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LOWER BOUNDS OF GENERALIZED NORMALIZED δ-CASORATI CURVATURE FOR REAL HYPERSURFACES IN COMPLEX QUADRIC ENDOWED WITH SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The main intention of the present paper is to develop two extremal inequalities involving the normalized δ-Casorati curvature and the extrinsic generalized normalized δ-Casorati curvature for real hypersurfaces in complex quadric Q^m admitting semi-symmetric metric connection. Further, we derive the necessary and sufficient condition for the equality in both cases.

1. Introduction

In 1993, B. Y. Chen [1] deliberated the notion of Chen invariants (or δ-invariants) and obtained some optimal inequalities consisting of intrinsic and some extrinsic invariants for any Riemannian submanifolds [2].

Moreover, Casorati introduced the theory of extrinsic invariant given by the normalized square of the second fundamental form which is called as Casorati curvature of submanifold in a Riemannian manifold [7]. This theory of Casorati curvature is the extended version of the notion of the principal curvatures of hypersurface in a Riemannian manifold. It is very fast growing area of research to obtain geometric relations or inequalities concerning the esteemed curvature. Thus, it is both important and very interesting to develop some extremal inequalities for the Casorati curvatures of submanifolds in any ambient Riemannian manifolds.

Additionally, many research articles involving optimal inequalities for the Casorati curvatures have been published by different authors for different submanifolds and ambient spaces in complex as well as in contact geometry. Some optimal inequalities containing Casorati

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curvatures were obtained for submanifolds of real space forms, complex space forms, Kenmotsu space forms, quaternionic space forms and also for hypersurfaces in complex two-plane Grassmannians, complex hyperbolic two-plane Grassmannians ([3]-[5],[9]-[10],[14]-[17]). Many geometers studied some geometric and fundamental results on real hypersurfaces of a complex quadric Q^m ([11]-[13],[18]-[20]).

However, Hayden [21] originated the idea of a semi-symmetric metric connection on a Riemannian manifold. Yano [23] deliberated this connection and found some properties of a Riemannian manifold with same connection. Also, A. Mihai and C. Özgür studied Chen extremities for submanifolds of real space forms with same connection [22].

In the present paper, we obtain optimal inequalities for real hypersurfaces of complex quadric Q^m involving the normalized δ-Casorati curvature and the extrinsic generalized normalized δ-Casorati curvatures admitting semi-symmetric metric connection. The equality cases are also considered.

As long as, by virtue of simpleness, throughout a paper we denote semi-symmetric metric connection and Levi-Civita connection by SSMC and LC connection respectively.

2. The complex quadric Q^m

The complex hypersurface of CP^{m+1} is said to be complex quadric Q^m defined by the relation $z_1^2+\cdots+z_{m+1}^2=0$, where $z_1,\ldots,z_{m+1}$ are homogeneous coordinates on CP^{m+1}. Then, naturally the Kähler structure on CP^{m+1} induces a standard Kähler structure (J,g) on Q^m [19]. The $1-$dimensional Q^1 and $2-$dimensional Q^2 are congruent to the round $2-$sphere S^2 and the Riemannian product $S^2\times S^2$ respectively. Thus, we will assume that m is greater than or equal to 3 throughout the paper.

Apart from J there is one more geometric structure on Q^m, known as complex conjugation A on the tangent spaces of Q^m which is a bundle of two vectors $\mathscr{U}$ containing S^1-bundle of real structures. For $p^{'}\in Q^m$, let $A_{\overline{p'}}$ be the shape operator of Q^m in $\mathbb{C}P^{m+1}$. Thus, $A_{\overline{p'}}W=W$ for $W\in T_{p'}Q^m$, that is, A is an involution or $A_{\overline{p'}}$ is complex conjugation restricted to $T_{p'}Q^m$. Now, $T_{p'}Q^m$ is decomposed as [19]:

$$T_{p'}Q^m=\mathscr{V}(A_{\overline{p'}})\oplus J\mathscr{V}(A_{\overline{p'}}),$$

such that $\mathscr{V}(A_{\overline{p'}})=(+1)$-eigenspace of $A_{\overline{p'}}$ and $J\mathscr{V}(A_{\overline{p'}})=(-1)$-eigenspace of $A_{\overline{p'}}$.

Now, $W\neq 0\in T_{p'}Q^m$ is known as *singular tangent vector* if it is tangent to more than one maximal flat in Q^m. Classification of singular tangent vectors for Q^m are given as:

1. If there exists $A\in\mathscr{U}$ such that W is an eigenvector corresponding to an eigenvalue $(+1)$, then the singular tangent vector W is known as $\mathscr{U}$*-principal*.

