equiv tg$ for $t\in\mathbb{C}$ such that $t^{n}=1$.*

In 2008, L. Liu [7] proved the following.

Theorem D. *Let $f,\ g$ be two non-constant meromorphic functions and let $n,\ m,\ k\in\mathbb{N}$ and $\lambda,\ \mu\in\mathbb{C}$ such that $|\lambda|+|\mu|\neq0$. If $E_{l}(1,(f^{n}(\lambda f^{m}+\mu))^{(k)})=E_{l}(1,(g^{n}(\lambda g^{m}+\mu))^{(k)})$ and one of the following conditions holds:*

(1) *$l\geq2$ and $n>3m^{*}+3k+8$;*
(2) *$l=1$ and $n>4m^{*}+5k+10$;*
(3) *$l=0$ and $n>6m^{*}+9k+14$.*

Then

(i) *when* $\lambda\mu \neq 0$, *if* $m \geq 2$ *and* $\delta(\infty; f) > \frac{3}{n+m}$, *then* $f \equiv g$; *if* $m = 1$ *and* $\Theta(\infty; f) > \frac{3}{n+1}$, *then* $f \equiv g$;

(ii) *when* $\lambda\mu = 0$, *if* $f, g \neq \infty$, *then either* $f \equiv tg$, *where* $t \in \mathbb{C}$ *such that* $t^{n+m^*} = 1$ *or* $f(z) = c_1 e^{cz}$ *and* $g(z) = c_2 e^{-cz}$, *where* c, c_1, $c_2 \in \mathbb{C}$ *such that*
$(-1)^k \lambda^2 (c_1c_2)^{n+m^*}((n+m^*)c)^{2k} = 1$ *or* $(-1)^k \mu^2 (c_1c_2)^{n+m^*}((n+m^*))^{2k} = 1$.

Regarding Theorem D, following question arises.

Question 1. How two meromorphic functions f and g are related, if the condition $E_l(1,(f^n(\mu f^m+\lambda))^{(k)}) = E_l(1,(g^n(\mu g^m+\lambda))^{(k)})$ in Theorem D is replaced with the condition $E_{l)}(1,(f^n(\mu f^m+\lambda))^{(k)}) = E_{l)}(1,(g^n(\mu g^m+\lambda))^{(k)})$?

In 2012, Xu et al. [13] answer the above question by proving the following results which also improve Theorem D in some sense.

Theorem E. *Let* f, g *be two non-constant meromorphic functions and let* n, m, $k \in \mathbb{N}$ *with* $n > \frac{13}{3}k + \frac{13}{3}m^* + \frac{28}{3}$ *and* λ, $\mu \in \mathbb{C}$ *such that* $|\lambda|+|\mu| \neq 0$. *If* $\overline{E}_{l)}(1,(f^n(\mu f^m+\lambda))^{(k)}) = \overline{E}_{l)}(1,(g^n(\mu g^m+\lambda))^{(k)})$ *and* $E_{1)}(1,(f^n(\mu f^m+\lambda))^{(k)}) = E_{1)}(1,(g^n(\mu g^m+\lambda))^{(k)})$, *where* $l \in \mathbb{N}\setminus\{1,2\}$, *then the conclusions of Theorem D still hold.*

Theorem F. *Let* f, g *be two non-constant meromorphic functions and let* n, m, $k \in \mathbb{N}$ *with* $n > 3k+3m^*+6$ *and* λ, $\mu \in \mathbb{C}$ *such that* $|\lambda|+|\mu| \neq 0$. *If* $\overline{E}_{l)}(1,(f^n(\mu f^m+\lambda))^{(k)}) = \overline{E}_{l)}(1,(g^n(\mu g^m+\lambda))^{(k)})$ *and* $E_{2)}(1,(f^n(\mu f^m+\lambda))^{(k)}) = E_{2)}(1,(g^n(\mu g^m+\lambda))^{(k)})$, *where* $l \in \mathbb{N}\setminus\{1,2,3\}$, *then the conclusions of Theorem D still hold.*

Observing Theorems E and F, Sahoo and Karmakar [10] asked the following question.

Question 2. What can be said about the relationship between two meromorphic functions f and g, if $(f^nP(f))^{(k)}$ and $(g^nP(g))^{(k)}$ share a non-zero polynomial, where $P(z) = \sum_{i=0}^{m} a_i z^i$ is any non-zero polynomial, $a_0, a_1, \ldots, a_m \in \mathbb{C}$?

Let us define $m^{**} = m$, if $P(z) \neq a_0$; $m^{**} = 0$, if $P(z) \equiv a_0$.

In the direction of the above question, Sahoo and Karmakar [10] obtained the following results.

Theorem G. *Let* f, g *be two transcendental meromorphic functions,* p *be a non-zero polynomial of degree* q *and* n, $k \in \mathbb{N}$, $m \in \mathbb{N}\cup\{0\}$ *with* $n > \max\{\frac{13}{3}k + \frac{11}{3}m^{**} + \frac{28}{3}, k+2q\}$. *Suppose that either* k, q *are co-prime or* $k > q$, *when* $q \geq 2$. *Let* $\overline{E}_{l)}(p,(f^nP(f))^{(k)}) = \overline{E}_{l)}(p,(g^nP(g))^{(k)})$ *and* $E_{1)}(p,(f^nP(f))^{(k)}) = E_{1)}(p,(g^nP(g))^{(k)})$, *where* $P(z) = \sum_{i=0}^{m} a_i z^i$ *is any non-zero polynomial and* $l \in \mathbb{N}\setminus\{1,2\}$. *Then the following conclusions hold.*

(i) *If* $P(z)=\sum_{i=0}^{m}a_iz^i$ *is not a monomial, then either* $f\equiv tg$, $t\in\mathbb{C}$ *such that* $t^d=1$, *where* $d=(n+m,\dots n+m-i\dots,n)$, $a_{m-i}\neq 0$ *for some* $i=0,1,2,\dots,m$ *or* f *and* g *satisfy the algebraic equation* $R(f,g)=0$, *where* $R(f,g)$ *is given by*

$$R(w_1,w_2)=w_1^n(a_mw_1^m+\ldots+a_1w_1+a_0)-w_2^n(a_mw_2^m+\ldots+a_1w_2+a_0).$$

(ii) *When* $P(z)=a_0$ *or* $P(z)=a_mz^m$, *then either* $f\equiv tg$, $t\in\mathbb{C}$ *such that* $t^{n+m^{**}}=1$ *or* $f(z)=b_1e^{bQ(z)}$ *and* $g(z)=b_2e^{-bQ(z)}$, *where* Q *is a polynomial without constant such that* $Q'=p$; b, b_1, $b_2\in\mathbb{C}$ *such that* $a_0^2(nb)^2(b_1b_2)^n=-1$ *or* $a_m^2((n+m)b)^2(b_1b_2)^{n+m}=-1$.

Theorem H. *Let* f, g *be two transcendental meromorphic functions,* p *be a non-zero polynomial of degree* q *and* n, $k\in\mathbb{N}$, $m\in\mathbb{N}\cup\{0\}$ *with* $n>\max\{3k+3m^{**}+6,k+2q\}$. *Suppose that either* k,q *are co-prime or* $k>q$, *when* $q\geq 2$. *Let* $\overline{E}_{l)}(p,(f^nP(f))^{(k)})=\overline{E}_{l)}(p,(g^nP(g))^{(k)})$ *and* $E_{2)}(p,(f^nP(f))^{(k)})=E_{2)}(p,(g^nP(g))^{(k)})$, *where* $l\in\mathbb{N}\setminus\{1,2,3\}$. *Then the conclusions of Theorem G still hold .*

Remark 1. In the proof of Theorem 1 [10], one can easily point out a number of gaps.

Firstly the authors [10] declare that Lemma 10 [10] can be proved in the line of the proof of Lemma 9 [21]. But this is not possible here. Actually in Lemma 9 [21], f, g share ∞ IM. But in Lemma 10 [10], authors did not consider the condition " f, g share ∞ IM". Therefore existence of Lemma 10 [10] is questionable here.

