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WARPED PRODUCT SUBMANIFOLDS IN METALLIC RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we study the existence of proper warped product submanifolds in metallic (or Golden) Riemannian manifolds and we discuss about semi-invariant, semi-slant and hemi-slant warped product submanifolds in metallic and Golden Riemannian manifolds. We also provide some examples of warped product submanifolds in Euclidean spaces.

1. Introduction

Warped products can be seen as a natural generalization of cartesian products. This concept appeared in mathematics starting with the J. F. Nash's studies, who proved an embedding theorem which states that every Riemannian manifold can be isometrically embedded into some Euclidean spaces. Also, Nash's theorem shows that every warped product $M_1 \times_f M_2$ can be embedded as a Riemannian submanifold in some Euclidean spaces ([26], 1956).

Then, the study of warped product manifolds was continued by R. L. Bishop and B. O'Neill in ([6], 1969), where they obtained fundamental properties of warped product manifolds and constructed a class of complete manifolds of negative curvature.

B. Y. Chen studied CR-submanifolds of a Kähler manifold which are warped products of holomorphic and totally real submanifolds, respectively ([13],[14],[15]). Also, in his new book ([12], 2017), he presents a multitude of properties for warped product manifolds and submanifolds, such as: warped products of Riemannian and Kähler manifolds, warped product submanifolds of Kähler manifolds (with the particular cases: warped product CR-submanifolds, warped product hemi-slant or semi-slant submanifolds of Kähler manifolds), CR-warped products in complex space forms and so on.

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As a generalization of contact CR-submanifolds, slant and semi-slant submanifolds, J. L. Cabrerizo et al. introduced the notion of bi-slant submanifolds of almost contact metric manifolds ([11]).

B. Sahin studied the properties of warped product submanifolds of Kähler manifolds with a slant factor and he proved that the warped product semi-slant submanifolds $M_T \times_f M_\theta$ and $M_\theta \times_f M_T$ in Kähler manifolds are simply Riemannian products of M_T and M_θ, where M_θ is a proper slant submanifold of the underlying Kähler manifold ([29]). Also, he studied warped product hemi-slant submanifolds of a Kähler manifold and he found some properties of warped product submanifolds of the form $M_\theta \times_f M_\perp$ ([30]).

Semi-invariant submanifolds in locally product Riemannian manifolds were studied in ([27],[2]). Semi-slant submanifolds in locally Riemannian product manifolds were studied by M. Atçeken which found that, in a locally Riemannian product manifold there does not exist any warped product semi-slant submanifold of the form $M_T \times_f M_\theta$ nor of the form $M_\perp \times_f M_\theta$ such that M_T is an invariant submanifold, M_θ is a proper slant submanifold and $M_\perp$ is an anti-invariant submanifold, but he found some examples of warped product semi-slant submanifolds of the form $M_\theta \times_f M_T$ and of the form $M_\theta \times_f M_\perp$ ([1]). Warped product submanifolds of the form $M_\theta \times_f M_T$ and $M_\theta \times_f M_\perp$ in locally Riemannian product manifolds were also studied by F. R. Al-Solamy and S. Uddin ([31]) and B. Sahin ([28]).

Warped product pseudo-slant (named also hemi-slant) submanifolds of the form $M_\theta \times_f M_\perp$, where M_θ and $M_\perp$ are proper slant and, respectively, anti-invariant submanifolds, in a locally product Riemannian manifold were studied by S. Uddin et al. in ([5],[16],[33],[35]).

Recently, warped product bi-slant submanifolds in Kähler manifolds were studied by S. Uddin et al. and some examples of this type of submanifolds in complex Euclidean spaces were constructed ([34]). Moreover, L. S. Alqahtani et al. have shown that there is no proper warped product bi-slant submanifold other than pseudo-slant warped product in cosymplectic manifolds ([4]).

The authors of the present paper studied some properties of invariant, anti-invariant and slant submanifolds ([8]), semi-slant submanifolds ([21]) and, respectively, hemi-slant submanifolds ([20]) in metallic and Golden Riemannian manifolds and they obtained integrability conditions for the distributions involved in these types of submanifolds. Moreover, properties of metallic and Golden warped product Riemannian manifolds were presented in the two previews works of the authors ([7],[10]).

In the present paper, we study the existence of proper warped product bi-slant submanifolds in locally metallic Riemannian manifolds. In Sections 2 and 3, we recall the main properties of metallic and Golden Riemannian manifolds and of their submanifolds. In Section 4, we discuss about slant and bi-slant submanifolds (with their particular cases: semi-slant and hemi-slant submanifolds) in locally metallic (or Golden) Riemannian manifolds. In Section 5, we find some properties of warped product bi-slant submanifolds in metallic (or Golden) Riemannian manifolds and, in particular, we discuss about warped product semi-invariant, semi-slant and hemi-slant submanifolds in locally metallic (or locally Golden) Riemannian manifolds. Moreover, we construct suitable examples.

2. Preliminaries

The metallic structure is a particular case of polynomial structure on a manifold, which was generally defined in ([19],[18]). The name of metallic number is given to the positive solution of the equation $x^2 - px - q = 0$ (where p and q are positive integer values), which is $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ ([32]).

Metallic Riemannian manifolds and their submanifolds were defined and studied by C. E. Hretcanu, M. Crasmareanu and A. M. Blaga in ([24],[22]), as a generalization of the Golden Riemannian manifolds studied in ([17],[23],[25]).

Let $\overline{M}$ be an m-dimensional manifold endowed with a tensor field J of type $(1,1)$ such that:

$$J^2 = pJ + qI, \tag{2.1}$$

for p, $q \in \mathbb{N}^*$, where I is the identity operator on $\Gamma(T\overline{M})$. Then the structure J is called a *metallic structure*. If the Riemannian metric $\overline{g}$ is J-compatible, i.e.:

$$\overline{g}(JX, Y) = \overline{g}(X, JY), \tag{2.2}$$

for any X, $Y \in \Gamma(T\overline{M})$, then $(\overline{M}, \overline{g}, J)$ is a *metallic Riemannian manifold* ([24]).

In particular, if $p = q = 1$ one obtains the *Golden structure* determined by a $(1,1)$-tensor field J which verifies $J^2 = J + I$. In this case, $(\overline{M}, J)$ is called *Golden manifold*. If $(\overline{M}, \overline{g})$ is a Riemannian manifold endowed with a Golden structure J such that the Riemannian metric $\overline{g}$ is J-compatible, then $(\overline{M}, \overline{g}, J)$ is a *Golden Riemannian manifold* ([17]).

From (2.1) and (2.2) we remark that the metric verifies:

$$\overline{g}(JX, JY) = \overline{g}(J^2X, Y) = p\overline{g}(JX, Y) + q\overline{g}(X, Y), \tag{2.3}$$

for any X, $Y \in \Gamma(T\overline{M})$.

Any almost product structure F on $\overline{M}$ induces two metallic structures on $\overline{M}$:

$$J = \frac{p}{2}I \pm \frac{2\sigma_{p,q} - p}{2}F, \tag{2.4}$$

where I is the identity operator on $\Gamma(T\overline{M})$. Also, on a metallic manifold $(\overline{M}, J)$, there exist two complementary distributions $\mathcal{D}_l$ and $\mathcal{D}_m$ corresponding to the projection operators l and m given by ([24]):

$$l = -\frac{1}{2\sigma_{p,q} - p} J + \frac{\sigma_{p,q}}{2\sigma_{p,q} - p} I, \; m = \frac{1}{2\sigma_{p,q} - p} J + \frac{\sigma_{p,q} - p}{2\sigma_{p,q} - p} I. \tag{2.5}$$

3. Submanifolds of metallic Riemannian manifolds

We recall the basic properties of a metallic Riemannian structure and prove some immediate consequences of the Gauss and Weingarten equations for an isometrically immersed submanifold in a metallic Riemannian manifold ([20],[21],[22]).