2. If there exists $A \in \mathscr{U}$ and orthonormal vectors $U, V \in \mathscr{V}(A)$ such that $W/||W|| = (U + JV)/\sqrt{2}$, then the singular tangent vector W is known as *$\mathscr{U}$-isotropic*.

3. Some general fundamental equations

Here, we first remind some notions for a real hypersurface $\mathscr{M}$ in Q^m.

Let $\mathscr{M}$ be a real hypersurface of Q^m with LC connection ∇ induced from the LC connection $\overline{\nabla}$ in Q^m. Then,

$$JU = \phi U + \eta(U)N,$$

where N is the unit normal vector field of $\mathscr{M}$ and ϕU denotes the tangential component of JU for $U \in \Gamma(T\mathscr{M})$. Here, $\mathscr{M}$ associates an induced *almost contact metric structure* (ϕ, ξ, η, g) satisfying [6]:

$$\begin{cases} \xi = -JN, \eta(\xi) = 1,\ \eta \circ \phi = 0,\ \phi^2 U = -U + \eta(U)\xi,\ \phi\xi = 0, \\ g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V). \end{cases}$$

On the other hand, the fundamental Gauss and Weingarten formulas for $\mathscr{M}$ are outlined as

$$\begin{aligned} \overline{\nabla}_U V &= \nabla_U V + h(U, V) = \nabla_U V + g(SU, V)N, \\ \overline{\nabla}_U N &= -SU, \end{aligned}$$

respectively, for $U, V \in \Gamma(T\mathscr{M})$ and $N \in \Gamma(T^{\perp}\mathscr{M})$, where the second fundamental form h and the shape operator S of $\mathscr{M}$ are correlated by

$$g(h(U, V), N) = g(S_N U, V) = g(SU, V).$$

Moreover, the almost contact metric structure (ϕ, ξ, η, g) satisfies

$$\nabla_U \xi = \phi SU.$$

Now, we can choose $A \in \mathscr{U}_{p'}$ with $N = \cos(\theta)Z_1 + \sin(\theta)JZ_2$ where $Z_1,\ Z_2$ are orthonormal vectors in $\mathscr{V}(A)$ and $0 \leq \theta \leq \frac{\pi}{4}$ is a function on $\mathscr{M}$ (see Proposition 3 in [8]). Since, we know that $\xi = -JN$, we can have the following relations,

$$\begin{aligned} N &= \cos(\theta)Z_1 + \sin(\theta)JZ_2,\ \ AN = \cos(\theta)Z_1 - \sin(\theta)JZ_2, \\ \xi &= \sin(\theta)Z_2 - \cos(\theta)JZ_1,\ \ A\xi = \sin(\theta)Z_2 + \cos(\theta)JZ_1, \end{aligned}$$

which follows that $g(\xi,\ AN) = 0$.

Now, the Gauss equation for $Q^m \subset CP^{m+1}$ is given as [18]

$$\overline{R}(U, V)W = R(U, V)W + g(SU, W)SV - g(SV, W)U.$$

Next, the Riemannian curvature tensor R of connection ∇ have following form [18]:

$$\begin{aligned} g(R(U,V)W,W') = & g(V,W)g(U,W') - g(U,W)g(V,W') + g(\phi V,W)g(\phi U,W') \\ & - g(\phi U,W)g(\phi V,W') - 2g(\phi U,V)g(\phi W,W') \\ & + (AV,W)g(AU,W') - g(AU,W)g(AV,W') \\ & + g(JAV,W)g(JAU,W') + g(SV,W)g(SU,W') \\ & - g(JAU,W)g(JAV,W') - g(SU,W)g(SV,W'), \end{aligned} \tag{3.1}$$

for all $U,V,W,W' \in \Gamma(T\mathscr{M})$.

4. Curvature tensor of real hypersurface $\mathscr{M}$ in Q^m admitting SSMC

In this section, we study SSMC and then we give the expression for the curvature tensor of real hypersurface $\mathscr{M}$ in Q^m with respect to SSMC.