Secondly in the proof of Lemma 11 [10] there is a big gap. From the relation

$$(a_mf^{n+m})'(a_mg^{n+m})'\equiv p^2$$

authors conclude that $f=e^{\alpha}$ and $g=e^{\beta}$. Again from the relation

$$(a_mf^{n+m})^{(k)}(a_mg^{n+m})^{(k)}\equiv p^2 \tag{1.1}$$

authors conclude that

$$N(r,\infty;a_mf^{n+m})+N(r,0;a_mf^{n+m})=O(\log r). \tag{1.2}$$

The calculations are not true. A question arises: When zeros of $f(g)$ are neutralized by the poles of $g(f)$? Actually the authors did not consider this case. As for example we consider the case. Suppose $k=4$, $m=1$, $q=1$ and $n=7$. Let z_0 be a zero of f of multiplicity 2. One can easily think that z_0 is a simple pole of g. It is clear that z_0 is a zero of $(a_mf^{n+m})^{(k)}$ of multiplicity 12 and a pole of $(a_mg^{n+m})^{(k)}$ of multiplicity 12. This shows that zeros of $f(g)$ can be neutralized by the poles of $g(f)$. Also poles of f can be neutralized by the zeros of $(a_mg^{n+m})^{(k)}$, but not the zeros of g. As a result from (1.1) we can not easily arrive at (1.2). Therefore existence of Lemma 11 [10] is questionable here.

Finally, since Lemmas 9 and 10 [10] play an important role in proving Theorems 1 and 2 [10], so existence of Theorem 1 [10] as well as Theorem 2 [10] are questionable here.

The above discussion is sufficient enough to make oneself inquisitive to investigate the accurate form of Theorems G and H. In this paper we study Theorem 1 [10] as well as Theorem 2 [10] again in more general form with out help of Lemma 10 [10] as well as Lemma 11 [10] .

Also it is quite natural to ask the following questions.

Question 3. Can the lower bound of n be further reduced in Theorems G and H ?

Question 4. Can one remove the condition "Suppose that either k, q are co-prime or $k > q$, when $q \geq 2$" in Theorems G and H ?

Question 5. Can one remove the condition $f \neq \infty$, $g \neq \infty$ keeping all the conclusions intact when $\lambda\mu = 0$ in Theorems D, E, F ?

2. Main results

In this paper, we always use $P(z)$ denoting an arbitrary polynomial of degree n as follows:

$$P(z) = \sum_{i=0}^{n} a_i z^i = a_n \prod_{i=1}^{s} (z - c_{l_i})^{l_i}, \tag{2.1}$$

where $a_0, a_1, \ldots, a_n (\neq 0) \in \mathbb{C}$ and $c_{l_j} \in \mathbb{C}$ $(j = 1, 2, \ldots, s)$ are distinct and $l_1, l_2, \ldots, l_s$, s, n, $k \in \mathbb{N}$ such that $\sum_{i=1}^{s} l_i = n$. Also we let $l = \max\{l_1, l_2, \ldots, l_s\}$ and e be the zero of $P(z)$ of multiplicity l. From (2.1) we have $P(z) = (z - e)^l P_*(z)$, where $P_*(z)$ is a polynomial in degree $n - l = m (\geq 0)$, say. We also use $P_1(z_1)$ as an arbitrary non-zero polynomial defined by

$$P_1(z_1) = a_n \prod_{\substack{i=1 \\ l_i \neq l}}^{s} (z_1 + e - c_{l_i})^{l_i} = \sum_{i=0}^{m} b_i z_1^i, \tag{2.2}$$

where $z_1 = z - e$ and $\deg(P_1(z_1)) = m \geq 0$. Obviously $P(z) = z_1^l P_1(z_1)$.

Taking the possible answers of the above questions into backdrop we obtain the following results which are not only rectify Theorems G, H, but also improve and generalize Theorems D-H.

Theorem 1. *Let f, g be two transcendental meromorphic functions and p be a non-zero polynomial with $\deg(p) \leq l - 1$, where $m \in \mathbb{N} \cup \{0\}$, k, $l \in \mathbb{N}$ such that $l > \frac{13}{3}k + \frac{11}{3}m + \frac{28}{3}$. Suppose $\overline{E}_{l)}(p, (P(f))^{(k)}) = \overline{E}_{l)}(p, (P(g))^{(k)})$ and $E_{1)}(p, (P(f))^{(k)}) = E_{1)}(p, (P(g))^{(k)})$, where $P(z)$ is defined as in (2.1) and $l \in \mathbb{N} \setminus \{1, 2\}$. Now*

(I) *when $P_1(z_1)$ is not a monomial, then one of the following three cases holds*

(I1) *$f(z)-e \equiv t(g(z)-e)$ for $t \in \mathbb{C}$ such that $t^{d_0}=1$, where $d_0 = GCD(l+m, \ldots, l+m-i, \ldots, l)$, $b_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$;*

(I2) *$f_1 = f-e$ and $g_1 = g-e$ satisfy the algebraic equation $R(f_1, g_1) = 0$, where $R(\omega_1, \omega_2) = \omega_1^l(b_m\omega_1^m + b_{m-1}\omega_1^{m-1} + \ldots + b_0) - \omega_2^l(b_m\omega_2^m + b_{m-1}\omega_2^{m-1} + \ldots + b_0)$;*

(I3) *$(P(f))^{(k)}(P(g))^{(k)} \equiv p^2$;*

(II) *when $P_1(z_1)$ is a monomial, say $P_1(z_1) = b_i z_1^i \not\equiv 0$, where $i \in \{0, 1, \ldots, m\}$, then one of the following two cases holds*

(II1) *$f-e \equiv t(g-e)$ for $t \in \mathbb{C}$ such that $t^{l+i} = 1$,*

(II2) *if $p \not\in \mathbb{C}$, then $f(z) = c_1 e^{cQ(z)} + e$ and $g(z) = c_2 e^{-cQ(z)} + e$, where $Q(z) = \int_0^z p(t)dt$, c, c_1, $c_2 \in \mathbb{C}$ such that $b_i^2(c_1c_2)^{l+i}\big((l+i)c\big)^2 = -1$; if $p(z) = b \in \mathbb{C}\setminus\{0\}$, then $f(z) = c_3e^{cz} + e$ and $g(z) = c_4 e^{-cz} + e$, where c, c_3, $c_4 \in \mathbb{C}$ such that $(-1)^k b_i^2 (c_3c_4)^{l+i}\big((l+i)c\big)^{2k} = b^2$.*

In particular when $\rho(f) > 2$ and $p \in \mathbb{C}\setminus\{0\}$, then (I3) does not hold.

Theorem 2. *Let f, g be two transcendental meromorphic functions and p be a non-zero polynomial with $\deg(p) \leq l-1$, where $m \in \mathbb{N}\cup\{0\}$, k, $l \in \mathbb{N}$ such that $l > 3k+3m+6$. Suppose $\overline{E}_{l)}(p, (P(f))^{(k)}) = \overline{E}_{l)}(p, (P(g))^{(k)})$ and $E_{2)}(p, (P(f))^{(k)}) = E_{2)}(p, (P(g))^{(k)})$, where $P(z)$ is defined as in (2.1) and $l \in \mathbb{N}\setminus\{1,2,3\}$. Then the conclusion of Theorem 1 holds.*

With the help of Theorem 1.5 [8] and Theorem 1 we get the following corollary immediately.

Corollary 1. *Let f, g be two transcendental meromorphic functions and p be a non-zero polynomial with $\deg(p) \leq l-1$, where $m \in \mathbb{N}\cup\{0\}$, k, $l \in \mathbb{N}$ such that $l > 3k+m+8$. Suppose $(P(f))^{(k)} - p$ and $(P(g))^{(k)} - p$ share $(0,2)$. Then the conclusion of Theorem 1 holds.*

We now explain some definitions and notations which are used in the paper.

Definition 1 ([6])**.** For $a \in \mathbb{C}\cup\{\infty\}$ we denote by $N(r,a;f\mid=1)$ the counting function of simple a points of f. For $m \in \mathbb{N}$ we denote by $N(r,a;f\mid\leq m)$ ($N(r,a;f\mid\geq m)$) the counting function of those a points of f whose multiplicities are not greater (less) than m where each a point is counted according to its multiplicity. $\overline{N}(r,a;f\mid\leq m)$ ($\overline{N}(r,a;f\mid\geq m)$) are defined similarly, where in counting the a-points of f we ignore the multiplicities. Also $N(r,a;f\mid< m)$, $N(r,a;f\mid> m)$, $\overline{N}(r,a;f\mid< m)$ and $\overline{N}(r,a;f\mid> m)$ are defined analogously.

Definition 2 ([17])**.** For $a \in \mathbb{C}\cup\{\infty\}$ and $p \in \mathbb{N}$ we let $N_p(r,a;f) = \sum_{i=1}^{p} \overline{N}(r,a;f\mid\geq i)$.

Definition 3. Let $a, b \in \mathbb{C} \cup \{\infty\}$ and $p \in \mathbb{N}$. We denote by $\overline{N}(r, a; f \mid \geq p \mid g = b)$ ($\overline{N}(r, a; f \mid \geq p \mid g \neq b)$) the reduced counting function of those a-points of f with multiplicities $\geq p$, which are the b-points (not the b-points) of g.