Let M be an isometrically immersed submanifold in the metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. Let T_xM be the tangent space of M in a point $x \in M$ and $T_x^{\perp}M$ the normal space of M in x. The tangent space $T_x\overline{M}$ can be decomposed into the direct sum $T_x\overline{M} = T_xM \oplus T_x^{\perp}M$, for any $x \in M$. Let i_* be the differential of the immersion $i : M \to \overline{M}$. Then the induced Riemannian metric g on M is given by $g(X, Y) = \overline{g}(i_*X, i_*Y)$, for any X, $Y \in \Gamma(TM)$. For the simplification of the notations, in the rest of the paper we shall denote by X the vector field i_*X, for any $X \in \Gamma(TM)$.

Let $TX := (JX)^T$ and $NX := (JX)^{\perp}$, respectively, be the tangential and normal components of JX, for any $X \in \Gamma(TM)$ and $tV := (JV)^T$, $nV := (JV)^{\perp}$ be the tangential and normal components of JV, for any $V \in \Gamma(T^{\perp}M)$. Then we get:

$$(i)\; JX = TX + NX, \; (ii)\; JV = tV + nV, \tag{3.1}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

The maps T and n are $\overline{g}$-symmetric ([8]):

$$\text{(i) } \overline{g}(TX, Y) = \overline{g}(X, TY), \qquad \text{(ii) } \overline{g}(nU, V) = \overline{g}(U, nV), \tag{3.2}$$

and

$$\overline{g}(NX, V) = \overline{g}(X, tV), \tag{3.3}$$

for any X, $Y \in \Gamma(TM)$ and U, $V \in \Gamma(T^{\perp}M)$.

If M is a submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then ([20]):

$$\text{(i) } T^2X = pTX + qX - tNX, \qquad \text{(ii) } pNX = NTX + nNX, \tag{3.4}$$

$$\text{(i) } n^2V = pnV + qV - NtV, \qquad \text{(ii) } ptV = TtV + tnV, \tag{3.5}$$

for any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

If $p = q = 1$ and M is a submanifold in a Golden Riemannian manifold $(\overline{M}, \overline{g}, J)$, then, for any $X \in \Gamma(TM)$ we get $T^2X = TX + X - tNX$, $NX = NTX + nNX$ and for any $V \in \Gamma(T^{\perp}M)$ we get $n^2V = nV + V - NtV$, $tV = TtV + tnV$.

Let $\overline{\nabla}$ and ∇ be the Levi-Civita connections on $(\overline{M}, \overline{g})$ and on its submanifold (M, g), respectively. The Gauss and Weingarten formulas are given by:

$$(i)\, \overline{\nabla}_X Y = \nabla_X Y + h(X, Y),\ (ii)\, \overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{3.6}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where h is the second fundamental form and A_V is the shape operator, which are related by:

$$\overline{g}(h(X, Y), V) = \overline{g}(A_V X, Y). \tag{3.7}$$

If $(\overline{M}, \overline{g}, J)$ is a metallic (or Golden) Riemannian manifold and J is parallel with respect to the Levi-Civita connection $\overline{\nabla}$ on $\overline{M}$ (i.e. $\overline{\nabla} J = 0$), then $(\overline{M}, \overline{g}, J)$ is called a *locally metallic (or locally Golden) Riemannian manifold* ([22]).

The covariant derivatives of the tangential and normal components of JX (and JV), T and N (t and n), respectively, are given by:

$$\text{(i) } (\nabla_X T)Y = \nabla_X TY - T(\nabla_X Y), \qquad \text{(ii) } (\overline{\nabla}_X N)Y = \nabla^{\perp}_X NY - N(\nabla_X Y), \tag{3.8}$$

$$\text{(i) } (\nabla_X t)V = \nabla_X tV - t(\nabla^{\perp}_X V), \qquad \text{(ii) } (\overline{\nabla}_X n)V = \nabla^{\perp}_X nV - n(\nabla^{\perp}_X V), \tag{3.9}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. From (2.1) it follows:

$$\overline{g}((\overline{\nabla}_X J)Y, Z) = \overline{g}(Y, (\overline{\nabla}_X J)Z), \tag{3.10}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$. Moreover, if M is an isometrically immersed submanifold in the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, then ([9]):

$$\overline{g}((\nabla_X T)Y, Z) = \overline{g}(Y, (\nabla_X T)Z), \tag{3.11}$$

for any $X, Y, Z \in \Gamma(TM)$.

Proposition 1. *If M is a submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, then the covariant derivatives of T and N verify:*

$$\text{(i) } (\nabla_X T)Y = A_{NY}X + th(X, Y), \qquad \text{(ii) } (\overline{\nabla}_X N)Y = nh(X, Y) - h(X, TY), \tag{3.12}$$

$$\text{(i) } (\nabla_X t)V = A_{nV}X - TA_V X, \qquad \text{(ii) } (\overline{\nabla}_X n)V = -h(X, tV) - NA_V X \tag{3.13}$$

and

$$\overline{g}((\overline{\nabla}_X N)Y, V) = \overline{g}((\nabla_X t)V, Y), \tag{3.14}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$ ([20]).

4. Bi-slant submanifolds in metallic or Golden Riemannian manifolds

Definition 1. ([8]) A submanifold M in a metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$ is called *slant submanifold* if the angle $\theta(X_x)$ between JX_x and T_xM is constant, for any $x\in M$ and $X_x\in T_xM$. In such a case, $\theta =: \theta(X_x)$ is called the *slant angle* of M in $\overline{M}$ and it verifies:

$$\cos\theta = \frac{\overline{g}(JX,TX)}{\|JX\|\cdot\|TX\|} = \frac{\|TX\|}{\|JX\|}. \tag{4.1}$$

The immersion $i : M \to \overline{M}$ is called *slant immersion* of M in $\overline{M}$.

Remark 1. The invariant and anti-invariant submanifolds in metallic (or Golden) Riemannian manifolds $(\overline{M},\overline{g},J)$ are particular cases of slant submanifolds with the slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. A slant submanifold M in $\overline{M}$, which is neither invariant nor anti-invariant, is called *proper slant submanifold* and the immersion $i : M \to \overline{M}$ is called *proper slant immersion* ([8]).

Definition 2. Let M be an immersed submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$. A differentiable distribution D on M is called *slant distribution* if the angle θ_D between JX_x and the vector subspace D_x is constant, for any $x\in M$ and any nonzero vector field $X_x\in\Gamma(D_x)$. The constant angle θ_D is called the *slant angle* of the distribution D ([21]).

Definition 3. A submanifold M in a metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$ is called *bi-slant* if there exist two orthogonal differentiable distributions D_1 and D_2 on M such that:

1. $TM = D_1 \oplus D_2$;
2. $JD_1 \perp D_2$ and $JD_2 \perp D_1$;
3. the distributions D_1 and D_2 are slant distributions with slant angles θ_1 and θ_2, respectively.

Moreover, M is called *proper bi-slant submanifold* of $\overline{M}$ if $dim(D_1)\cdot dim(D_2) \neq 0$ and $\theta_1,\theta_2 \in (0,\frac{\pi}{2})$.

Definition 4. ([20]) An immersed submanifold M in a metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$ is *semi-slant* submanifold (*hemi-slant* submanifold, respectively) if there exist two orthogonal distributions D_1 and D_2 on M such that:

1. TM admits the orthogonal direct decomposition $TM = D_1 \oplus D_2$;
2. the distribution D_1 is an invariant distribution, i.e. $J(D_1) = D_1$ (D_1 is an anti-invariant distribution, i.e. $J(D_1) \subseteq \Gamma(T^{\perp}M)$, respectively);
3. the distribution D_2 is slant with the angle $\theta \in [0,\frac{\pi}{2}]$.