Consider a Riemannian manifold $(\mathscr{M}^n, g)$ with linear connection $\hat{\nabla}$. Then, $\hat{\nabla}$ is called *semi-symmetric connection* [23] if its torsion tensor $\hat{\mathscr{T}}$, defined by

$$\hat{\mathscr{T}}(U,V) = \hat{\nabla}_U V - \hat{\nabla}_V U - [U,V], \tag{4.1}$$

satisfy

$$\hat{\mathscr{T}}(U,V) = \eta(V)U - \eta(U)V, \tag{4.2}$$

for $U,V \in \Gamma(T\mathscr{M})$ and a 1-form η. In addition, a semi-symmetric linear connection $\hat{\nabla}$ is said to be SSMC if it holds

$$\hat{\nabla} g = 0, \tag{4.3}$$

otherwise it is said to be a *semi-symmetric non-metric connection*. A SSMC $\hat{\nabla}$ in terms of the LC connection ∇ on $\mathscr{M}$ is defined by

$$\hat{\nabla}_U V = \nabla_U V + \eta(V)U - g(U,V)\xi \tag{4.4}$$

for $U,V \in \Gamma(T\mathscr{M})$.

Let us consider complex quadric Q^m admitting SSMC $\hat{\overline{\nabla}}$ and the LC connection $\overline{\nabla}$. Now, let $\mathscr{M}$ be a real hypersurface of Q^m with induced SSMC $\hat{\nabla}$ and the induced LC connection ∇. Assume that $\hat{\overline{R}}$ and $\overline{R}$ be the curvature tensors of Q^m with respect to the connections $\hat{\overline{\nabla}}$ and $\overline{\nabla}$ respectively. Put $\hat{R}$ as the curvature tensor field of $\hat{\nabla}$ and R as the curvature tensor field of ∇ on $\mathscr{M}$. The Gauss formulae with respect to $\hat{\nabla}$ and ∇ respectively, has the expression

$$\hat{\overline{\nabla}}_U V = \hat{\nabla}_U V + \hat{h}(U,V) \;\; and \;\; \overline{\nabla}_U V = \nabla_U V + h(U,V)$$

where $\hat{h}$ and h are (0,2)-tensor and the second fundamental form respectively of $\mathcal{M}$ in Q^m and from these two relations, one can easily get $\hat{h}(U,V) = h(U,V)$. Furthermore, using (4.4) for $U, V \in \Gamma(T\mathcal{M})$, we have

$$\begin{aligned}(\hat{\nabla}_U\eta)(V) &= (\nabla_U\eta)(V) + g(\phi U,\phi V) = g(\phi SU,V) + g(\phi U,\phi V),\\ (\hat{\nabla}_U\phi)(V) &= (\nabla_U\phi)(V) - g(U,\phi V)\xi - \eta(V)\phi U\\ &= \eta(V)SU - \eta(V)\phi U - g(SU,V)\xi + g(\phi U,V)\xi.\end{aligned}$$

Now, we know the curvature tensor of $\mathcal{M}$ with respect to induced SSMC $\hat{\nabla}$ can be calculated by

$$\hat{R}(U,V)W = \hat{\nabla}_U\hat{\nabla}_V W - \hat{\nabla}_V\hat{\nabla}_U W - \hat{\nabla}_{[U,V]}W.$$

Using (4.4), the relation between curvature tensor vector $\hat{R}$ and R of $\mathcal{M}$ in Q^m admitting SSMC $\hat{\nabla}$ and LC connection ∇ is given by

$$\begin{aligned}\hat{R}(U,V)W =& R(U,V)W + g(\phi SU,W)V - g(\phi SV,W)U\\ &+ \eta(W)[\eta(V)U - \eta(U)V] - g(V,W)[\phi SU + U - \eta(U)\xi]\\ &+ g(U,W)[\phi SV + V - \eta(V)\xi]\end{aligned}$$

Above relation can be rewritten as

$$\begin{aligned}g(\hat{R}(U,V)W,W') =& g(R(U,V)W,W') + \eta(W)[\eta(V)g(U,W') - \eta(U)g(V,W')]\\ &+ g(\phi SU,W)g(V,W') - g(V,W)[g(\phi SU,W') + g(U,W')\\ &- \eta(U)\eta(W')] - g(\phi SV,W)g(U,W')\\ &+ g(U,W)[g(\phi SV,W') + g(V,W') - \eta(V)\eta(W')]. \end{aligned}\tag{4.5}$$

Consider an orthonormal basis $\{e_i\}_1^{2m-1}$ of the tangent space of $\mathcal{M}$ and an orthonormal basis $\{e_{2m} = N\}$ of the normal space of $\mathcal{M}$. The *scalar curvature* τ of $\mathcal{M}$ is formulated as

$$\hat{\tau} = \sum_{1\le i<j\le 2m-1} \hat{K}(e_i \wedge e_j),$$

where $\hat{K}(\pi)$ stands for the sectional curvature of $\mathcal{M}$ associated with a plane section $\pi \subset \Gamma(T\mathcal{M})$ and is spanned by tangent vectors $\{e_i, e_j\}$ and $\hat{K}(e_i \wedge e_j) = g(\hat{R}(e_i,e_j)e_j,e_i)$ for $1 \le i < j \le 2m-1$.