Definition 4 ([5])**.** Let $a \in \mathbb{C} \cup \{\infty\}$ and $k \in \mathbb{N}$. We define

$$\Theta(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)} \text{ and } \delta_k(a; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{N_k(r, a; f)}{T(r, f)}.$$

3. Lemmas

Let h be a meromorphic function in $\mathbb{C}$. Then h is called a normal function if there exists a positive real number M such that $h^{\#}(z) \leq M$, for all $z \in \mathbb{C}$, where

$$h^{\#}(z) = \frac{|h'(z)|}{1 + |h(z)|^2}$$

denotes the spherical derivative of h. Let $\mathscr{F}$ be a family of meromorphic functions in a domain $D \subset \mathbb{C}$. We say that $\mathscr{F}$ is normal in D if every sequence $\{f_n\}_n \subseteq \mathscr{F}$ contains a subsequence which converges spherically and uniformly on the compact subsets of D (see [11]).

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{3.1}$$

Lemma 1 ([14])**.** *Let f be a non-constant meromorphic function and let $a_n(z) (\not\equiv 0)$, $a_{n-1}(z), \ldots,$ $a_0(z)$ be meromorphic functions such that $T(r, a_i(z)) = S(r, f)$ for $i = 0, 1, 2, \ldots, n$. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2 ([20])**.** *Let f be a non-constant meromorphic function and p, $k \in \mathbb{N}$. Then*

$$N_p\left(r, 0; f^{(k)}\right) \leq T\left(r, f^{(k)}\right) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f),$$
$$N_p\left(r, 0; f^{(k)}\right) \leq k\overline{N}(r, \infty; f) + N_{p+k}(r, 0; f) + S(r, f).$$

Lemma 3 ([12])**.** *Let f, g be two non-constant meromorphic functions and k, $n \in \mathbb{N}$ such that $n > 2k+1$. If $(f^n)^{(k)} \equiv (g^n)^{(k)}$, then $f \equiv tg$ for $t \in \mathbb{C}$ such that $t^n = 1$.*

Lemma 4. *Let f, g be two non-constant meromorphic functions. Let l, $k \in \mathbb{N}$ and $m \in \mathbb{N} \cup \{0\}$ such that $l > m + 3k$. Suppose $(P(f))^{(k)} \equiv (P(g))^{(k)}$, where $P(z)$ be defined as in (2.1). Now*

(I) *when $P_1(z_1)$ is not a monomial, then one of the following two cases holds:*

(Ii) *$f(z)-e\equiv t(g(z)-e)$ for $t\in\mathbb{C}$ such that $t^{d_0}=1$, where $d_0=GCD(l+m,\ldots,l+m-i,\ldots,l)$, $b_{m-i}\neq 0$ for some $i=0,1,\ldots,m$;*

(Iii) *$f_1=f-e$ and $g_1=g-e$ satisfy the equation $R(f_1,g_1)=0$, where $R(\omega_1,\omega_2)=\omega_1^l(b_m\omega_1^m+b_{m-1}\omega_1^{m-1}+\ldots+b_0)-\omega_2^l(b_m\omega_2^m+b_{m-1}\omega_2^{m-1}+\ldots+b_0)$.*

(II) *when $P_1(z_1)$ is a monomial, say $P_1(z_1)=b_iz_1^i\not\equiv 0$, where $i\in\{0,1,\ldots,m\}$, then $f-e\equiv t(g-e)$ for $t\in\mathbb{C}$ such that $t^{l+i}=1$.*

Proof. We have $(P(f))^{(k)}\equiv(P(g))^{(k)}$. Integrating we get $(P(f))^{(k-1)}\equiv(P(g))^{(k-1)}+c_{k-1}$. If possible suppose $c_{k-1}\neq 0$. Now in view of Lemma 2 for $p=1$ and using the second fundamental theorem we get

$$\begin{aligned}
n\,T(r,f) &= T(r,P(f))+O(1)\\
&\le T(r,(P(f))^{(k-1)})-\overline{N}(r,0;(P(f))^{(k-1)})+N_k(r,0;P(f))+S(r,f)\\
&\le \overline{N}(r,0;(P(f))^{(k-1)})+\overline{N}(r,\infty;f)+\overline{N}(r,c_{k-1};(P(f))^{(k-1)})-\overline{N}(r,0;(P(f))^{(k-1)})\\
&\quad +N_k(r,0;P(f))+S(r,f)\\
&\le \overline{N}(r,\infty;f)+\overline{N}(r,0;(P(g))^{(k-1)})+N_k(r,0;P(f))+S(r,f)\\
&\le \overline{N}(r,\infty;f)+(k-1)\overline{N}(r,\infty;g)+N_k(r,0;P(g))+N_k(r,0;P(f))+S(r,f)\\
&\le \overline{N}(r,\infty;f)+(k-1)\overline{N}(r,\infty;g)+k\overline{N}(r,e;g)+N(r,0;P(g)\mid g\neq e)+k\overline{N}(r,e;f)\\
&\quad +N(r,0;P(f)\mid f\neq e)+S(r,f)\\
&\le (n-l+k+1)\,T(r,f)+(n-l+2k-1)\,T(r,g)+S(r,f)+S(r,g)\\
&\le (2n-2l+3k)\,T(r)+S(r).
\end{aligned}$$

Similarly we get $n\,T(r,g)\le(2n-2l+3k)\,T(r)+S(r)$. Combining these we get $(2l-n-3k)\,T(r)\le S(r)$, which is a contradiction since $l>m+3k$. Therefore $c_{k-1}=0$. So $(P(f))^{(k-1)}\equiv(P(g))^{(k-1)}$. Proceeding in this way we get $(P(f))'\equiv(P(g))'$. Integrating we get $P(f)\equiv P(g)+c_0$. If possible let $c_0\neq 0$. Using the second fundamental theorem we get

$$\begin{aligned}
n\,T(r,f) &= T(r,P(f))+O(1)\\
&\le \overline{N}(r,0;P(f))+\overline{N}(r,\infty;P(f))+\overline{N}(r,c_0;P(f))\\
&\le \overline{N}(r,0;P(f))+\overline{N}(r,\infty;f)+\overline{N}(r,0;P(g))\\
&\le (n-l+2)\,T(r,f)+(n-l+1)\,T(r,g)+S(r,f)\\
&\le (2n-2l+3)\,T(r)+S(r).
\end{aligned}$$

Similarly we get $n\,T(r,g)\le(2n-2l+3)\,T(r)+S(r)$. Combining these we get $(2l-n-3)\,T(r)\le S(r)$, which is a contradiction since $l>m+3$. Therefore $c_0=0$ and so $P(f)\equiv P(g)$, i.e.,

$$f_1^l(b_mf_1^m+b_{m-1}f_1^{m-1}+\ldots+b_0)\equiv g_1^l(b_mg_1^m+b_{m-1}g_1^{m-1}+\ldots+b_0), \tag{3.2}$$

where $f_1 = f - e$ and $g_1 = g - e$. Suppose $P_1(z_1)$ is not a monomial.

Let $h = \frac{f_1}{g_1}$. If h is a constant, then substituting $f_1 = g_1 h$ into (3.2) we deduce that

$$b_m g_1^{l+m}(h^{l+m}-1) + b_{m-1} g_1^{b+m-1}(h^{l+m-1}-1) + \ldots + c_0 g_1^l (h^l - 1) \equiv 0,$$

which implies $h^{d_0} = 1$, where $d_0 = GCD(l+m, \ldots, l+m-i, \ldots, l)$, $b_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f_1 \equiv t g_1$, i.e., $f(z) - e \equiv t(g(z) - e)$ for $t \in \mathbb{C}$ such that $t^{d_0} = 1$, where $d_0 = GCD(l+m, \ldots, l+m-i, \ldots, l)$, $b_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. If h is not a constant, then we know by (3.2) that f_1 and g_1 satisfying the equation $R(f_1, g_1) = 0$, where $R(\omega_1, \omega_2) = \omega_1^l (b_m \omega_1^m + b_{m-1}\omega_1^{m-1} + \ldots + b_0) - \omega_2^l (b_m \omega_2^m + b_{m-1}\omega_2^{m-1} + \ldots + b_0)$.