If $dim(D_1)\cdot dim(D_2)\neq 0$ and $\theta\in(0,\frac{\pi}{2})$, then M is a proper semi-slant submanifold (hemi-slant submanifold, respectively) in the metallic (or Golden) Riemannian manifold $(\overline{M},\overline{g},J)$.

Now we provide an example of bi-slant submanifold in a metallic and Golden Riemannian manifold $(\mathbb{R}^4,\langle\cdot,\cdot\rangle,J)$.

Example 1. Let $\mathbb{R}^4$ be the Euclidean space endowed with the usual Euclidean metric $\langle\cdot,\cdot\rangle$ and the immersion $i:M\to\mathbb{R}^4$, given by:

$$i(f_1,f_2):=\left(f_1\cos t,\frac{\sigma}{\sqrt{q}}f_1\sin t,f_2,f_2\right),$$

where $M:=\{(f_1,f_2)\mid f_1,f_2>0\}$, $t\in[0,\frac{\pi}{2}]$ and $\sigma:=\sigma_{p,q}=\frac{p+\sqrt{p^2+4q}}{2}$ is a metallic number ($p,q\in N^*$).

We can find a local orthogonal frame on TM given by:

$$Z_1=\cos t\frac{\partial}{\partial x_1}+\frac{\sigma}{\sqrt{q}}\sin t\frac{\partial}{\partial x_2},\ Z_2=\frac{\partial}{\partial x_3}+\frac{\partial}{\partial x_4}.$$

We define the metallic structure $J:\mathbb{R}^4\to\mathbb{R}^4$ by:

$$J(X_1,X_2,X_3,X_4):=(\sigma X_1,\overline{\sigma}X_2,\sigma X_3,\overline{\sigma}X_4),$$

which verifies $J^2=pJ+qI$ and $\langle JX,Y\rangle=\langle X,JY\rangle$, for any X, $Y\in\mathbb{R}^4$, where $\overline{\sigma}:=p-\sigma$. Since

$$JZ_1=\sigma\cos t\frac{\partial}{\partial x_1}-\sqrt{q}\sin t\frac{\partial}{\partial x_2},\ JZ_2=\sigma\frac{\partial}{\partial x_3}+\overline{\sigma}\frac{\partial}{\partial x_4},$$

we remark that $\langle JZ_1,Z_1\rangle=\sigma\cos 2t$ and $\langle JZ_2,Z_2\rangle=\sigma+\overline{\sigma}$. Moreover, $\|Z_1\|^2=\frac{p\sigma\sin^2 t+q}{q}$, $\|JZ_1\|^2=p\sigma\cos^2 t+q$, $\|Z_2\|^2=2$, $\|JZ_2\|^2=\sigma^2+\overline{\sigma}^2$.

Let us consider $D_1:=span\{Z_1\}$ and $D_2:=span\{Z_2\}$. Then, the distributions D_1 and D_2 satisfy the conditions from Definition 3, where $\cos\theta_1=\frac{2\sqrt{q}\cos 2t}{\sqrt{p^2\sin^2 2t+4q}}$ and $\cos\theta_2=\frac{\sigma+\overline{\sigma}}{\sqrt{2(\sigma^2+\overline{\sigma}^2)}}$.

Consequently, the submanifold M_{θ_1,θ_2} with $TM_{\theta_1,\theta_2}=D_1\oplus D_2$ and the metric $g:=(\frac{p\sigma}{q}\sin^2 t+1)g_1+2g_2$ is a bi-slant submanifold in the metallic Riemannian manifold $(\mathbb{R}^4,\langle\cdot,\cdot\rangle,J)$, where g_1 and g_2 are the metric tensors of the integral manifolds M_1 and M_2 of the distributions D_1 and D_2.

Moreover, for $t=0$ ($t=\frac{\pi}{4}$) we obtain $\theta_1=0$ ($\theta_1=\frac{\pi}{2}$, respectively). Then, the submanifold M_{0,θ_2} ($M_{\frac{\pi}{2},\theta_2}$) is a semi-slant (a hemi-slant, respectively) submanifold in the metallic Riemannian manifold $(\mathbb{R}^4,\langle\cdot,\cdot\rangle,J)$.

In particular, for $p=q=1$, the immersion $i:M\to\mathbb{R}^4$ is given by $i(f_1,f_2):=(f_1\cos t,\phi f_1\sin t,f_2,f_2)$ and the Golden structure has the form $J(X_1,X_2,X_3,X_4):=(\phi X_1,\overline{\phi}X_2,\phi X_3,\overline{\phi}X_4)$, where $\phi := \sigma_{1,1} = \frac{1+\sqrt{5}}{2}$ is the Golden number and $\overline{\phi} := 1-\phi$. Since $JZ_1 = \phi \cos t \frac{\partial}{\partial x_1} - \sin t \frac{\partial}{\partial x_2}$ and $JZ_2 = \phi \frac{\partial}{\partial x_3} + \overline{\phi} \frac{\partial}{\partial x_4}$, we remark that $\cos\theta_1 = \frac{2\cos 2t}{\sqrt{\sin^2 2t+4}}$ and $\cos\theta_2 = \frac{\phi+\overline{\phi}}{\sqrt{2(\phi^2+\overline{\phi}^2)}}$.

The distributions $D_1 := span\{Z_1\}$ and $D_2 := span\{Z_2\}$ satisfy the conditions from Definition 3. Thus, the submanifold M_{θ_1,θ_2} with $TM_{\theta_1,\theta_2} = D_1 \oplus D_2$ and the metric $g := (\phi \sin^2 t + 1)g_1 + 2g_2$ is a bi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^4, \langle\cdot,\cdot\rangle, J)$, where g_1 and g_2 are the metric tensors of the integral manifolds M_1 and M_2 of the distributions D_1 and D_2. Moreover, for $t = 0$ ($t = \frac{\pi}{4}$) we obtain $\theta_1 = 0$ ($\theta_1 = \frac{\pi}{2}$, respectively). Then, the submanifold M_{0,θ_2} ($M_{\frac{\pi}{2},\theta_2}$) is a semi-slant (a hemi-slant, respectively) submanifold in the Golden Riemannian manifold $(\mathbb{R}^4, \langle\cdot,\cdot\rangle, J)$.

Remark 2. If M is a bi-slant submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the orthogonal distribution D_1 and D_2 and the slant angles θ_1 and θ_2, respectively, then

$$JX = P_1TX + P_2TX + NX = TP_1X + TP_2X + NP_1X + NP_2X,$$

for any $X \in \Gamma(TM)$, where P_1 and P_2 are the projection operators on $\Gamma(D_1)$ and $\Gamma(D_2)$, respectively.

Proposition 2. *([21]) If M is a bi-slant submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, with the slant angles $\theta_1 = \theta_2 = \theta$ and $g(JX,Y) = 0$, for any $X \in \Gamma(D_1)$, $Y \in \Gamma(D_2)$, then M is a slant submanifold in the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ with the slant angle θ.*

Remark 3. If M is a bi-slant submanifold in a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ such that $TM = D_1 \oplus D_2$, $dim(D_1) \cdot dim(D_2) \neq 0$, where D_2 is the slant distribution (with the slant angle θ), then we get:

(1) M is an invariant submanifold if $\theta = 0$ and D_1 is invariant;
(2) M is an anti-invariant submanifold if $\theta = \frac{\pi}{2}$ and D_1 is anti-invariant;
(3) M is a semi-invariant submanifold if D_1 is invariant and D_2 is anti-invariant. The semi-invariant submanifold is a particular case of semi-slant submanifold (hemi-slant submanifold), with the slant angle $\theta = \frac{\pi}{2}$ ($\theta = 0$, respectively).