The *normalized scalar curvature* $\hat{\rho}$ of $\mathcal{M}$ is given by

$$\hat{\rho} = \frac{2\hat{\tau}}{(2m-1)(2m-2)}.$$

We denote the *mean curvature vector field* by $\hat{\mathscr{H}}$ and it is determined by

$$\hat{\mathscr{H}} = \frac{1}{2m-1}\sum_{i=1}^{2m-1} h(e_i, e_i).$$

Conveniently, we take as read $h^{\alpha}_{ij} = g(h(e_i, e_j), e_\alpha) = g(h(e_i, e_j), N)$ for $i, j \in \{1, \ldots, 2m-1\}$ and $\alpha = 2m$. Then, we possess the squared mean curvature $||\hat{\mathscr{H}}||^2$ of M in Q^m and the squared norm $||h||^2$ of h as

$$||\hat{\mathscr{H}}||^2 = \frac{1}{(2m-1)^2}\Big(\sum_{i,j=1}^{2m-1} h^{\alpha}_{ij}\Big)^2 \ \ and \ \ ||h||^2 = \sum_{i,j=1}^{2m-1}(h^{\alpha}_{ij})^2,$$

respectively, where $\alpha = 2m,\ h^{\alpha}_{ij} = g(h(e_i, e_j),\ N)$.

It is well known that the squared norm of h over dimension $2m-1$ is called the *Casorati curvature* $\mathscr{C}$ of $\mathscr{M}$ in Q^m [7]. Thus, we have

$$\mathscr{C} = \frac{||h||^2}{2m-1} = \frac{1}{2m-1}\sum_{i,j=1}^{n}(h^{\alpha}_{ij})^2$$

Since $||h||^2 = tr(S^2)$, the above expession can be reexpressed by

$$\mathscr{C} = \frac{1}{2m-1}\,\mathrm{tr}(S^2).$$

The real hypersurface $\mathscr{M}$ of Q^m is known to be *invariantly quasi-umbilical* if $\exists$ a local orthonormal normal frame $\{e_{2m}\}$ of $\mathscr{M}$ in Q^m such that the shape operators $S_{e_{2m}}$ have an eigenvalue of multiplicity $2m$ - 2 for $\alpha = 2m$ and the distinguished eigendirection of $S_{e_{2m}}$ is the same for $\alpha = 2m$ [14].

Now, let us suppose that L be a k-dimensional subspace of $\Gamma(T\mathscr{M})$, $k \geq 2$, such that $\{e_1, \ldots, e_k\}$ is an orthonormal basis of L. Then, the scalar curvature $\hat{\tau}(L)$ and the Casorati curvature $\mathscr{C}(L)$ of the k-plane L are respectively given by

$$\hat{\tau}(L) = \sum_{1\leq i<j\leq k} K(e_i \wedge e_j) \ \ and \ \ \mathscr{C}(L) = \frac{1}{k}\sum_{i,j=1}^{k}(h^{\alpha}_{ij})^2.$$

The *normalized* δ*-Casorati curvatures* $\delta_c(2m-2)$ and $\hat{\delta}_c(2m-2)$ of $\mathscr{M}$ in Q^m are given by [9]

$$[\delta_c(2m-2)]_p = \frac{1}{2}\mathscr{C}_p + \frac{2m}{2(2m-1)} inf(B) \tag{4.6}$$

and

$$[\tilde{\delta}_c(2m-2)]_p = 2\mathscr{C}_p - \frac{[2(2m-1)-1]}{2(2m-1)} sup(B), \tag{4.7}$$

where the set B is defined as $B = \{\mathscr{C}(L)|L : hyperplane\, of\ \Gamma(T\mathscr{M})\}$.

The *generalized normalized δ-Casorati curvatures* $\delta_c(r;s)$ and $\tilde{\delta}_c(r;s)$ of $\mathcal{M}$ in Q^m for any positive real number $r \neq s(s+1)$ and $A(r,s) = \frac{s(s+1+r)[(s+1)^2-(s+1)-r]}{r(s+1)}$ where $s = 2m-2$ are given as [9]:

$$[\delta_c(r;s)]_p = r\mathscr{C}_p + A(r,s)inf(B),\ if\ 0 < r < s(s+1),$$
$$[\tilde{\delta}_c(r;s)]_p = r\mathscr{C}_p + A(r,s)sup(B),\ if\ r > s(s+1).$$

