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A NOTE ON TAUBERIAN THEOREMS FOR REGULARLY GENERATED SEQUENCES

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Abstract. We prove some Tauberian theorems which generalize results in [5, Theorems 1 and 2] and [4, Theorem 3.2.2].

1. Introduction

Let (A) denote the Abel method of summability of a sequence (s_n) of real numbers and $\mathscr{S}$ the class of slowly oscillating sequences in the sense of Stanojević [7]. Çanak [1] proved the following theorem known as the generalized Littlewood tauberian theorem [6].

Theorem 1.1. *If $s_n \to L(A)$ and $(s_n) \in \mathscr{S}$, then $s_n \to L$.*

Dik, Dik and Çanak [5] have generalized Theorem 1.1 by means of the concept of regularly generated sequence. Recently, Çanak and Totur [2, 3] have proved some tauberian theorems for which tauberian conditions are given in terms of control modulo of oscillatory behavior of a sequence.

The object of this work is to show that the proof of the main results of Section 3 below and their generalizations are essentially based on the following theorems.

Theorem 1.2. ([3]) *If $s_n \to L(A)$ and $(\omega_n^{(m)}(s))$ is $(C,1)$ slowly oscillating for any integer $m \geq 1$, then $s_n \to L$.*

Theorem 1.3. ([2]) *If $s_n \to L(A)$ and $\omega_n^{(m)}(s) \geq -C$ for some $C \geq 0$ and for any integer $m \geq 1$, then $s_n \to L$*

(for relevant definitions, see Section 2.)

2. Definitions and basic properties

Suppose throughout that $s = (s_n)$ is a sequence of real numbers and any term with

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a negative index is zero. For a sequence (s_n), denote

$$\sigma_n^{(m)}(s) = \begin{cases} \dfrac{1}{n+1}\sum_{k=0}^{n}\sigma_k^{(m-1)}(s) = s_0 + \sum_{k=1}^{n}\dfrac{V_k^{(m-1)}(\Delta s)}{k}, & m \geq 1 \\ s_n, & m = 0 \end{cases}$$

where

$$V_n^{(m)}(\Delta s) = \begin{cases} \dfrac{1}{n+1}\sum_{k=0}^{n} V_k^{(m-1)}(\Delta s), & m \geq 1 \\ \dfrac{1}{n+1}\sum_{k=0}^{n} k\Delta s_k, & m = 0 \end{cases}$$

and $\Delta s_n = \begin{cases} s_n - s_{n-1}, & n \geq 1 \\ s_0, & n = 0 \end{cases}$. Note that for any integer $m \geq 0$, $V_n^{(m)}(\Delta\sigma^{(1)}(s)) = V_n^{(m+1)}(\Delta s)$. For the sequence (s_n),

$$s_n - \sigma_n^{(1)}(s) = V_n^{(0)}(\Delta s). \tag{2.1}$$

Since $\sigma_n^{(1)}(s) = s_0 + \sum_{k=1}^{n}\frac{V_k^{(0)}(\Delta s)}{k}$, we may write (2.1) as

$$s_n = V_n^{(0)}(\Delta s) + \sum_{k=1}^{n}\frac{V_k^{(0)}(\Delta s)}{k} + s_0. \tag{2.2}$$

A sequence (s_n) is said to be Abel summable to L, and we write $s_n \to L(A)$ if $\sum_{n=0}^{\infty}(s_n - s_{n-1})x^n$ converges for $0 < x < 1$ and tends to L as $x \to 1^-$. A sequence (s_n) is said to be slowly oscillating if $\lim_{\lambda\to 1^+}\overline{\lim_n}\max_{n+1\leq k\leq[\lambda n]}|s_k - s_n| = 0$. A sequence (s_n) is said to be $(C,1)$ slowly oscillating if $(\sigma_n^{(1)}(s))$ is slowly oscillating. A sequence (s_n) is said to be moderately oscillating if, for $\lambda > 1$, $\overline{\lim_n}\max_{n+1\leq k\leq[\lambda n]}|s_k - s_n| < \infty$. Denote by $\mathscr{M}$ the class of moderately oscillating sequences.

It is shown in [4] that if (s_n) is slowly oscillating, then $(V_n^{(0)}(\Delta s))$ is bounded. It is clear, by (2.2), that a sequence (s_n) is slowly oscillating if and only if $(V_n^{(0)}(\Delta s))$ is bounded and slowly oscillating.

Denote by $\omega_n^{(0)}(s) = n\Delta s_n$ the classical control modulo of the oscillatory behavior of (s_n). For each integer $m \geq 1$, define recursively $\omega_n^{(m)}(s) = \omega_n^{(m-1)}(s) - \sigma_n^{(1)}(\omega^{(m-1)}(s))$, the general control modulo of the oscillatory behavior of the sequence (s_n) of order m.