Proposition 3 ([21])**.** *Let M be an isometrically immersed submanifold in the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$. If M is a slant submanifold with the slant angle θ, then:*

$$\overline{g}(TX,TY) = \cos^2\theta[p\overline{g}(X,TY) + q\overline{g}(X,Y)] \tag{4.2}$$

$$\overline{g}(NX,NY) = \sin^2\theta[p\overline{g}(X,TY) + q\overline{g}(X,Y)], \tag{4.3}$$

for any X, $Y \in \Gamma(TM)$ and

$$(i)\ T^2 = \cos^2\theta(pT + qI),\ (ii)\nabla(T^2) = p\cos^2\theta(\nabla T). \tag{4.4}$$

where I is the identity on $\Gamma(TM)$.

Proposition 4 ([20],[21])**.** *If M is a semi-slant submanifold (hemi-slant submanifold, respectively) in the metallic Riemannian manifold $(\overline{M},\overline{g},J)$ with the slant angle θ of the distribution D_2, then:*

$$\overline{g}(TP_2X,TP_2Y)=\cos^2\theta[p\overline{g}(TP_2X,P_2Y)+q\overline{g}(P_2X,P_2Y)] \tag{4.5}$$

$$\overline{g}(NX,NY)=\sin^2\theta[p\overline{g}(TP_2X,P_2Y)+q\overline{g}(P_2X,P_2Y)], \tag{4.6}$$

for any X, $Y\in\Gamma(TM)$.

5. Semi-invariant, semi-slant and hemi-slant warped product submanifolds in metallic Riemannian manifolds

In this section we present some results regarding the existence and nonexistence of semi-invariant, semi-slant and hemi-slant warped product submanifolds in metallic Riemannian manifolds $(\overline{M},\overline{g},J)$ and we give some examples of these types of submanifolds in metallic (or Golden) Riemannian manifolds.

In ([7]), the authors of this paper introduced the Golden warped product Riemannian manifold and provided a necessary and sufficient condition for the warped product of two locally Golden Riemannian manifolds to be locally Golden. Moreover, the subject was continued in the paper ([10]), where the authors characterized the metallic structure on the product of two metallic manifolds in terms of metallic maps and provided a necessary and sufficient condition for the warped product of two locally metallic Riemannian manifolds to be locally metallic.

Let (M_1,g_1) and (M_2,g_2) be two Riemannian manifolds of dimensions $n_1>0$ and $n_2>0$, respectively. We denote by π_1 and π_2 the projection maps from the product manifold $M_1\times M_2$ onto M_1 and M_2, respectively and by $\widetilde{\varphi}:=\varphi\circ\pi_1$ the lift to $M_1\times M_2$ of a smooth function φ on M_1. M_1 is called *the base* and M_2 is *the fiber* of $M_1\times M_2$. The unique element $\widetilde{X}$ of $\Gamma(T(M_1\times M_2))$ that is π_1-related to $X\in\Gamma(TM_1)$ and to the zero vector field on M_2 will be called the *horizontal lift of* X and the unique element $\widetilde{V}$ of $\Gamma(T(M_1\times M_2))$ that is π_2-related to $V\in\Gamma(TM_2)$ and to the zero vector field on M_1 will be called the *vertical lift of* V. We denote by $\mathscr{L}(M_1)$ the set of all horizontal lifts of vector fields on M_1 and by $\mathscr{L}(M_2)$ the set of all vertical lifts of vector fields on M_2.

For $f:M_1\to(0,\infty)$ a smooth function on M_1, we consider the Riemannian metric g on $M:=M_1\times M_2$:

$$g:=\pi_1^*g_1+(f\circ\pi_1)^2\pi_2^*g_2. \tag{5.1}$$

Definition 5. ([6]) The product manifold of M_1 and M_2 together with the Riemannian metric g defined by (5.1) is called *the warped product* of M_1 and M_2 by the warping function f.

In the next considerations we shall denote by $(f \circ \pi_1)^2 =: f^2$, $\pi_1^* g_1 =: g_1$ and $\pi_2^* g_2 =: g_2$, respectively.

Definition 6. ([12]) A warped product manifold $M := M_1 \times_f M_2$ is called *trivial* if the warping function f is constant. In this case, $M_1 \times_f M_2$ is the Riemannian product $M_1 \times M_{2f}$, where M_{2f} is the manifold M_2 equipped with the metric $f^2 g_2$ (which is homothetic to g_2).

The Levi-Civita connection ∇ on the warped product of M_1 and M_2 by the warping function f, $M := M_1 \times_f M_2$ is related to the Levi-Civita connections on M_1 and M_2, as follows:

Lemma 1 ([1] and [12], p. 49)**.** *For X, $Y \in \Gamma(TM_1)$ and Z, $W \in \Gamma(TM_2)$, we have on $M := M_1 \times_f M_2$ that:*

(1) $\nabla_X Y \in \mathscr{L}(M_1)$;

(2) $\nabla_X Z = \nabla_Z X = X(\ln f)Z$;

(3) $\nabla_Z W = \nabla_Z^{M_2} W - \frac{grad f}{f} g(Z,W)$,

where ∇ and ∇^{M_2} denote the Levi-Civita connections on M and M_2, respectively.

Proposition 5. *([6]) The warped product manifold $M := M_1 \times_f M_2$ is characterized by the fact that M_1 is totally geodesic and M_2 is a totally umbilical submanifold of M, respectively.*

Definition 7. A warped product $M_1 \times_f M_2$ of two slant submanifolds M_1 and M_2 in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ is called a *warped product bi-slant submanifold.* A warped product bi-slant submanifold $M_1 \times_f M_2$ is called *proper* if both submanifolds M_1 and M_2 are proper slant in $(\overline{M}, \overline{g}, J)$.

Definition 8. If $M := M_1 \times_f M_2$ is a warped product submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ such that one of the components M_i $(i \in \{1,2\})$ is an invariant submanifold (respectively, anti-invariant submanifold) in $\overline{M}$ and the other one is a slant submanifold in $\overline{M}$, with the slant angle $\theta \in [0, \frac{\pi}{2}]$, then we call the submanifold M *warped product semi-slant* (respectively, *hemi-slant*) *submanifold* in the metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$.

Remark 4. In particular, if $M := M_1 \times_f M_2$ is a warped product semi-slant (respectively, hemi-slant) submanifold in a metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ such that the slant angle $\theta = \frac{\pi}{2}$ (respectively, $\theta = 0$), then we obtain $M := M_T \times_f M_\perp$ or $M := M_\perp \times_f M_T$ and M is a warped product semi-invariant submanifold in $\overline{M}$.

Lemma 2. *Let $M := M_1 \times_f M_2$ be a warped product bi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$. Then, for any X, $Y \in \Gamma(TM_1)$ and Z, $W \in \Gamma(TM_2)$, we have:*

$$\text{(i) } \overline{g}(h(X,Y), NZ) = -\overline{g}(h(X,Z), NY); \qquad \text{(ii) } \overline{g}(h(X,Z), NW) = 0; \tag{5.2}$$

$$\overline{g}(h(Z,W), NX) = TX(\ln f)\overline{g}(Z,W) - X(\ln f)\overline{g}(Z, TW). \tag{5.3}$$

Proof. For any $X, Y \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$ we get

$$\overline{g}(h(X,Y),NZ) = \overline{g}(\overline{\nabla}_X Y, JZ) - \overline{g}(\overline{\nabla}_X Y, TZ).$$

By using $(\overline{\nabla}_X J)Y = 0$ and (2.2) we obtain

$$\overline{g}(\overline{\nabla}_X Y, JZ) = \overline{g}(\overline{\nabla}_X JY, Z) = \overline{g}(\overline{\nabla}_X TY, Z) + \overline{g}(\overline{\nabla}_X NY, Z).$$

Moreover, using $\overline{g}(\overline{\nabla}_X Y, TZ) = -\overline{g}(\overline{\nabla}_X TZ, Y)$ and (3.6), we obtain

$$\overline{g}(h(X,Y),NZ) = \overline{g}(\nabla_X TY, Z) - \overline{g}(A_{NY}X, Z) + \overline{g}(Y, \nabla_X TZ).$$

By using Lemma 1(2), we have

$$\overline{g}(\nabla_X TY, Z) = -\overline{g}(\nabla_X Z, TY) = -X(\ln f)\overline{g}(TY, Z)$$

and $\overline{g}(Y, \nabla_X TZ) = X(\ln f)\overline{g}(Y, TZ)$.