From the above two relations, one can note that the generalized normalized δ-Casorati curvatures $\delta_c(r;2m-2)$ and $\tilde{\delta}_c(r;2m-2)$ are the generalized versions of the normalized δ-Casorati curvatures $\delta_c(2m-2)$ and $\tilde{\delta}_c(2m-2)$ respectively by substituting r to $\frac{(2m-1)(2m-2)}{2}$ as

$$\big[\delta_c(\frac{(2m-1)(2m-2)}{2};2m-2)\big]_p = (2m-1)(2m-2)[\delta_c(2m-2)]_p, \tag{4.8}$$
$$\big[\tilde{\delta}_c(\frac{(2m-1)(2m-2)}{2};2m-2)\big]_p = (2m-1)(2m-2)[\tilde{\delta}_c(2m-2)]_p, \tag{4.9}$$

for $p \in \mathcal{M}$.

5. Main results

In this section, we will obtain some extremal inequalities consisting of the scalar curvature, the normalized scalar curvature and the generalized normalized δ-Casorati curvature for real hypersurfaces $\mathcal{M}$ of complex quadric Q^m endowed with SSMC.

By setting $U = W' = e_i$ and $V = W = e_j$ in (4.5), the scalar curvature $\hat{\tau}$ with respect to SSMC has the form

$$2\hat{\tau} = 3(2m-2) - 2 + g^2(AN,N) + (2m-1)^2||\hat{\mathscr{H}}||^2 - ||h||^2 - 2(2m-2)\,\mathrm{tr}(\phi S). \tag{5.1}$$

Case 1: If the unit normal vector field N is $\mathscr{U}$-principal (i.e. $AN = N$ or $A\xi = -\xi$), then the scalar curvature given by (5.1) has the reduced form

$$2\hat{\tau} = 3(2m-2) - 1 + (2m-1)^2||\hat{\mathscr{H}}||^2 - ||h||^2 - 2(2m-2)\,\mathrm{tr}(\phi S).$$

From this, we have an inequality

$$(2m-1)^2||\hat{\mathscr{H}}||^2 \geq 2\hat{\tau} - 3(2m-2) + 1 + 2(2m-2)\,\mathrm{tr}(\phi S).$$

Case 2: If the unit normal vector field N is $\mathscr{U}$-isotropic i.e. $g(AN,N) = 0$, then (5.1) reduces to

$$2\hat{\tau} = 3(2m-2) - 2 + (2m-1)^2||\hat{\mathscr{H}}||^2 - ||h||^2 - 2(2m-2)\,\mathrm{tr}(\phi S)$$

and thus, we arrive

$$(2m-1)^2||\hat{\mathscr{H}}||^2 \geq 2\hat{\tau} - 3(2m-2) + 2 + 2(2m-2)\,\mathrm{tr}(\phi S).$$

Using above arguments, we can state the following theorem

Theorem 5.1. *Let $\mathscr{M}$ be a real hypersurface in Q^m with SSMC. Then, the scalar curvature and the mean curvature vector field satisfies*

(i) *If N is $\mathscr{U}$-principal vector field*

$$||\hat{\mathscr{H}}||^2 \geq \frac{2\hat{\tau}}{(2m-1)^2} - \frac{3(2m-2)-1}{(2m-1)^2} + \frac{2(2m-2)}{(2m-1)^2}\operatorname{tr}(\phi S)$$

(ii) *If N is $\mathscr{U}$-isotropic vector field*

$$||\hat{\mathscr{H}}||^2 \geq \frac{2\hat{\tau}}{(2m-1)^2} - \frac{3(2m-2)-2}{(2m-1)^2} + \frac{2(2m-2)}{(2m-1)^2}\operatorname{tr}(\phi S)$$

and equality holds in both (i) *and* (ii) *if and only if the real hypersurface $\mathscr{M}$ is totally geodesic.*

Now, we move to obtain extremal inequalities consisting of the normalized scalar curvature and the generalized normalized δ-Casorati curvature.

Theorem 5.2. *Let $\mathscr{M}$ be a real hypersurface of Q^m with SSMC. Then, the generalized normalized δ-Casorati curvatures $\delta_c(r;2m-2)$ and $\tilde{\delta}_c(r;2m-2)$ respectively holds*

$$\text{(i)} \quad \hat{\rho} \leq \frac{\delta_c(r;2m-2)}{(2m-1)(2m-2)} + \frac{g(AN,N)^2}{(2m-1)(2m-2)} + \frac{3(2m-2)-2}{(2m-1)(2m-2)} - \frac{2\operatorname{tr}(\phi S)}{2m-1}$$

$$\text{(ii)} \quad \hat{\rho} \leq \frac{\tilde{\delta}_c(r;2m-2)}{(2m-1)(2m-2)} + \frac{g(AN,N)^2}{(2m-1)(2m-2)} + \frac{3(2m-2)-2}{(2m-1)(2m-2)} - \frac{2\operatorname{tr}(\phi S)}{2m-1}.$$