For each integer $m \geq 1$, all nonnegative integers n and for a sequence $s = (s_n)$ we define inductively

$$(n\Delta)_0 s_n = s_n, \quad (n\Delta)_m s_n = n\Delta((n\Delta)_{m-1}s_n).$$

Lemma 2.1. ([2]) *For each integer $m \geq 1$, $\omega_n^{(m)}(s) = (n\Delta)_m V_n^{(m-1)}(\Delta s)$.*

Let $\mathscr{L}$ be any linear space of sequences and $\mathscr{B}$ be a subclass of $\mathscr{L}$. For each integer $m \geq 1$, define the class $\mathscr{B}^{(m)} = \{(b_n^{(m)}) | b_n^{(m)} = \sum_{k=1}^{n}\frac{b_k^{(m-1)}}{k}\}$, where $(b_n^{(0)}) := (b_n) \in \mathscr{B}$. Let $s = (s_n) \in \mathscr{L}$. If

$$s_n = b_n^{(m)} + \sum_{k=1}^{n}\frac{b_k^{(m)}}{k}$$

for some $b^{(m)} = (b_n^{(m)}) \in \mathscr{B}^{(m)}$, we say that the sequence (s_n) is regularly generated by the sequence $(b_n^{(m)})$ and $b^{(m)}$ is called a generator of (s_n). The class of all sequences regularly generated by sequences in $\mathscr{B}^{(m)}$ is denoted by $U(\mathscr{B}^{(m)})$.

Let $\mathscr{B}_{>}$ denote the class of all sequences $b = (b_n)$ such that for every $(b_n) \in \mathscr{B}_{>}$ there exists $C_b \geq 0$ such that $b_n \geq -C_b$. $U(\mathscr{B}_{>}^{(m)})$ can be defined in the same manner as in definition above.

Let $\mathscr{B} = \mathscr{S}$. It follows from the definition that if $(s_n) \in U(\mathscr{S})$, then $(V_n^{(0)}(\Delta s)) \in U(\mathscr{S})$ and $(\sigma_n^{(1)}(s)) \in U(\mathscr{S}^{(1)})$. If $\mathscr{B}$ is the class of all bounded and slowly sequences, then $U(\mathscr{B})$ is the class of all slowly oscillating sequences.

3. Main result

For the results in this section, we require the following lemma.

Lemma 3.1. *Let $s = (s_n) \in \mathscr{L}$ and $k, m \geq 0$ be any integers. If $(V_n^{(k)}(\Delta s)) \in U(\mathscr{B}^{(m)})$, then $(n\Delta)_{k+1} V_n^{(k+1)}(\Delta s) = b_n$.*

((b_n) is as in Definition in Section 2.)

Proof. If $(V_n^{(k)}(\Delta s)) \in U(\mathscr{B}^{(m)})$, it then follows that

$$V_n^{(k)}(\Delta s) = \sigma_n^{(k-1)}(s) - \sigma_n^{(k)}(s) = b_n^{(m)} + \sum_{j=1}^{n} \frac{b_j^{(m)}}{j} \tag{3.1}$$

for some $(b_n^{(m)}) \in \mathscr{B}^{(m)}$. From (3.1), we obtain

$$V_n^{(k-1)}(\Delta s) - V_n^{(k)}(\Delta s) = n\Delta b_n^{(m)} + b_n^{(m)}. \tag{3.2}$$

Subtracting (3.2) from the arithmetic mean of (3.2), we have

$$(V_n^{(k-1)}(\Delta s) - V_n^{(k)}(\Delta s)) - (V_n^{(k)}(\Delta s) - V_n^{(k+1)}(\Delta s)) = b_n^{(m-1)}. \tag{3.3}$$

(3.3) can be expressed as

$$n\Delta V_n^{(k)}(\Delta s) - n\Delta V_n^{(k+1)}(\Delta s) = b_n^{(m-1)},$$

which implies $(n\Delta)_2 V_n^{(k+1)}(\Delta s) = b_n^{(m-1)}$. By repeating the same reasoning, we have

$$\sigma_n^{(1)}(\omega^{(k+1)}(s)) = (n\Delta)_{k+1} V_n^{(k+1)}(\Delta s) = b_n^{(0)} = b_n.$$

Theorem 3.2. *If $s_n \to L(A)$ and $(V_n^{(m)}(\Delta s)) \in U(\mathscr{S}^{(m)})$ for any integer $m \geq 1$, then $s_n \to L$.*

Proof. In Lemma 3.1, take $\mathscr{B} = \mathscr{S}$ and $k = m$. Then $\sigma_n^{(1)}(\omega^{(m+1)}(s)) = b_n$. If $(b_n) \in \mathscr{S}$, $(\omega^{(m+1)}(s))$ is $(C,1)$ slowly oscillating. By Theorem 1.2, we have $s_n \to L$.