Thus, from (3.2)(i) we get $\overline{g}(h(X,Y),NZ) = -\overline{g}(A_{NY}X, Z)$ which implies (5.2)(i).

For any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$ we get

$$\overline{g}(h(X,Z),NW) = \overline{g}(\overline{\nabla}_X Z, JW) - \overline{g}(\overline{\nabla}_X Z, TW).$$

By using $(\overline{\nabla}_X J)Z = 0$, (2.2), (3.1)(i) and (3.6)(ii) we obtain

$$\overline{g}(h(X,Z),NW) = \overline{g}(\nabla_X TZ, W) - \overline{g}(A_{NZ}X, W) - \overline{g}(\nabla_X Z, TW)$$

and using (2) from Lemma 1, we have

$$\overline{g}(h(X,Z),NW) = X(\ln f)[\overline{g}(TZ,W) - \overline{g}(Z,TW)] - \overline{g}(h(X,W),NZ).$$

Thus, from (3.2)(i) we get $\overline{g}(h(X,Z),NW) = -\overline{g}(h(X,W),NZ)$.

On the other hand, we have

$$\overline{g}(h(X,Z),NW) = \overline{g}(\nabla_Z TX, W) - \overline{g}(A_{NX}Z, W) - \overline{g}(\nabla_Z X, TW)$$

and we obtain

$$\overline{g}(h(X,Z),NW) = TX(\ln f)\overline{g}(Z,W) - X(\ln f)\overline{g}(Z,TW) - \overline{g}(h(Z,W),NX).$$

After interchanging Z by W and using (3.2)(i), we have

$$\overline{g}(h(X,Z),NW) = \overline{g}(h(X,W),NZ) = -\overline{g}(h(X,Z),NW),$$

which implies (5.2)(ii).

For any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$ we get

$$\overline{g}(h(Z,W), NX) = \overline{g}(\overline{\nabla}_Z W, JX) - \overline{g}(\overline{\nabla}_Z W, TX).$$

By using $(\overline{\nabla}_Z J)W = 0$, (2.2) and (3.6) we obtain

$$\overline{g}(h(Z,W), NX) = \overline{g}(\nabla_Z TW, X) - \overline{g}(A_{NW}Z, X) - \overline{g}(\nabla_Z W, TX).$$

By using (5.2)(ii), we have $\overline{g}(A_{NW}Z, X) = \overline{g}(h(Z,X), NW) = 0$ and we get

$$\overline{g}(h(Z,W), NX) = -\overline{g}(TW, \nabla_Z X) + \overline{g}(W, \nabla_Z TX)$$

and using Lemma 1(2) we obtain (5.3). □

Proposition 6. *Let $M := M_T \times_f M_\perp$ be a semi-invariant submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ (i.e. M_T is invariant and $M_\perp$ is an anti-invariant submanifold in $\overline{M}$). Then, we have:*

$$TX(\ln f) = -\frac{q}{p} X(\ln f), \tag{5.4}$$

for any $X \in \Gamma(TM_T)$. Moreover, if M is a semi-invariant submanifold in a locally Golden Riemannian manifold, then we get $TX(\ln f) = -X(\ln f)$.

Proof. If $M := M_1 \times_f M_2$ is a warped product submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$, by using (2.2) and (2.3) we obtain

$$q\overline{g}(\nabla_Z X, W) = q\overline{g}(\overline{\nabla}_Z X, W) = \overline{g}(J\overline{\nabla}_Z X, JW) - p\overline{g}(J\overline{\nabla}_Z X, W).$$

Moreover, from $(\overline{\nabla}_Z J)X = 0$, we get

$$q\overline{g}(\nabla_Z X, W) = \overline{g}(\overline{\nabla}_Z JX, JW) - p\overline{g}(\overline{\nabla}_Z JX, W), \tag{5.5}$$

for any $X \in \Gamma(TM_1)$ and $Z, W \in \Gamma(TM_2)$.

If $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\perp)$ (i.e. $JX = TX$ and $JW = NW$), we get

$$\begin{aligned} q\overline{g}(\nabla_Z X, W) &= \overline{g}(\overline{\nabla}_Z TX, NW) - p\overline{g}(\overline{\nabla}_Z TX, W) \\ &= \overline{g}(h(TX, Z), NW) - p\overline{g}(\nabla_Z TX, W). \end{aligned}$$

By using (5.2)(ii), we get $\overline{g}(h(TX,Z), NW) = 0$ and from Lemma 1(2) we obtain

$$qX(\ln f)\overline{g}(Z,W) = -pTX(\ln f)\overline{g}(Z,W).$$

Thus, for the non-null vector field $Z = W \in \Gamma(TM_\perp)$, we obtain (5.4). □

Using a similar idea as in ([1], Theorem 3.1), we get a property of a warped product semi-invariant submanifold in a locally metallic (locally Golden) Riemannian manifold of the form $M_T \times_f M_\perp$.

Theorem 1. *Let $M := M_T \times_f M_\perp$ be a warped product semi-invariant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ (i.e. M_T is invariant and $M_\perp$ is an anti-invariant submanifold in $\overline{M}$). Then $M := M_T \times_f M_\perp$ is a non proper warped product submanifold in $\overline{M}$ (i.e. the warping function f is constant on the connected components of M_T).*

Proof. We have $JX = TX$, for any $X \in \Gamma(TM_T)$ and $JW = NW$, for any Z, $W \in \Gamma(TM_\perp)$. By using (2.3) and $\overline{\nabla}J = 0$, we get:

$$\overline{g}(\nabla_Z X, W) = \overline{g}(\overline{\nabla}_Z X, W) = \frac{1}{q}\overline{g}(\overline{\nabla}_Z JX, JW) - \frac{p}{q}\overline{g}(\overline{\nabla}_Z X, JW),$$

where ∇ and $\overline{\nabla}$ denote the Levi-Civita connections on M and $\overline{M}$, respectively. Thus, from (5.2)(ii) we get

$$q\overline{g}(\nabla_Z X, W) = \overline{g}(h(Z, TX), NW) - p\overline{g}(h(X, Z), NW) = 0,$$

for any $X \in \Gamma(TM_T)$ and Z, $W \in \Gamma(TM_\perp)$. By using Lemma 1(2) we have $qX(\ln f)\overline{g}(Z, W) = 0$ (where $q \in N^*$). Thus, for any non-null $Z = W \in \Gamma(TM_\perp)$, we have $X(\ln f)\|Z\|^2 = 0$ and it follows $X(\ln f) = 0$, for any $X \in \Gamma(TM_T)$, which implies that f is a constant function on the connected components of M_T. ☐

Now we provide examples of warped product semi-invariant submanifold of the type $M := M_\perp \times_f M_T$ in a metallic (and Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$.

Example 2. Let $\mathbb{R}^5$ be the Euclidean space endowed with the usual Euclidean metric $\langle \cdot, \cdot \rangle$. Let $i : M \to \mathbb{R}^5$ be the immersion given by:

$$i(f, \alpha, \beta) := \left(f\sin\alpha, f\cos\alpha, f\sin\beta, f\cos\beta, \sqrt{\frac{p\sigma}{q}}f \right),$$

where $M := \{(f, \alpha, \beta) \mid f > 0, \alpha, \beta \in (0, \frac{\pi}{2})\}$ and $\sigma := \sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is a metallic number ($p, q \in N^*$).