Moreover, necessary and sufficient condition for equality in both relations (i) *and* (ii)*:*
$\mathscr{M}$ is an invariantly quasi-umbilical real hypersurface having flat normal connection $\nabla^{\perp}$ in Q^m such that with some orthonormal basis $\{e_1,\dots,e_{2m-1}\}$ and $\{e_{2m}=N\}$ of $\Gamma(T\mathscr{M})$ and $\Gamma(T^{\perp}\mathscr{M})$ respectively, the shape operator S_N takes the matrix form

$$S_N = \begin{pmatrix} \mathscr{G} & 0 \\ 0 & \dfrac{(2m-1)(2m-2)}{r}a \end{pmatrix}, \tag{5.2}$$

where $\mathscr{G}$ is the diagonal matrix of order $2m-2$ with entries a.

Proof. Consider the polynomial $\mathscr{P}$ in terms of h with $n=2m-1$

$$\mathscr{P} = r\mathscr{C} + \frac{(n-1)(n+r)(n^2-n-r)}{rn}\mathscr{C}(L) - 2\hat{\tau}(p) + 3(n-1) - 2 + g(AN,N)^2 - 2(n-1)\operatorname{tr}(\phi S). \tag{5.3}$$

Let us assume that L is spanned by $\{e_1,\dots,e_{n-1}\}$ and take $e_\alpha = N = e_{n+1}$ for $\alpha = n+1$. Then, we have

$$\mathscr{P} = \frac{r}{n}\sum_{i,j=1}^{n}(h_{ij}^{\alpha})^2 + \frac{(n+r)(n^2-n-r)}{rn}\sum_{i,j=1}^{n-1}(h_{ij}^{\alpha})^2 - 2\hat{\tau}(p) + 3(n-1) - 2$$

$$-2(n-1)\operatorname{tr}(\phi S)+g(AN,N)^2.$$

Incorporating above relation with (5.1) yields

$$\begin{aligned}\mathscr{P}&=\frac{r}{n}\sum_{i,j=1}^{n}(h^{\alpha}_{ij})^2+\frac{(n+r)(n^2-n-r)}{rn}\sum_{i,j=1}^{n-1}(h^{\alpha}_{ij})^2-n^2\|\hat{H}\|^2+n\mathscr{C}\\&=\frac{n+r}{n}\sum_{i,j=1}^{n}(h^{\alpha}_{ij})^2+\frac{(n+r)(n^2-n-r)}{rn}\sum_{i,j=1}^{n-1}(h^{\alpha}_{ij})^2-\Big(\sum_{i=1}^{n}h^{\alpha}_{ii}\Big)^2.\end{aligned}$$

Then, with some computations, we derive

$$\begin{aligned}\mathscr{P}=&\sum_{i=1}^{n-1}\Big[(h^{\alpha}_{ii})^2\Big(\frac{n^2+n(r-1)-2r}{r}\Big)+\Big(\frac{n+r}{n}\Big)\Big((h^{\alpha}_{in})^2+(h^{\alpha}_{ni})^2\Big)\Big]\\&+\frac{(n-1)(n+r)}{n}\sum_{1\le i\ne j\le n-1}(h^{\alpha}_{ij})^2-2\sum_{1\le i\ne j\le n}h^{\alpha}_{ii}h^{\alpha}_{jj}+\frac{r}{n}(h^{\alpha}_{nn})^2.\end{aligned}\tag{5.4}$$

Now, the critical points $h^c=(h^{\alpha}_{11},h^{\alpha}_{22},\dots,h^{\alpha}_{nn})$ of $\mathscr{P}$ are the solutions of the system of linear homogeneous equations:

$$\begin{cases}\dfrac{\partial\mathscr{P}}{\partial h^{\alpha}_{ii}}=\dfrac{2(n+r)(n-1)}{r}h^{\alpha}_{ii}-2\sum_{k=1}^{n}h^{\alpha}_{kk}=0,\\[2mm]\dfrac{\partial\mathscr{P}}{\partial h^{\alpha}_{in}}=\dfrac{2(n+r)}{n}h^{\alpha}_{in}=0,\\[2mm]\dfrac{\partial\mathscr{P}}{\partial h^{\alpha}_{ni}}=\dfrac{2(n+r)}{n}h^{\alpha}_{ni}=0,\\[2mm]\dfrac{\partial\mathscr{P}}{\partial h^{\alpha}_{ij}}=\dfrac{2(n+r)(n-1)}{n}h^{\alpha}_{ij}=0,\\[2mm]\dfrac{\partial\mathscr{P}}{\partial h^{\alpha}_{nn}}=\dfrac{2r}{n}h^{\alpha}_{nn}-2\sum_{k=1}^{n-1}h^{\alpha}_{kk}=0,\end{cases}\tag{5.5}$$

for $i,j\in\{1,2,\dots,n-1\}$ with $i\ne j$.