Suppose $P_1(z_1)$ is a monomial, say $P_1(z_1) = b_i z_1^i \not\equiv 0$, where $i \in \{0, 1, \ldots, m\}$. Then by Lemma 3 we have $f - e \equiv t(g - e)$ for $t \in \mathbb{C}$ such that $t^{l+i} = 1$. This proves the proof. □

Lemma 5. *Let f, g be two non-constant meromorphic functions and let $F = (P(f))^{(k)}/\alpha$, $G = (P(g))^{(k)}/\alpha$, where $P(z)$ be defined as in (2.1), α be a small function with respect to f, g and $m \in \mathbb{N} \cup \{0\}$, k, $l \in \mathbb{N}$ such that $l > m + 3k + 3$. Suppose $H \equiv 0$. Then either $(P(f))^{(k)}(P(g))^{(k)} \equiv \alpha^2$, where $(P(f))^{(k)} - \alpha$ and $(P(g))^{(k)} - \alpha$ share 0 CM or $(P(f))^{(k)} \equiv (P(g))^{(k)}$.*

Proof. We have $H \equiv 0$. By integration, we get $\frac{F'}{(F-1)^2} \equiv d \frac{G'}{(G-1)^2}$, where $d \in \mathbb{C} \setminus \{0\}$, i.e.,

$$\frac{\left(\frac{F_1 - \alpha}{\alpha}\right)'}{\left(\frac{F_1 - \alpha}{\alpha}\right)^2} \equiv d \frac{\left(\frac{G_1 - \alpha}{\alpha}\right)'}{\left(\frac{G_1 - \alpha}{\alpha}\right)^2},$$

where $F_1 = (P(f))^{(k)}$ and $G_1 = (P(g))^{(k)}$. This shows that $\frac{F_1 - \alpha}{\alpha}$ and $\frac{G_1 - \alpha}{\alpha}$ share 0 CM and so $F_1 - \alpha$ and $G_1 - \alpha$ share 0 CM. Finally, by integration we get

$$\frac{1}{F-1} \equiv \frac{bG + a - b}{G - 1}, \tag{3.3}$$

where $a(\neq 0), b \in \mathbb{C}$. We now consider the following cases.

Case 1. Let $b \neq 0$ and $a \neq b$. If $b = -1$, from (3.3) we have $F \equiv \frac{-a}{G-a-1}$. Therefore $\overline{N}(r, a+1; G) = \overline{N}(r, \infty; F) = \overline{N}(r, \infty; f) + S(r, f)$. So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned}
n\, T(r,g) &= T(r, P(f)) + O(1) \\
&\leq T(r,G) + N_{k+1}(r,0;P(g)) - \overline{N}(r,0;G) \\
&\leq \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,a+1;G) + N_{k+1}(r,0;P(g)) - \overline{N}(r,0;G) + S(r,g) \\
&\leq \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,e;g) + N(r,0;P(g) \mid g \neq e) + \overline{N}(r,\infty;f) + S(r,g) \\
&\leq T(r,f) + (n - l + k + 2)\, T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$. So for $r \in I$ we have $(l-k-3)\, T(r,g) \leq S(r,g)$, which is a contradiction since $l > k+3$.

If $b \neq -1$, from (3.3) we obtain that $F-(1+\frac{1}{b}) \equiv \frac{-a}{b^2\left(G+\frac{a-b}{b}\right)}$. So $\overline{N}(r,\frac{b-a}{b};G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f)+S(r,f)$. Using Lemma 2 and the same argument as used in the case when $b=-1$ we can get a contradiction.

Case 2. Let $b \neq 0$ and $a = b$. If $b=-1$, then from (3.3) we have $FG \equiv \alpha^2$, i.e., $(P(f))^{(k)}(P(g))^{(k)} \equiv \alpha^2$, where $(P(f))^k - \alpha$ and $(P(g))^k - \alpha$ share 0 CM.
If $b \neq -1$, from (3.3) we have $\frac{1}{F} \equiv \frac{bG}{(1+b)G-1}$. Therefore $\overline{N}(r,\frac{1}{1+b};G) = \overline{N}(r,0;F)$. So in view of Lemma 2 and the second fundamental theorem we get

$$\begin{aligned}
n\,T(r,g) &\leq T(r,G)+N_{k+1}(r,0;P(g))-\overline{N}(r,0;G)+S(r,g)\\
&\leq \overline{N}(r,\infty;G)+\overline{N}(r,0;G)+\overline{N}(r,\frac{1}{1+b};G)+N_{k+1}(r,0;P(g))-\overline{N}(r,0;G)+S(r,g)\\
&\leq \overline{N}(r,\infty;g)+N_{k+1}(r,0;P(g))+\overline{N}(r,0;F)+S(r,g)\\
&\leq \overline{N}(r,\infty;g)+N_{k+1}(r,0;P(g))+N_{k+1}(r,0;P(f))+k\overline{N}(r,\infty;f)+S(r,f)+S(r,g)\\
&\leq (n-l+k+2)\,T(r,g)+(n-l+2k+1)\,T(r,f)+S(r,f)+S(r,g).
\end{aligned}$$

So for $r \in I$ we have $(2l-n-3k-3)\,T(r,g) \leq S(r,g)$, which is a contradiction since $l > m+3k+3$.

Case 3. Let $b=0$. From (3.3) we obtain $F \equiv \frac{G+a-1}{a}$. If $a \neq 1$ then we obtain $\overline{N}(r,1-a;G) = \overline{N}(r,0;F)$. We can similarly deduce a contradiction as in Case 2. Therefore $a=1$ and so $F \equiv G$, i.e., $(P(f))^{(k)} \equiv (P(g))^{(k)}$. This completes the proof. □

Lemma 6 ([15], Theorem 1.24)**.** *Suppose that f is a non-constant meromorphic function in the complex plane and $k \in \mathbb{N}$. Then*

$$N(r,0;f^{(k)}) \leq N(r,0;f)+k\overline{N}(r,\infty;f)+O(\log T(r,f)+\log r),$$

as $r \to \infty$, outside of a possible exceptional set of finite linear measure.

Lemma 7 ([5], Lemma 3.5)**.** *Suppose that F is meromorphic in a domain D and set $f = \frac{F'}{F}$. Then for $n \in \mathbb{N}$,*

$$\frac{F^{(n)}}{F} = f^n + \frac{n(n-1)}{2} f^{n-2}f' + a_n f^{n-3}f'' + b_n f^{n-4}(f')^2 + P_{n-3}(f),$$

where $a_n = \frac{1}{6}n(n-1)(n-2)$, $b_n = \frac{1}{8}n(n-1)(n-2)(n-3)$ and $P_{n-3}(f)$ is a differential polynomial with constant coefficients, which vanishes identically for $n \leq 3$ and has degree $n-3$ when $n > 3$.

Lemma 8 ([2], Lemma 1)**.** *Let f be a meromorphic function on $\mathbb{C}$. If f has bounded spherical derivative on $\mathbb{C}$, then $\rho(f) \leq 2$. If in addition f is entire, then $\rho(f) \leq 1$.*

Lemma 9 ([15], Theorem 2.11)**.** *Let f be a transcendental meromorphic function in the complex plane such that $\rho(f) > 0$. If f has two distinct Borel exceptional values in the extended complex plane, then $\mu(f) = \rho(f)$ and $\rho(f) \in \mathbb{N} \cup \{\infty\}$.*

Lemma 10 ([18])**.** *Let F be a family of meromorphic functions in the unit disc Δ such that all zeros of functions in F have multiplicity greater than or equal to l and all poles of functions in F have multiplicity greater than or equal to j and α be a real number satisfying $-l < \alpha < j$. Then F is not normal in any neighborhood of $z_0 \in \Delta$, if and only if there exist*

(i) *points $z_n \in \Delta$, $z_n \to z_0$,*
(ii) *positive numbers ρ_n, $\rho_n \to 0^+$ and*
(iii) *functions $f_n \in F$,*

such that $\rho_n^{-\alpha} f_n(z_n + \rho_n\zeta) \to g(\zeta)$ spherically locally uniformly in $\mathbb{C}$, where g is a non-constant meromorphic function. The function g may be taken to satisfy the normalisation $g^{\#}(\zeta) \le g^{\#}(0) = 1 (\zeta \in \mathbb{C})$.

Lemma 11. *Let f, g be two transcendental entire functions and p be a non-zero polynomial with $\deg(p) \le n-1$, where $n,\ k \in \mathbb{N}$ such that $n > \max\{2k, k+2\}$. Suppose $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$, where $(f^n)^{(k)} - p$ and $(g^n)^{(k)} - p$ share 0 CM. Now (i) if $p \not\in \mathbb{C}$, then $f(z) = c_1 e^{cQ(z)}$ and $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, $c,\ c_1,\ c_2 \in \mathbb{C}$ such that $(nc)^2(c_1c_2)^n = -1$; (ii) if $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = c_3 e^{dz}$ and $g(z) = c_4 e^{-dz}$, where $c_3,\ c_4,\ d \in \mathbb{C}$ such that $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$.*

Proof. The proof of lemma follows from the proof of Lemma 11 [1]. □

Lemma 12. *Let f, g be two transcendental meromorphic functions and p be a non-zero polynomial with $\deg(p) \le n-1$, where $n,\ k \in \mathbb{N}$ such that $n > \max\{2k, k+2\}$. Suppose $(f^n)^{(k)}(g^n)^{(k)} \equiv p^2$, where $(f^n)^{(k)} - p$ and $(g^n)^{(k)} - p$ share 0 CM. Now (i) if $p \not\in \mathbb{C}$, then $f(z) = c_1 e^{cQ(z)}$ and $g(z) = c_2 e^{-cQ(z)}$, where $Q(z) = \int_0^z p(t)dt$, $c,\ c_1,\ c_2 \in \mathbb{C}$ such that $(nc)^2(c_1c_2)^n = -1$; (ii) if $p(z) = b \in \mathbb{C} \setminus \{0\}$, then $f(z) = c_3 e^{dz}$ and $g(z) = c_4 e^{-dz}$, where $c_3,\ c_4,\ d \in \mathbb{C}$ such that $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$.*

Proof. Suppose

$$(f^n)^{(k)}(g^n)^{(k)} \equiv p^2. \tag{3.4}$$

We consider the following cases.