We can find a local orthogonal frame on TM given by:

$$Z_1 = \sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_2} + \sin\beta\frac{\partial}{\partial x_3} + \cos\beta\frac{\partial}{\partial x_4} + \sqrt{\frac{p\sigma}{q}}\frac{\partial}{\partial x_5},$$

$$Z_2 = f\cos\alpha\frac{\partial}{\partial x_1} - f\sin\alpha\frac{\partial}{\partial x_2}, \; Z_3 = f\cos\beta\frac{\partial}{\partial x_3} - f\sin\beta\frac{\partial}{\partial x_4}.$$

We define the metallic structure $J : \mathbb{R}^5 \to \mathbb{R}^5$ by:

$$J(X_1, X_2, X_3, X_4, X_5) := (\sigma X_1, \sigma X_2, \overline{\sigma} X_3, \overline{\sigma} X_4, \overline{\sigma} X_5),$$

which verifies $J^2X = pJ + qI$ and $\langle JX, Y\rangle = \langle X, JY\rangle$, for any $X, Y \in \mathbb{R}^5$. Since

$$JZ_1 = \sigma \sin\alpha \frac{\partial}{\partial x_1} + \sigma \cos\alpha \frac{\partial}{\partial x_2} + \overline{\sigma} \sin\beta \frac{\partial}{\partial x_3} + \overline{\sigma} \cos\beta \frac{\partial}{\partial x_4} + \overline{\sigma}\sqrt{\frac{p\sigma}{q}} \frac{\partial}{\partial x_5},$$

$$JZ_2 = \sigma Z_2,\ JZ_3 = \overline{\sigma} Z_3,$$

we remark that $JZ_1 \perp span\{Z_1, Z_2, Z_3\} = TM$ (i.e. $\langle JZ_1, Z_i\rangle = 0$, for any $i \in \{1,2,3\}$), and $JZ_2, JZ_3 \subseteq span\{Z_2, Z_3\}$. We find that $\|Z_1\|^2 = 1 + \frac{\sigma^2}{q}$ and $\|Z_2\|^2 = \|Z_3\|^2 = f^2$.

Let us consider $D_1 = span\{Z_1\}$ and $D_2 = span\{Z_2, Z_3\}$. The distributions D_1 and D_2 satisfy the conditions from Definition 3 and they are completely integrable. Moreover, D_1 is an anti-invariant distribution and D_2 is an invariant distribution with respect to J.

Let $M_\perp$ and M_T be the integral manifolds of D_1 and D_2, respectively. Therefore, $M := M_\perp \times_f M_T$ with the Riemannian metric tensor

$$g := \left(1 + \frac{\sigma^2}{q}\right) df^2 + f^2(d\alpha^2 + d\beta^2) = g_{M_\perp} + f^2 g_{M_T}$$

is a warped product semi-invariant submanifold in the metallic Riemannian manifold $(\mathbb{R}^5, \langle\cdot,\cdot\rangle, J)$.

In particular, for $p = q = 1$ and $\phi := \sigma_{1,1} = \frac{1+\sqrt{5}}{2}$ the Golden number, the immersion $i : M \to \mathbb{R}^5$ is given by

$$i(f, \alpha, \beta) := (f \sin\alpha, f \cos\alpha, f \sin\beta, f \cos\beta, \sqrt{\phi} f),$$

and the Golden structure $J : \mathbb{R}^5 \to \mathbb{R}^5$ is defined by

$$J(X_1, X_2, X_3, X_4, X_5) := (\phi X_1, \phi X_2, \overline{\phi} X_3, \overline{\phi} X_4, \overline{\phi} X_5),$$

where $\overline{\phi} = 1 - \phi$. If $M_\perp$ and M_T are the integral manifolds of the distributions $D_1 := span\{Z_1\}$ and $D_2 := span\{Z_2, Z_3\}$, respectively and the metric on $M := M_\perp \times_f M_T$ is given by

$$g := (1 + \phi^2) df^2 + f^2(d\alpha^2 + d\beta^2) = g_{M_\perp} + f^2 g_{M_T},$$

then we obtain a warped product semi-invariant submanifold (M, g) in the Golden Riemannian manifold $(\mathbb{R}^5, \langle\cdot,\cdot\rangle, J)$.

Proposition 7. *Let $M := M_\perp \times_f M_T$ be a warped product semi-invariant submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ (i.e. M_T is invariant and $M_\perp$ is an anti-invariant submanifold in $\overline{M}$). Then, f is constant on the connected components of $M_\perp$ if and only if we have:*

$$(T - pI) A_{JX} Z = -t \nabla^\perp_Z JX, \tag{5.6}$$

for any $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_T)$, where I is the identity on $\Gamma(TM)$ and $\nabla^\perp$ is the normal connection on $\Gamma(T^\perp M)$.

Proof. By using (5.5) and $(\overline{\nabla}_Z J)X = 0$ we obtain

$$q\overline{g}(\nabla_Z X, \overline{W}) = \overline{g}(\overline{\nabla}_Z JX, J\overline{W}) - p\overline{g}(\overline{\nabla}_Z JX, \overline{W}),$$

for any $X \in \Gamma(TM_\perp)$, $Z \in \Gamma(TM_T)$ (i.e. $JX = NX$, $JZ = TZ$) and $\overline{W} \in \Gamma(TM)$. Thus, we get

$$q\overline{g}(\nabla_Z X, \overline{W}) = \overline{g}(J\overline{\nabla}_Z NX, \overline{W}) - p\overline{g}(\overline{\nabla}_Z NX, \overline{W})$$

and from here we get

$$q\overline{g}(\nabla_Z X, \overline{W}) = -\overline{g}(TA_{NX}Z + t\nabla^\perp_Z NX, \overline{W}) + p\overline{g}(A_{NX}Z, \overline{W}).$$

Thus, by using Lemma 1(2), we obtain

$$\overline{g}(qX(\ln f)Z, \overline{W}) = -\overline{g}((T - pI)A_{NX}Z + t\nabla^\perp_Z NX, \overline{W}),$$

for any $X \in \Gamma(TM_\perp)$, $Z \in \Gamma(TM_T)$ and $\overline{W} \in \Gamma(TM)$, which implies

$$qX(\ln f)Z = -(T - pI)A_{JX}Z - t\nabla^\perp_Z JX.$$

Thus f is constant on the connected components of $M_\perp$ if and only if (5.6) occurs. ☐

Proposition 8. *Let $M := M_{1T} \times_f M_{2T}$ be a warped product submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, where M_{1T} and M_{2T} are invariant submanifolds in $\overline{M}$. If $X(\ln f) \neq 0$, for any $X \in \Gamma(TM_{1T})$, then we have:*

$$TX(\ln f) = \tilde{\sigma} X(\ln f), \tag{5.7}$$

for any $X \in \Gamma(TM_{1T})$ and $Z \in \Gamma(TM_{2T})$, where $\tilde{\sigma} \in \{\sigma, \overline{\sigma}\}$, $\sigma := \sigma_{p,q}$ is a metallic number ($p, q \in N^*$) and $\overline{\sigma} := p - \sigma$.