Theorem 3.3. *If $s_n \to L(A)$ and $(V_n^{(m-1)}(\Delta s)) \in U(\mathscr{S}^{(m)})$ for any integer $m \geq 1$, then $s_n \to L$.*

Proof. In Lemma 3.1, take $\mathscr{B} = \mathscr{S}$ and $k = m-1$. Then $\omega^{(m+1)}(s) = b_n$. If $(b_n) \in S$, $(\omega^{(m+1)}(s))$ is $(C,1)$ slowly oscillating. By Theorem 1.2, we have $s_n \to L$.

Similar results can be given for one-sidedly regularly generated sequences.

Theorem 3.4. *If $s_n \to L(A)$ and $(V_n^{(m)}(\Delta s)) \in U(\mathscr{S}_>^{(m)})$ for any integer $m \geq 1$, then $s_n \to L$.*

Proof. In Lemma 3.1, take $\mathscr{B} = \mathscr{S}$ and $k = m-1$. Then $\omega^{(m+1)}(s) = b_n$. Since $\sigma_n^{(1)}(\omega^{(m+1)}(s)) = \omega_n^{(m+1)}(\sigma^{(1)}(s))$ and $s_n \to L(A)$ implies $\sigma_n^{(1)}(s) \to L(A)$, $\sigma_n^{(1)}(s) \to L$ by Theorem 1.3.

Theorem 3.5. *If $s_n \to L(A)$ and $(V_n^{(m-1)}(\Delta s)) \in U(\mathscr{S}_>^{(m)})$ for any integer $m \geq 1$, then $s_n \to L$.*

Proof. In Lemma 3.1, take $\mathscr{B} = \mathscr{S}$ and $k = m-1$. Then $\omega^{(m+1)}(s) = b_n$. We have $s_n \to L$ by Theorem 1.2.

Corollary 3.6. *If $\sigma_n^{(1)}(s) \to L(A)$ and $(V_n^{(0)}(\Delta s)) \in U(\mathscr{S})$, then $s_n \to L$.*

Proof. If $(V_n^{(0)}(\Delta s)) \in U(\mathscr{S})$, we have $(V_n^{(1)}(\Delta s)) \in U(\mathscr{S}^{(1)})$. Recalling $V_n^{(1)}(\Delta s) = V_n^{(0)}(\Delta\sigma^{(1)}(s))$, by Theorem 3.2 we have $\sigma_n^{(1)}(s) \to L$, which implies $s_n \to L(A)$. Again by Theorem 3.2, $s_n \to L$.

Corollary 3.6. is Theorem 1 in [5].

For the next corollary which is Theorem 2 in [5], we need the following generalization of Theorem 1.2.

Theorem 3.7. *If $\sigma_n^{(1)}(s) \to L(A)$ and $(\omega_n^{(m)}(s))$ is $(C,1)$ slowly oscillating for any integer $m \geq 1$, then $s_n \to L$.*

Proof. It is clear that $\sigma_n^{(1)}(\omega^{(m)}(s)) = \omega_n^{(m)}(\sigma^{(1)}(s))$. By Theorem 1.2 $\sigma_n^{(1)}(s) \to L$, which implies $s_n \to L(A)$. Again by Theorem 1.2, $s_n \to L$.

Corollary 3.8. *If $\sigma_n^{(1)}(s) \to L(A)$ and $(V_n^{(0)}(\Delta s)) \in U(\mathscr{M}^{(1)})$, then $s_n \to L$.*

Proof. If $(V_n^{(0)}(\Delta s)) \in U(\mathscr{M}^{(1)})$, we have $(\sigma_n^{(1)}(\omega^{(2)}(s))) \in \mathscr{S}$. By Theorem 3.7, we have $s_n \to L$.

Corollary 3.9. *If $\sigma_n^{(1)}(s) \to L(A)$ and $(s_n) \in U(\mathscr{B}_>^{(1)})$, then $s_n \to L$.*

Proof. If $(s_n) \in U(\mathscr{B}_>^{(1)})$, we have $s_n = b_n^{(1)} + \sum_{k=1}^{n} \frac{b_k^{(1)}}{k}$ for some $(b_n^{(1)}) \in \mathscr{B}^{(1)}$. Hence, we obtain

$$n\Delta s_n = b_n + \sum_{k=1}^{n} \frac{b_k}{k}. \tag{3.4}$$

It follows from (3.4) and (2.1) that

$$\omega_n^{(1)}(s) = b_n = n\Delta V_n^{(0)}(\Delta s) \geq -C$$

for some $C \geq 0$, which implies $\sigma_n^{(1)}(\omega^{(1)}(s)) = \omega_n^{(1)}(\sigma^{(1)}(s)) = n\Delta V_n^{(1)}(\Delta s) \geq -C$. Since $s_n \to L(A)$ implies $\sigma_n^{(1)}(s) \to L(A)$, by Theorem 1.3, we have $\sigma_n^{(1)}(s) \to L$, which implies $s_n \to L(A)$. Since $\omega_n^{(1)}(s) \geq -C$ for some $C \geq 0$, again by Theorem 1.3, we have $s_n \to L$.

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