Case 1. Suppose ∞ is a Picard exceptional value of both f and g. Then f and g are transcendental entire functions. Remaining part follows from Lemma 11.

Case 2. Suppose ∞ is not a Picard exceptional value of either f or g, or both of f and g.

First we suppose ∞ is a Picard exceptional value of g. Let z_0 be a zero of f with multiplicity q_0. Clearly z_0 is a zero of $(f^n)^{(k)}$ with multiplicity $nq_0 - k$. Now from (3.4) we observe that

z_0 must be a zero of p with multiplicity $nq_0 - k$. Since $\deg(p) \le n-1$ and $n > 2k$, from (3.4) we conclude that either f has no zeros or f has finitely many zeros. Therefore in either cases we have $N(r,0;f) = O(\log r)$ as $r \to \infty$.

Next we suppose ∞ is not a Picard exceptional value of g. Let z_1 ($p(z_1) \ne 0$) be a zero of f with multiplicity q_1. Clearly z_1 is a zero of $(f^n)^{(k)}$ with multiplicity $nq_1 - k$. From (3.4) we observe that z_1 must be a pole of g with multiplicity r_1, say. Note that z_1 is a pole of $(g^n)^{(k)}$ with multiplicity $nr_1 + k$. Therefore $nq_1 - k = nr_1 + k$ and so $q_1 > r_1$. Now $nq_1 - k = nr_1 + k$ implies that $n(q_1 - r_1) = 2k$. Since $n > 2k$, we arrive at a contradiction. This shows that z_1 is a zero of p and so we have $N(r,0;f) = O(\log r)$ as $r \to \infty$. Similarly we can prove that $N(r,0;g) = O(\log r)$ as $r \to \infty$.

Let $H = f^n$, $\hat{H} = g^n$, $F = \frac{H}{p}$ and $G = \frac{\hat{H}}{p}$. Let $\mathscr{F} = \{F_\omega\}$ and $\mathscr{G} = \{G_\omega\}$, where $F_\omega(z) = F(z+\omega) = \frac{H(z+\omega)}{p(z+\omega)}$ and $G_\omega(z) = G(z+\omega) = \frac{\hat{H}(z+\omega)}{p(z+\omega)}$, $z \in \mathbb{C}$. Clearly $\mathscr{F}$ and $\mathscr{G}$ are two families of meromorphic functions defined on $\mathbb{C}$. We now consider following two sub-cases.

Sub-case 2.1. Suppose one of the families $\mathscr{F}$ and $\mathscr{G}$, say $\mathscr{F}$, is normal on $\mathbb{C}$. Then by Marty's theorem $F^{\#}(\omega) = F^{\#}_{\omega}(0) \le M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. Hence by Lemma 8 we have $\rho(F) \le 2$. Now from (3.4) we have

$$\rho(f) = \rho\Big(\frac{f^n}{p}\Big) = \rho(f^n) = \rho((f^n)^{(k)}) = \rho((g^n)^{(k)}) = \rho(g^n) = \rho\Big(\frac{g^n}{p}\Big) = \rho(g) \le 2. \tag{3.5}$$

From (3.4), (3.5) and Lemma 6 we have

$$\begin{aligned}(n+k)\,\overline{N}(r,\infty;f) &\le N(r,\infty;(f^n)^{(k)}) = N\Big(r,\infty;\frac{p^2}{(g^n)^{(k)}}\Big)\\ &\le N(r,0;(g^n)^{(k)}) + O(\log r)\\ &\le N(r,0;g^n) + k\,\overline{N}(r,\infty;g^n) + O(\log r) \le k\,\overline{N}(r,\infty;g) + O(\log r),\end{aligned}$$

as $r \to \infty$. Similarly $(n+k)\,\overline{N}(r,\infty;g) \le k\,\overline{N}(r,\infty;f) + O(\log r)$, as $r \to \infty$. Therefore we have $\overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) \le O(\log r)$, as $r \to \infty$. This shows that f and g have at most finitely many poles. On the other hand f and g have at most finitely many zeros. Let

$$f = \frac{P_1}{Q_1}e^{\alpha} \text{ and } g = \frac{P_2}{Q_2}e^{\beta}, \tag{3.6}$$

where P_1, P_2, Q_1 and Q_2 are non-zero polynomials and α and β are non-constant polynomials. Since every zeros of $f(z)$ are the poles of $g(z)$ as well as the zeros of $p(z)$, it follows that $\deg(P_1) < \deg(Q_2)$. Similarly $\deg(P_2) < \deg(Q_1)$. Let $R_1 = \frac{P_1}{Q_1}$ and $R_2 = \frac{P_2}{Q_2}$. Since f and g are transcendental meromorphic functions, from (3.6) we have $\rho(f) > 0$ and $\rho(g) > 0$. We observe from (3.5) and Lemma 9 that $\mu(f) = \rho(f) = 1$ or $\mu(f) = \rho(f) = 2$ and so $\deg(\alpha) \le 2$. Similarly we have $\deg(\beta) \le 2$. Now from (3.6) and Lemma 7 we have

$$(f^n)^{(k)} = \Big(\Big(\alpha' + \frac{R_1'}{R_1}\Big)^k + P^*_{k-1}\Big(\alpha' + \frac{R_1'}{R_1}\Big)\Big)AR_1^n e^{n\alpha}, \tag{3.7}$$

$$(g^n)^{(k)} = \left(\left(\beta' + \frac{R_2'}{R_2}\right)^k + P_{k-1}^*\left(\beta' + \frac{R_2'}{R_2}\right)\right)BR_2^n e^{n\beta}, \tag{3.8}$$

where $A,\ B \in \mathbb{C}\setminus\{0\}$ and $P_{k-1}^*\big(\alpha' + \frac{R_1'}{R_1}\big)\Big(P_{k-1}^*\Big(\beta' + \frac{R_2'}{R_2}\Big)\Big)$ is a differential polynomial of degree at most $k-1$ in $\alpha' + \frac{R_1'}{R_1}\Big(\beta' + \frac{R_2'}{R_2}\Big)$. Now from (3.4), (3.7) and (3.8)

$$AB\left(\left(\alpha' + \frac{R_1'}{R_1}\right)^k + P_{k-1}^*\left(\alpha' + \frac{R_1'}{R_1}\right)\right)\left(\left(\beta' + \frac{R_2'}{R_2}\right)^k + P_{k-1}^*\left(\beta' + \frac{R_2'}{R_2}\right)\right)e^{n(\alpha+\beta)} = \frac{p^2}{(R_1R_2)^n},$$

i.e.,

$$AB\left(\left(\alpha' + \frac{R_1'}{R_1}\right)^k + P_{k-1}^*\left(\alpha' + \frac{R_1'}{R_1}\right)\right)\left(\left(\beta' + \frac{R_2'}{R_2}\right)^k + P_{k-1}^*\left(\beta' + \frac{R_2'}{R_2}\right)\right)e^{n(\alpha+\beta)} = (P^*Q^*)^n p_*^2, \tag{3.9}$$

where P^*, Q^* are non-constant polynomials and p_* is a non-zero polynomial. Since P^*, Q^*, α and β are non-constant polynomials, from (3.9) we have $\alpha+\beta = d_1$, where $d_1 \in \mathbb{C}$. Therefore $\alpha' + \beta' = 0$. Now from (3.9) we have

$$AB\left(\left(\alpha' + \frac{R_1'}{R_1}\right)^k + P_{k-1}^*\left(\alpha' + \frac{R_1'}{R_1}\right)\right)\left(\left(-\alpha' + \frac{R_2'}{R_2}\right)^k + P_{k-1}^*\left(-\alpha' + \frac{R_2'}{R_2}\right)\right)e^{nd_1} = (P^*Q^*)^n p_*^2. \tag{3.10}$$

Letting $|z| \to \infty$, we see that $2k\deg(\alpha') = n\deg(P^*Q^*) + 2\deg(p_*)$. Since $\deg(\alpha') \le 1$ and $n > 2k$, we arrive at a contradiction.