Proof. From (5.5) we get $q\overline{g}(\nabla_Z X, \overline{W}) = \overline{g}(\overline{\nabla}_Z JX, J\overline{W}) - p\overline{g}(\overline{\nabla}_Z X, J\overline{W})$, for any $X \in \Gamma(TM_{1T})$, $Z \in \Gamma(TM_{2T})$ (i.e. $JX = TX$, $JZ = TZ$) and $\overline{W} \in \Gamma(TM)$. Thus, we have

$$q\overline{g}(\nabla_Z X, \overline{W}) = \overline{g}(\nabla_Z TX, T\overline{W}) + \overline{g}(h(Z, TX), N\overline{W}) - p\overline{g}(\nabla_Z X, J\overline{W}).$$

For $\overline{W} \in \Gamma(TM)$ we have $\overline{W} = W_1 + W_2$, where $W_1 \in \Gamma(TM_{1T})$ and $W_2 \in \Gamma(TM_{2T})$. Thus, $J\overline{W} = JW_1 + JW_2 = TW_1 + TW_2 = T\overline{W}$ and $N\overline{W} = 0$. By using Lemma 1(2), we obtain

$$\overline{g}(qX(\ln f)Z, \overline{W}) = \overline{g}(TX(\ln f)TZ, \overline{W}) - p\overline{g}(X(\ln f)Z, T\overline{W}),$$

which implies

$$\overline{g}(qX(\ln f)Z, \overline{W}) = \overline{g}((TX(\ln f) - pX(\ln f))JZ, \overline{W}),$$

for any $X \in \Gamma(TM_{1T})$, $Z \in \Gamma(TM_{2T})$ and $\overline{W} \in \Gamma(TM)$, which implies

$$(TX(\ln f) - pX(\ln f))JZ = qX(\ln f)Z. \tag{5.8}$$

Applying J in (5.8) and using (2.1), we obtain

$$[qX(\ln f) - p(TX(\ln f) - pX(\ln f))]JZ = q(TX(\ln f) - pX(\ln f)Z). \tag{5.9}$$

Using the proportionality of the coefficients of JZ and Z, respectively, from the equalities (5.8) and (5.9), we obtain

$$\left(\frac{TX(\ln f)}{X(\ln f)}\right)^2 - p \cdot \frac{TX(\ln f)}{X(\ln f)} - q = 0. \tag{5.10}$$

Denoting by $\alpha =: \dfrac{TX(\ln f)}{X(\ln f)}$ in (5.10), we obtain the equation verified by the metallic number, $\alpha^2 - p\alpha - q = 0$, with the solutions $\sigma = \dfrac{p + \sqrt{p^2 + 4q}}{2}$, $\overline{\sigma} := p - \sigma$ and from here we obtain (5.7). ☐

Proposition 9. *Let $M := M_{1\perp} \times_f M_{2\perp}$ be a warped product submanifold in a locally metallic (or locally Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$, where $M_{1\perp}$ and $M_{2\perp}$ are anti-invariant submanifolds in $\overline{M}$. Then, the warped function f is constant on the connected components of $M_{1\perp}$ if and only if*

$$t\nabla^{\perp}_{Z} NX = pth(X, Z), \tag{5.11}$$

for any $X \in \Gamma(TM_{1\perp})$ and $Z \in \Gamma(TM_{2\perp})$, where $\nabla^{\perp}$ is the normal connection on $\Gamma(T^{\perp}M)$.

Proof. For $\overline{W} \in \Gamma(TM)$ we have $\overline{W} = W_1 + W_2$, where $W_1 \in \Gamma(TM_{1\perp})$ and $W_2 \in \Gamma(TM_{2\perp})$. Thus, $J\overline{W} = JW_1 + JW_2 = NW_1 + NW_2 = N\overline{W}$ and $T\overline{W} = 0$.

By using (5.5) we obtain $q\overline{g}(\nabla_Z X, \overline{W}) = \overline{g}(\overline{\nabla}_Z JX, J\overline{W}) - p\overline{g}(\overline{\nabla}_Z X, J\overline{W})$, for any $X \in \Gamma(TM_{1\perp})$ and $Z \in \Gamma(TM_{2\perp})$ (i.e. $JX = NX$, $JZ = NZ$).

Thus, we get

$$q\overline{g}(\nabla_Z X, \overline{W}) = \overline{g}(\overline{\nabla}_Z NX, J\overline{W}) - p\overline{g}(\overline{\nabla}_Z X, N\overline{W})$$

and, from here we get

$$q\overline{g}(\nabla_Z X, \overline{W}) = \overline{g}(\nabla^{\perp}_Z NX, N\overline{W}) - p\overline{g}(h(X, Z), N\overline{W}).$$

Thus, by using Lemma 1(2) and (3.3), we obtain

$$\overline{g}(qX(\ln f)Z, \overline{W}) = \overline{g}(t\nabla^{\perp}_Z NX - pth(X, Z), \overline{W}),$$

for any $X \in \Gamma(TM_{1\perp})$, $Z \in \Gamma(TM_{2\perp})$ and $\overline{W} \in \Gamma(TM)$, which implies

$$qX(\ln f)Z = t\nabla^{\perp}_Z NX - pth(X, Z).$$

Therefore, the warped function f is constant on the connected components of $M_{1\perp}$ if and only if (5.11) occurs. ☐

Theorem 2. *Let* $M := M_T \times_f M_\theta$ *be a warped product semi-slant submanifold in a locally metallic (or locally Golden) Riemannian manifold* $(\overline{M}, \overline{g}, J)$ *(i.e.* M_T *is invariant and* M_θ *is a proper slant submanifold in* $\overline{M}$*, with the slant angle* $\theta \in (0, \frac{\pi}{2})$*). Then* $M := M_T \times_f M_\theta$ *is a non proper warped product submanifold in* $\overline{M}$ *(i.e. the warping function* f *is constant on the connected components of* M_T*).*

Proof. For any $X \in \Gamma(TM_T)$ (i.e. $JX = TX$) and $Z \in \Gamma(TM_\theta)$, from $\overline{\nabla} J = 0$, we get $\overline{\nabla}_Z JX = J\overline{\nabla}_Z X$ (where $\overline{\nabla}$ denotes the Levi-Civita connection on $\overline{M}$) and by using Lemma 1(2), (3.1) and (3.6)(i) we obtain:

$$TX(\ln f)Z + h(TX, Z) = T\nabla_Z X + N\nabla_Z X + th(X, Z) + nh(X, Z)$$

Thus, from the equality of the normal parts of the last equation, it follows

$$h(TX, Z) = X(\ln f)NZ + nh(X, Z). \tag{5.12}$$

By using $JX = TX$, for any $X \in \Gamma(TM_T)$ and replacing X with TX in (5.12), we get $h(J^2X, Z) = TX(\ln f)NZ + nh(TX, Z)$ and applying $\overline{g}(\cdot, NZ)$ in the last equality, we have

$$TX(\ln f)\overline{g}(NZ, NZ) = \overline{g}(h(J^2X, Z), NZ) - \overline{g}(nh(TX, Z), NZ)$$

and using (2.1), we get

$$TX(\ln f)\overline{g}(NZ, NZ) = p\overline{g}(h(TX, Z), NZ) + q\overline{g}(h(X, Z), NZ) - \overline{g}(nh(TX, Z), NZ),$$

for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$.

By using (5.2)(ii) we have $\overline{g}(h(TX, Z), NZ) = 0$ and $\overline{g}(h(X, Z), NZ) = 0$, for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$ and from (4.6) we obtain

$$TX(\ln f)\sin^2\theta[p\overline{g}(TZ, Z) + q\overline{g}(Z, Z)] = -\overline{g}(nh(TX, Z), NZ). \tag{5.13}$$

Moreover, by using (2.1) and (3.2)(ii), we have

$$\begin{aligned}\overline{g}(nh(TX, Z), NZ) &= \overline{g}(h(TX, Z), nNZ) = \overline{g}(h(TX, Z), J^2Z - JTZ)\\ &= p\overline{g}(h(TX, Z), NZ) + q\overline{g}(h(TX, Z), Z) - \overline{g}(h(TX, Z), NTZ) = 0\end{aligned}$$

because $\overline{g}(h(TX, Z), NZ) = \overline{g}(h(TX, Z), NTZ) = 0$ (from (5.2)(ii)) and $TX \in \Gamma(TM_T)$ and Z, $TZ \in \Gamma(TM_\theta)$.