So, from (5.5) it follows that any solution satisfies $h^{\alpha}_{ij}=0$ for $i,j\in\{1,2,\dots,n\},i\ne j$. Moreover, Hessian matrix of system (5.5) has the following form

$$H(p)=\begin{pmatrix}H_1&\mathbf{0}&\mathbf{0}\\\mathbf{0}&H_2&\mathbf{0}\\\mathbf{0}&\mathbf{0}&H_3\end{pmatrix},$$

whose diagonal block submatrices are given by

$$H_1=\begin{pmatrix}\frac{2(n+r)(n-1)}{r}-2&-2&\dots&-2&-2\\-2&\frac{2(n+r)(n-1)}{r}-2&\dots&-2&-2\\.&.&&.&.\\.&.&&.&.\\.&.&&.&.\\-2&-2&\dots&\frac{2(n+r)(n-1)}{r}-2&-2\\-2&-2&\dots&-2&\frac{2r}{n}\end{pmatrix},$$

$$H_2 = \begin{pmatrix} \frac{2(n+r)(n-1)}{r} & \mathbf{0} \\ \mathbf{0} & \frac{2(n+r)(n-1)}{r} \end{pmatrix}, \quad H_3 = \begin{pmatrix} \frac{2(n+r)}{n} & \mathbf{0} \\ \mathbf{0} & \frac{2(n+r)}{n} \end{pmatrix}.$$

Since the Hessian $H(p)$ is positive semidefinite for all points, the function or polynomial $\mathscr{P}$ is convex. Due to the convexity of $\mathscr{P}$, the critical point h^c is a minimum and in fact a global minimum. Thus, the polynomial $\mathscr{P}$ is of parabolic type and has a minimum at any solution h^c of (5.5).

After this, applying (5.5) on (5.4) follows that h^c is a solution of $\mathscr{P}$ i.e. $\mathscr{P}(h^c) = 0$.

So, $\mathscr{P} \geq 0$ and thus from (5.3), we obtain

$$2\hat{\tau}(p) \leq r\mathscr{C} + \frac{(n-1)(n+r)(n^2-n-r)}{rn}\mathscr{C}(L) + 3(n-1) - 2 + g(AN,N)^2 - 2(n-1)\,\mathrm{tr}(\phi S).$$

Finally, we have

$$\hat{\rho} \leq \frac{r}{n(n-1)}\mathscr{C} + \frac{(n+r)(n^2-n-r)}{rn^2}\mathscr{C}(L) + \frac{3(n-1)-2}{n(n-1)} + \frac{g(AN,N)^2}{n(n-1)} - \frac{2}{n}\mathrm{tr}(\phi S).$$

which is the desired inequality (i). Similarly, one can easily get the inequality (ii).

Furthermore, we can easily check that the equality arises in (i) and (ii) if and only if

$$\begin{aligned} &h_{ij} = 0 \quad \text{for} \quad i,j \in \{1,2,\dots,n\} \quad \text{with} \quad i \neq j, \\ &h_{nn} = \frac{n(n-1)}{r}h_{11} = \frac{n(n-1)}{r}h_{22} = \dots = \frac{n(n-1)}{r}h_{n-1\,n-1}. \end{aligned}$$

Thus, we get the equalities case if and only if the real hypersurface M is invariantly quasi-umbilical having flat $\nabla^{\perp}$ in Q^m such that the shape operator takes the form (5.2). ☐

Immediate consequences of above theorem can be stated as

Corollary 5.3. *Let $\mathscr{M}$ be a real hypersurface of Q^m with SSMC. Then, the generalized normalized δ-Casorati curvature $\delta_c(r;2m-2)$ and $\tilde{\delta}_c(r;2m-2)$ holds*