Sub-case 2.2. Suppose that one of the families $\mathscr{F}$ and $\mathscr{G}$, say $\mathscr{F}$ is not normal on $\mathbb{C}$. Then there exists at least one $z_0 \in \Delta$ such that $\mathscr{F}$ is not normal z_0, we assume that $z_0 = 0$. Now by Marty's theorem there exists a sequence of meromorphic functions $\{F(z+\omega_j)\} \subset \mathscr{F}$, where $z \in \{z : |z| < 1\}$ and $\{\omega_j\} \subset \mathbb{C}$ is some sequence of complex numbers such that $F^{\#}(\omega_j) \to \infty$, as $|\omega_j| \to \infty$. Note that p has only finitely many zeros. So there exists a $r > 0$ such that $p(z) \neq 0$ in $D = \{z : |z| \ge r\}$. Since $p(z)$ is a polynomial, for all $z \in \mathbb{C}$ satisfying $|z| \ge r$, we have

$$0 \leftarrow \left|\frac{p'(z)}{p(z)}\right| \le \frac{M_1}{|z|} < 1, \quad p(z) \neq 0. \tag{3.11}$$

Also since $w_j \to \infty$ as $j \to \infty$, without loss of generality we may assume that $|w_j| \ge r+1$ for all j. Let $D_1 = \{z : |z| < 1\}$ and

$$F(w_j + z) = \frac{H(w_j + z)}{p(w_j + z)}.$$

Since $|w_j + z| \ge |w_j| - |z|$, it follows that $w_j + z \in D$ for all $z \in D_1$. Also since $p(z) \neq 0$ in D, it follows that $p(\omega_j + z) \neq 0$ in D_1 for all j. Then by Lemma 10 there exist

(i) points z_j, $|z_j| < 1$,
(ii) positive numbers ρ_j, $\rho_j \to 0^+$,
(iii) a subsequence $\{F(\omega_j + z_j + \rho_j\zeta)\}$ of $\{F(\omega_j + z)\}$

such that

$$h_j(\zeta) = \rho_j^{-k} F(\omega_j + z_j + \rho_j\zeta) \to h(\zeta),$$

i.e.,

$$h_j(\zeta) = \rho_j^{-k} \frac{H(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h(\zeta) \tag{3.12}$$

spherically locally uniformly in $\mathbb{C}$, where $h(\zeta)$ is some non-constant meromorphic function such that $h^{\#}(\zeta) \le h^{\#}(0) = 1$. Now from Lemma 8 we see that $\rho(h) \le 2$. In the proof of Zalcman's lemma (see [9, 19]) we see that

$$\rho_j = \frac{1}{F^{\#}(b_j)} \text{ and } F^{\#}(b_j) \ge F^{\#}(\omega_j), \tag{3.13}$$

where $b_j = \omega_j + z_j$. Note that

$$\frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to 0, \tag{3.14}$$

as $j \to \infty$. By Hurwitz's theorem we can see that the multiplicity of every zero and pole of $h(\zeta)$ is a multiple of n. Therefore we can deduce that $h = \bar{h}^n$, where $\bar{h}$ is some non-constant meromorphic function in the complex plane. We now prove that

$$(h_j(\zeta))^{(k)} = \frac{H^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(k)}(\zeta). \tag{3.15}$$

Note that from (3.12)

$$\begin{aligned}\rho_j^{-k+1} \frac{H'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} &= h_j'(\zeta) + \rho_j^{-k+1} \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p^2(\omega_j + z_j + \rho_j\zeta)} H(\omega_j + z_j + \rho_j\zeta) \\ &= h_j'(\zeta) + \rho_j \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} h_j(\zeta).\end{aligned} \tag{3.16}$$

Now from (3.12), (3.14) and (3.16) we observe that $\rho_j^{-k+1} \frac{H'(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)} \to h'(\zeta)$. Suppose

$$\rho_j^{-k+l} \frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h^{(l)}(\zeta).$$

Let

$$G_j(\zeta) = \rho_j^{-k+l} \frac{H^{(l)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}.$$

Then $G_j(\zeta) \to h^{(l)}(\zeta)$. Note that

$$\rho_j^{-k+l+1} \frac{H^{(l+1)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} = G_j'(\zeta) + \rho_j^{-k+l+1} \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p^2(\omega_j + z_j + \rho_j\zeta)} H^{(l)}(\omega_j + z_j + \rho_j\zeta)$$

$$= G_j'(\zeta) + \rho_j \frac{p'(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} G_j(\zeta). \tag{3.17}$$

So from (3.14) and (3.17) we see that $\rho_j^{-k+l+1} \frac{H^{(l+1)}(\omega_j+z_j+\rho_j\zeta)}{p(\omega_j+z_j+\rho_j\zeta)} \to G_j'(\zeta)$, i.e.,

$$\rho_j^{-k+l+1} \frac{H^{(l+1)}(\omega_j + z_n + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to h_j^{(l+1)}(\zeta).$$

Then by mathematical induction we get desired result (3.15). Let

$$(\hat{h}_j(\zeta))^{(k)} = \frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)}. \tag{3.18}$$

From (3.4) we have

$$\frac{H^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \frac{\hat{H}^{(k)}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} = 1$$

and so from (3.15) and (3.18) we get

$$(h_j(\zeta))^{(k)} (\hat{h}_j(\zeta))^{(k)} = 1. \tag{3.19}$$

From (3.15), (3.19) and formula of higher derivatives we can deduce that $\hat{h}_j(\zeta) \to \hat{h}_1(\zeta)$, i.e.,

$$\frac{\hat{H}(\omega_j + z_j + \rho_j\zeta)}{p(\omega_j + z_j + \rho_j\zeta)} \to \hat{h}_1(\zeta), \tag{3.20}$$

spherically locally uniformly in $\mathbb{C}$, where $\hat{h}_1(\zeta)$ is some non-constant meromorphic function in the complex plane. By Hurwitz's theorem we can see that the multiplicity of every zero and pole of $\hat{h}_1(\zeta)$ is a multiple of n. Therefore we can deduce that $\hat{h}_1 = \hat{h}^n$, where $\hat{h}$ is some non-constant meromorphic function in the complex plane. Therefore (3.20) can be rewritten as $\hat{h}_j(\zeta) \to \hat{h}^n(\zeta)$, spherically locally uniformly in $\mathbb{C}$ and so

$$(\hat{h}_j(\zeta))^{(k)} \to (\hat{h}^n(\zeta))^{(k)} \tag{3.21}$$

spherically locally uniformly in $\mathbb{C}$. From (3.15), (3.19) and (3.21) we get

$$(\bar{h}^n(\zeta))^{(k)} (\hat{h}^n(\zeta))^{(k)} \equiv 1. \tag{3.22}$$

Since $\rho(h) \le 2$, from (3.22) we see that

$$\rho(h) = \rho(\bar{h}^n) = \rho((\bar{h}^n)^{(k)}) = \rho((\hat{h}^n)^{(k)}) = \rho(\hat{h}^n) = \rho(\hat{h}) \le 2. \tag{3.23}$$

Let ζ_0 be a zero of $\bar{h}$ with multiplicity q_0. Note that ζ_0 is a zero of $(\bar{h}^n)^{(k)}$ with multiplicity $nq_0 - k$. From (3.22) we see that ζ_0 must be a pole of $\hat{h}$ with multiplicity r_0, say. Note that

ζ_0 is a pole of $(\hat{h}^n)^{(k)}$ with multiplicity nr_0+k. Therefore $nq_0-k=nr_0+k$ and so $q_0>r_0$. Now $nq_0-k=nr_0+k$ implies that $n(q_0-r_0)=2k$. Since $n>2k$, we arrive at a contradiction. Therefore $\bar{h}\neq 0$. Similarly we can prove that $\hat{h}\neq 0$. From (3.22), (3.23) and Lemma 6 we have

$$\begin{aligned}(n+k)\,\overline{N}(r,\infty;\bar{h}) &\leq N(r,\infty;(\bar{h}^n)^{(k)}) = N\Big(r,\infty;\frac{1}{(\hat{h}^n)^{(k)}}\Big)\\ &\leq N(r,0;(\hat{h}^n)^{(k)})+O(\log r)\\ &\leq N(r,0;\hat{h}^n)+k\,\overline{N}(r,\infty;\hat{h}^n)+O(\log r)\leq k\,\overline{N}(r,\infty;\hat{h})+O(\log r),\end{aligned}$$

as $r\to\infty$. Similarly $(n+k)\,\overline{N}(r,\infty;\hat{h})\leq k\,\overline{N}(r,\infty;\bar{h})+O(\log r)$, as $r\to\infty$. Therefore we have $\overline{N}(r,\infty;\bar{h})+\overline{N}(r,\infty;\hat{h})\leq O(\log r)$, as $r\to\infty$. This shows that $\bar{h}$ and $\hat{h}$ have at most finitely many poles. Let

$$\bar{h}=\frac{1}{P_3}e^{\alpha_2}\text{ and }\hat{h}=\frac{1}{Q_3}e^{\beta_2}, \tag{3.24}$$

where P_3, Q_3 are non-zero polynomials and α_2, β_2 are non-constant polynomials. Since $\bar{h}$ and $\hat{h}$ are transcendental meromorphic functions, from (3.24) we have $\rho(\bar{h})>0$ and $\rho(\hat{h})>0$. We observe from (3.23) and Lemma 9 that $\mu(\bar{h})=\rho(\bar{h})=1$ or $\mu(\bar{h})=\rho(\bar{h})=2$ and so $\deg(\bar{h})\leq 2$. Similarly we have $\deg(\hat{h})\leq 2$. Next in the same manner as in Sub-case 2.1, we get $2k\deg(\alpha_2')=n\deg(P_3Q_3)$. Since $\deg(\alpha_2')\leq 1$ and $n>2k$, we can deduce that P_3, $Q_3\in\mathbb{C}$. This shows that ∞ is a Picard exceptional value of both $\bar{h}$ and $\hat{h}$. Combining this with Theorem 1 in Fang [4] and the assumption $n>2k$, we get

$$\bar{h}(z)=\bar{c}_1e^{cz}\text{ and }\hat{h}(z)=\hat{c}_2e^{-cz}, \tag{3.25}$$

where c, $\bar{c}_1$, $\hat{c}_2\in\mathbb{C}$ such that $(-1)^k(\bar{c}_1\hat{c}_2)^n(nc)^{2k}=1$. Since $h=\bar{h}^n$, from (3.12) and (3.25) we have

$$\frac{h_j'(\zeta)}{h_j(\zeta)}=\rho_j\frac{F'(w_j+z_j+\rho_j\zeta)}{F(w_j+z_j+\rho_j\zeta)}\to\frac{h'(\zeta)}{h(\zeta)}=nc, \tag{3.26}$$

spherically locally uniformly in $\mathbb{C}$. From (3.13) and (3.26) we get

$$\rho_j\Big|\frac{F'(\omega_j+z_j)}{F(\omega_j+z_j)}\Big|=\frac{1+|F(\omega_j+z_j)|^2}{|F'(\omega_j+z_j)|}\frac{|F'(\omega_j+z_j)|}{|F(\omega_j+z_j)|}=\frac{1+|F(\omega_j+z_j)|^2}{|F(\omega_j+z_j)|}\to\Big|\frac{h'(0)}{h(0)}\Big|=n|c|,$$

which implies that

$$\lim_{j\to\infty}F(\omega_j+z_j)\neq 0,\infty. \tag{3.27}$$

From (3.12) and (3.27) we see that

$$h_j(0)=\rho_j^{-k}F(\omega_j+z_j)\to\infty. \tag{3.28}$$

Again from (3.12) and (3.25) we have

$$h_j(0) \to h(0) = \bar{c}_1^n. \tag{3.29}$$

Now from (3.28) and (3.29) we arrive at a contradiction. This completes the proof. □

From Lemma 12 we have the following lemma.

Lemma 13. *Let f, g be two transcendental meromorphic functions and p be a non-zero polynomial with $\deg(p) \le n-1$, where $n,\ k \in \mathbb{N}$ with $n > \max\{2k, k+2\}$. Let $((f-a)^n)^{(k)} - p$, $((g-a)^n)^{(k)} - p$ share 0 CM and $((f-a)^n)^{(k)}((g-a)^n)^{(k)} \equiv p^2$. Now* (i) *if $p \not\subset \mathbb{C}$, then $f(z) = c_1 e^{cQ(z)} + a$ and $g(z) = c_2 e^{-cQ(z)} + a$, where $Q(z) = \int_0^z p(t)\,dt$, c, c_1, $c_2 \in \mathbb{C}$ such that $(nc)^2(c_1c_2)^n = -1$;* (ii) *if $p(z) = b \in \mathbb{C}\setminus\{0\}$, then $f(z) = c_3 e^{dz} + a$ and $g(z) = c_4 e^{-dz} + a$, where d, c_3, $c_4 \in \mathbb{C}$ such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$.*

Lemma 14. *Let f, g be two transcendental meromorphic functions and let $m,\ n,\ k \in \mathbb{N}$ such that $n > 2k$. Let $P(z) = \sum_{i=0}^m a_i z^i$ be a non-zero polynomial such that $P(z)$ is not a monomial. If $(f^nP(f))^{(k)}(g^nP(g))^{(k)} \equiv 1$, then f is of order at most 2.*

Proof. We have

$$(f^nP(f))^{(k)}(g^nP(g))^{(k)} \equiv 1. \tag{3.30}$$

Without loss of generality we may assume that $a_m, a_0 \ne 0$. Let $\mathscr{F} = \{f_\omega\}$ and $\mathscr{G} = \{g_\omega\}$, where $f_\omega(z) = f(z+\omega)$ and $g_\omega(z) = g(z+\omega)$, $z \in \mathbb{C}$. Clearly $\mathscr{F}$ and $\mathscr{G}$ are two families of meromorphic functions defined on $\mathbb{C}$. We now consider following two sub-cases.

Sub-case 2.1. Suppose one of the families $\mathscr{F}$ and $\mathscr{G}$, say $\mathscr{F}$, is normal on $\mathbb{C}$. Then by Marty's theorem $f^{\#}(\omega) = f_\omega^{\#}(0) \le M$ for some $M > 0$ and for all $\omega \in \mathbb{C}$. By Lemma 8, $\rho(f) \le 2$.

Sub-case 2.2. Suppose one of the families $\mathscr{F}$ and $\mathscr{G}$, say $\mathscr{F}$ is not normal on $\mathbb{C}$. Then there exists at least one $z_0 \in \Delta$ such that $\mathscr{F}$ is not normal z_0, we assume that $z_0 = 0$. Now by Marty's theorem there exists a sequence of meromorphic functions $\{f(z+\omega_j)\} \subset \mathscr{F}$, where $z \in \{z : |z| < 1\}$ and $\{\omega_j\} \subset \mathbb{C}$ is some sequence of complex numbers such that $f^{\#}(\omega_j) \to \infty$, as $|\omega_j| \to \infty$. Then by Lemma 10 there exist

(i) points z_j, $|z_j| < 1$,
(ii) positive numbers ρ_j, $\rho_j \to 0^+$,
(iii) a subsequence $\{f(\omega_j + z_j + \rho_j\zeta) = f_j(z_j + \rho_j\zeta)\}$ of $\{f(\omega_j + z)\}$

such that

$$h_j(\zeta) = \rho_j^{-\frac{k}{n}} f_j(z_j + \rho_j\zeta) \to h(\zeta) \tag{3.31}$$

spherically locally uniformly in $\mathbb{C}$, where $h(\zeta)$ is some non-constant meromorphic function such that $h^{\#}(\zeta) \le h^{\#}(0) = 1$. Now from Lemma 8 we see that $\rho(h) \le 2$. In the proof of Zalcman's lemma (see [9, 19]) we see that

$$\rho_j = \frac{1}{f^{\#}(b_j)} \text{ and } f^{\#}(b_j) \ge f^{\#}(\omega_j), \tag{3.32}$$

where $b_j = \omega_j + z_j$. Now (3.31) yields

$$\begin{aligned}&\left(a_m \rho_j^{\frac{km}{n}} h_j^{n+m}(\zeta) + \ldots + a_1 \rho_j^{\frac{k}{n}} h_j^{n+1}(\zeta) + a_0 h_j^n(\zeta)\right)^{(k)} \\ &\quad = \left(a_m f_j^{n+m}(z_j + \rho_j\zeta) + \ldots + a_1 f_j^{n+1}(z_j + \rho_j\zeta) + a_0 f_j^n(z_j + \rho_j\zeta)\right)^{(k)} \to a_0 (h^n(\zeta))^{(k)},\end{aligned} \tag{3.33}$$

spherically locally uniformly in $\mathbb{C}$. Next we suppose $\hat{h}_j(\zeta) = \rho_j^{-\frac{k}{n}} g_j(z_j + \rho_j\zeta)$. Therefore

$$\begin{aligned}&\left(a_m \rho_j^{\frac{km}{n}} \hat{h}_j^{n+m}(\zeta) + \ldots + a_1 \rho_j^{\frac{k}{n}} \hat{h}_j^{n+1}(\zeta) + a_0 \hat{h}_j^n(\zeta)\right)^{(k)} \\ &\quad = \left(a_m g_j^{n+m}(z_j + \rho_j\zeta) + \ldots + a_1 g_j^{n+1}(z_j + \rho_j\zeta) + a_0 g_j^n(z_j + \rho_j\zeta)\right)^{(k)}.\end{aligned} \tag{3.34}$$