Normal vector field N	*Inequalities*
$\mathscr{U}$-principal	$\hat{\rho} \leq \frac{\delta_c(r;2m-2)}{(2m-1)(2m-2)} + \frac{3(2m-2)-1}{(2m-1)(2m-2)} - \frac{2}{2m-1}\mathrm{tr}(\phi S)$, $\hat{\rho} \leq \frac{\tilde{\delta}_c(r;2m-2)}{(2m-1)(2m-2)} + \frac{3(2m-2)-1}{(2m-1)(2m-2)} - \frac{2}{2m-1}\mathrm{tr}(\phi S)$
$\mathscr{U}$-isotropic	$\hat{\rho} \leq \frac{\delta_c(r;2m-2)}{(2m-1)(2m-2)} + \frac{3(2m-2)-2}{(2m-1)(2m-2)} - \frac{2}{2m-1}\mathrm{tr}(\phi S)$, $\hat{\rho} \leq \frac{\tilde{\delta}_c(r;2m-2)}{(2m-1)(2m-2)} + \frac{3(2m-2)-2}{(2m-1)(2m-2)} - \frac{2}{2m-1}\mathrm{tr}(\phi S)$

Proposition 5.4. *For a real hypersurface $\mathscr{M}$ of Q^m with SSMC, we have*

(i) *the normalized δ-Casorati curvature $\delta_c(2m-2)$ holds*

$$\hat{\rho} \le \delta_c(2m-2) + \frac{g(AN,N)^2}{(2m-1)(2m-2)} + \frac{3(2m-2)-2}{(2m-1)(2m-2)} - \frac{2}{2m-1}\operatorname{tr}(\phi S).$$

Moreover, necessary and sufficient condition for relation (i) to become an equality: $\mathcal{M}$ is an invariantly quasi-umbilical real hypersurface having flat $\nabla^\perp$ in Q^m such that with some orthonormal basis $\{e_1,\dots,e_{2m-1}\}$ and $\{e_{2m}=N\}$ of $\Gamma(TM)$ and $\Gamma(T^\perp\mathcal{M})$ respectively, the shape operator S_N takes the following form

$$S_N = \begin{pmatrix} \mathcal{G} & 0 \\ 0 & a \end{pmatrix},$$

where $\mathcal{G}$ is the diagonal matrix of order $2m-2$ with entries $2a$.

(ii) *the normalized δ-Casorati curvature $\tilde{\delta}_c(2m-2)$ holds*

$$\hat{\rho} \le \tilde{\delta}_c(2m-2) + \frac{g(AN,N)^2}{(2m-1)(2m-2)} + \frac{3(2m-2)-2}{(2m-1)(2m-2)} - \frac{2}{2m-1}\operatorname{tr}(\phi S).$$

Moreover, necessary and sufficient condition for relation (ii) to become an equality: $\mathcal{M}$ is an invariantly quasi-umbilical real hypersurface having flat $\nabla^\perp$ in Q^m such that with some orthonormal basis $\{e_1,\dots,e_{2m-1}\}$ and $\{e_{2m}=N\}$ of $\Gamma(T\mathcal{M})$ and $\Gamma(T^\perp\mathcal{M})$ respectively, the shape operator S_N takes the form

$$S_N = \begin{pmatrix} \mathcal{G} & 0 \\ 0 & 2a \end{pmatrix},$$

where $\mathcal{M}$ is the diagonal matrix of order $2m-2$ with entries a.

Proof. From (4.8) (resp. (4.9)) by taking $r = \frac{(2m-1)(2m-2)}{2}$ and using (4.6) (resp. (4.7)), we obtain our result for normalized δ-Casorati curvature of a real hypersurface $\mathcal{M}$ in Q^m. □

Corollary 5.5. *Let $\mathcal{M}$ be a real hypersurface of Q^m with SSMC. Then, the normalized δ-Casorati curvatures $\delta_c(2m-2)$ and $\tilde{\delta}_c(2m-2)$ holds*

Normal vector field N	*Inequalities*
$\mathcal{U}$-principal	$\hat{\rho} \le \delta_c(2m-2) + \frac{3(2m-2)-1}{(2m-1)(2m-2)} - \frac{2}{2m-1}\operatorname{tr}(\phi S)$, $\hat{\rho} \le \tilde{\delta}_c(2m-2) + \frac{3(2m-2)-1}{(2m-1)(2m-2)} - \frac{2}{2m-1}\operatorname{tr}(\phi S)$
$\mathcal{U}$-isotropic	$\hat{\rho} \le \delta_c(2m-2) + \frac{3(2m-2)-2}{(2m-1)(2m-2)} - \frac{2}{2m-1}\operatorname{tr}(\phi S)$, $\hat{\rho} \le \tilde{\delta}_c(2m-2) + \frac{3(2m-2)-2}{(2m-1)(2m-2)} - \frac{2}{2m-1}\operatorname{tr}(\phi S)$

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