Now from (3.30), (3.33) and (3.34) we have

$$\left(a_m \rho_j^{\frac{km}{n}} h_j^{n+m}(\zeta) + \ldots + a_0 h_j^n(\zeta)\right)^{(k)} \left(a_m \rho_j^{\frac{km}{n}} \hat{h}_j^{n+m}(\zeta) + \ldots + a_0 \hat{h}_j^n(\zeta)\right)^{(k)} \equiv 1. \tag{3.35}$$

Letting $j \to \infty$, from (3.33), (3.35) and the formula of higher derivatives we can deduce that

$$a_m \rho_j^{\frac{km}{n}} \hat{h}_j^{n+m}(\zeta) + \ldots + a_0 \hat{h}_j^n(\zeta) \to a_0 \hat{h}_1(\zeta), \tag{3.36}$$

spherically locally uniformly in $\mathbb{C}$, where $\hat{h}_1(\zeta)$ is some non-constant meromorphic function in the complex plane. Now from (3.33), (3.35) and (3.36) we observe that

$$a_0^2 (h^n(\zeta))^{(k)} (\hat{h}_1(\zeta))^{(k)} \equiv 1. \tag{3.37}$$

By Hurwitz's theorem we can see that the multiplicity of every zero and pole of $\hat{h}_1(\zeta)$ is a multiple of n. Therefore we can deduce that $\hat{h}_1 = \hat{h}^n$, where $\hat{h}$ is some non-constant meromorphic function in the complex plane. Therefore (3.36) can be rewritten as

$$a_m \rho_j^{\frac{km}{n}} \hat{h}_j^{n+m}(\zeta) + \ldots + a_0 \hat{h}_j^n(\zeta) \to a_0 \hat{h}^n(\zeta),$$

spherically locally uniformly in $\mathbb{C}$ and so $\left(a_m \rho_j^{\frac{km}{n}} \hat{h}_j^{n+m}(\zeta) + \ldots + a_0 \hat{h}_j^n(\zeta)\right)^{(k)} \to a_0 (\hat{h}^n(\zeta))^{(k)}$ spherically locally uniformly in $\mathbb{C}$. From (3.37) we see that $a_0^2 (h^n(\zeta))^{(k)} (\hat{h}^n(\zeta))^{(k)} \equiv 1$. Now by applying Sub-case 2.2 of Lemma 12, we arrive at a contradiction.

This completes the proof. □

Lemma 15. *Let f and g be two transcendental meromorphic functions such that $\overline{E}_{l)}(p;(P(f))^{(k)}) = \overline{E}_{l)}(p;(P(g))^{(k)})$ and $E_{1)}(p;(P(f))^{(k)}) = E_{1)}(p;(P(g))^{(k)})$, where $l \in \mathbb{N}\setminus\{1,2\}$, $P(z)$ be defined as in (2.1) and $p(z)(\not\equiv 0)$ is a polynomial. If*

$$\Delta_{1l} = \left(\frac{8}{3}+\frac{2}{3}k\right)\Theta(\infty,P(f)) + (k+2)\,\Theta(\infty,P(g)) + \Theta(0,P(f)) + \Theta(0,P(g)) + \frac{5}{3}\delta_{k+1}(0,P(f)) + \delta_{k+1}(0,P(g)) > \frac{5}{3}(k+5), \tag{3.38}$$

then $H \equiv 0$.

Proof. Proof of lemma follows from Lemma 8 [10]. □

Lemma 16. *Let f and g be two transcendental meromorphic functions such that $\overline{E}_{l)}(p;(P(f))^{(k)}) = \overline{E}_{l)}(p;(P(g))^{(k)})$ and $E_{2)}(p;(P(f))^{(k)}) = E_{2)}(p;(P(g))^{(k)})$, where $l \in \mathbb{N}\setminus\{1,2,3\}$, $P(z)$ be defined as in (2.1) and $p(z)(\not\equiv 0)$ is a polynomial. If*

$$\Delta_{2l} = \left(\frac{1}{2}k+2\right)[\Theta(\infty,f)+\Theta(\infty,g)] + \Theta(0,f) + \Theta(0,g) + \delta_{k+1}(0,f) + \delta_{k+1}(0,g) > (k+7), \tag{3.39}$$

then $H \equiv 0$.

Proof. Proof of lemma follows from Lemma 9 [10]. □

4. Proof of the Theorem

Proof of Theorem 1. Let $F = P(f)$ and $G = P(g)$. Then we see that $\overline{E}_{l)}(p;F^{(k)}) = \overline{E}_{l)}(p;G^{(k)})$ and $\overline{E}_{1)}(p;F^{(k)}) = \overline{E}_{1)}(p;G^{(k)})$, where $l \in \mathbb{N}\setminus\{1,2\}$. Note that

$$\begin{aligned}\Theta(\infty;P(f)) &= 1 - \limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,\infty;P(f))}{T(r,P(f))}\\ &= 1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,\infty;f)}{nT(r,f)} \geq 1-\limsup_{r\longrightarrow\infty}\frac{T(r,f)}{nT(r,f)} \geq 1-\frac{1}{n} = \frac{n-1}{n},\end{aligned} \tag{4.1}$$

$$\begin{aligned}\delta_{k+1}(0;P(f)) &= 1-\limsup_{r\longrightarrow\infty}\frac{N_{k+1}(r,0;P(f))}{T(r,P(f))}\\ &\geq 1-\limsup_{r\longrightarrow\infty}\frac{\sum_{\substack{j=1\\ l_j\neq l}}^{s} N_{k+1}(r,0;(f-c_{l_j})^{l_j}) + N_{k+1}(r,0;(f-c_l)^l)}{nT(r,f)}\\ &\geq 1-\limsup_{r\longrightarrow\infty}\frac{(s-1)T(r,f)+(k+1)T(r,f)+S(r,f)}{nT(r,f)}\\ &\geq 1-\frac{s+k}{n} \geq \frac{l-k-1}{n},\end{aligned} \tag{4.2}$$

$$\begin{aligned}\Theta(0;P(f)) &= 1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,0;P(f))}{T(r,P(f))}\\ &\geq 1-\limsup_{r\longrightarrow\infty}\frac{\sum_{\substack{j=1\\ l_j\neq l}}^{s}\overline{N}(r,0;(f-c_{l_j})^{l_j})+\overline{N}(r,0;(f-c_l)^{l})}{nT(r,f)}\\ &\geq 1-\limsup_{r\longrightarrow\infty}\frac{(s-1)T(r,f)+T(r,f)+S(r,f)}{nT(r,f)}\geq 1-\frac{s}{n}\geq\frac{l-1}{n}. \end{aligned}\tag{4.3}$$

Similarly we have

$$\Theta(\infty;P(g))\geq\frac{n-1}{n},\quad \delta_{k+1}(0;P(g))\geq\frac{l-k-1}{n}\ \text{ and }\ \Theta(0;P(g))\geq\frac{l-1}{n}.\tag{4.4}$$

Now in view (3.38) and (4.1)-(4.4) we obtain

$$\begin{aligned}\Delta_{1l} &\geq\left(\frac{14}{3}+\frac{5}{3}k\right)\frac{n-1}{n}+2\,\frac{l-1}{n}+\frac{8}{3}\left(\frac{l-k-1}{n}\right)\\ &=\left(\frac{14}{3}+\frac{5}{3}k\right)\frac{l+r-1}{l+r}+2\,\frac{l-1}{l+r}+\frac{8}{3}\left(\frac{l-k-1}{l+r}\right).\end{aligned}$$

Since $l>\frac{13}{3}k+\frac{11}{3}r+\frac{28}{3}$, we get $\Delta_{1l}>\frac{5}{3}(k+5)$. Therefore by Lemma 15 we have $H\equiv 0$.

Then theorem follows from Lemmas 4, 5, 13 and 14. □

Proof of Theorem 2. Using Lemma 16, theorem can be proved in the line of the proof of Theorem 1. So we omit the details. □

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Department of Mathematics, Raiganj University, Raiganj, West Bengal-733134, India.

E-mail: sm05math@gmail.com

Department of Mathematics, Raiganj University, Raiganj, West Bengal-733134, India.

E-mail: rajib547mandal@gmail.com; rajib_math@yahoo.in