Thus, from (5.13) we obtain $TX(\ln f)\sin^2\theta[p\overline{g}(TZ, Z) + q\overline{g}(Z, Z)] = 0$ and using (4.4)(i) we get

$$TX(\ln f)\sin^2\theta\overline{g}\left(\frac{1}{\cos^2\theta}T^2Z, Z\right) = 0.$$

Therefore, from (3.2)(i) we obtain $TX(\ln f)\tan^2\theta\overline{g}(TZ, TZ) = 0$, for any $Z \in \Gamma(TM_\theta)$.

Moreover, for $\theta \in (0, \frac{\pi}{2})$ we get $TZ \neq 0$ and $\tan^2\theta \neq 0$ which imply $TX(\ln f) = 0$, for any $X \in \Gamma(TM_T)$. Therefore, the warping function f is constant on M_T. $\square$

We shall construct some examples for the non trivial case of a warped product submanifold $M := M_\theta \times_f M_T$ in a metallic (and Golden) Riemannian manifold.

Example 3. Let $\mathbb{R}^4$ be the Euclidean space endowed with the usual Euclidean metric $\langle\cdot,\cdot\rangle$. Let $i : M \to \mathbb{R}^4$ be the immersion given by:

$$i(f,\alpha,\beta) := (f\sin\alpha, f\cos\alpha, f\sin\beta, f\cos\beta),$$

where $M := \{(f,\alpha,\beta) \mid f > 0, \alpha,\beta \in (0,\frac{\pi}{2})\}$ and $\sigma := \sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the metallic number $(p,q \in N^*)$.

We can find a local orthogonal frame on TM given by:

$$Z_1 = \sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_2} + \sin\beta\frac{\partial}{\partial x_3} + \cos\beta\frac{\partial}{\partial x_4},$$

$$Z_2 = f\cos\alpha\frac{\partial}{\partial x_1} - f\sin\alpha\frac{\partial}{\partial x_2}, \qquad Z_3 = f\cos\beta\frac{\partial}{\partial x_3} - f\sin\beta\frac{\partial}{\partial x_4}.$$

We define the metallic structure $J : \mathbb{R}^4 \to \mathbb{R}^4$ by:

$$J(X_1,X_2,X_3,X_4) := (\sigma X_1, \sigma X_2, \overline{\sigma}X_3, \overline{\sigma}X_4),$$

which verifies $J^2X = pJ + qI$ and $\langle JX, Y\rangle = \langle X, JY\rangle$, for any X, $Y \in \mathbb{R}^4$. Since

$$JZ_1 = \sigma\sin\alpha\frac{\partial}{\partial x_1} + \sigma\cos\alpha\frac{\partial}{\partial x_2} + \overline{\sigma}\sin\beta\frac{\partial}{\partial x_3} + \overline{\sigma}\cos\beta\frac{\partial}{\partial x_4},$$

$$JZ_2 = \sigma Z_2, \qquad JZ_3 = \overline{\sigma}Z_3,$$

we remark that $\langle JZ_1, Z_1\rangle = \sigma + \overline{\sigma} = p$ and $JZ_2, JZ_3 \subseteq span\{Z_2, Z_3\}$. We find that $\|Z_1\|^2 = 2$ and $\|Z_2\|^2 = \|Z_3\|^2 = f^2$.

Let us consider $D_1 = span\{Z_1\}$ the slant distribution with the slant angle $\theta = \arccos\frac{\sigma+\overline{\sigma}}{\sqrt{2(\sigma^2+\overline{\sigma}^2)}}$ and $D_2 = span\{Z_2, Z_3\}$ the invariant distribution with respect to J. The distributions D_1 and D_2 satisfy the conditions from Definition 3 and they are completely integrable. Let M_θ and M_T be the integral manifolds of D_1 and D_2, respectively.

Therefore, $M := M_\theta \times_f M_T$, with the Riemannian metric tensor

$$g := 2df^2 + f^2(d\alpha^2 + d\beta^2) = g_{M_\theta} + f^2 g_{M_T}$$

is a warped product semi-slant submanifold in the metallic Riemannian manifold $(\mathbb{R}^4, \langle\cdot,\cdot\rangle, J)$.

In particular, for $p = q = 1$ and $\phi := \sigma_{1,1} = \frac{1+\sqrt{5}}{2}$ the Golden number, the immersion $i : M \to \mathbb{R}^4$ is given by

$$i(f,\alpha,\beta) := (f\sin\alpha, f\cos\alpha, f\sin\beta, f\cos\beta),$$

and the Golden structure $J : \mathbb{R}^4 \to \mathbb{R}^4$ is defined by

$$J(X_1, X_2, X_3, X_4) := (\phi X_1, \phi X_2, \overline{\phi} X_3, \overline{\phi} X_4),$$

where $\overline{\phi} = 1 - \phi$.

For M_θ the integral manifold of the slant distribution $D_1 := span\{Z_1\}$, with the slant angle $\theta = \arccos\frac{1}{\sqrt{6}}$ and M_T the integral manifold of the invariant distribution $D_2 := span\{Z_2, Z_3\}$, then $M := M_\theta \times_f M_T$ is a warped product semi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^4, \langle\cdot,\cdot\rangle, J)$, with the metric

$$g := 2df^2 + f^2(d\alpha^2 + d\beta^2) = g_{M_\theta} + f^2 g_{M_T}.$$

Example 4. Let $\mathbb{R}^8$ be the Euclidean space endowed with the usual Euclidean metric $\langle\cdot,\cdot\rangle$. Let $i : M \to \mathbb{R}^8$ be the immersion given by:

$$i(f_1, f_2, \alpha, \beta) := (f_1\cos\alpha, f_2\cos\alpha, f_1\cos\beta, f_2\cos\beta, f_1\sin\alpha, f_2\sin\alpha, f_1\sin\beta, f_2\sin\beta),$$

where $M := \{(f_1, f_2, \alpha, \beta) \mid f_1, f_2 > 0, \alpha, \beta \in (0, \frac{\pi}{2})\}$.

We can find a local orthogonal frame on TM given by:

$$\begin{aligned}
Z_1 &= \cos\alpha\frac{\partial}{\partial x_1} + \cos\beta\frac{\partial}{\partial x_3} + \sin\alpha\frac{\partial}{\partial x_5} + \sin\beta\frac{\partial}{\partial x_7},\\
Z_2 &= \cos\alpha\frac{\partial}{\partial x_2} + \cos\beta\frac{\partial}{\partial x_4} + \sin\alpha\frac{\partial}{\partial x_6} + \sin\beta\frac{\partial}{\partial x_8},\\
Z_3 &= -f_1\sin\alpha\frac{\partial}{\partial x_1} - f_2\sin\alpha\frac{\partial}{\partial x_2} + f_1\cos\alpha\frac{\partial}{\partial x_5} + f_2\cos\alpha\frac{\partial}{\partial x_6},\\
Z_4 &= -f_1\sin\beta\frac{\partial}{\partial x_3} - f_2\sin\beta\frac{\partial}{\partial x_4} + f_1\cos\beta\frac{\partial}{\partial x_7} + f_2\cos\beta\frac{\partial}{\partial x_8}.
\end{aligned}$$

We define the metallic structure $J : \mathbb{R}^8 \to \mathbb{R}^8$ by:

$$J(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) := (\sigma X_1, \sigma X_2, \overline{\sigma} X_3, \overline{\sigma} X_4, \sigma X_5, \sigma X_6, \overline{\sigma} X_7, \overline{\sigma} X_8),$$

which verifies $J^2 = pJ + qI$ and $\langle JX, Y\rangle = \langle X, JY\rangle$, for any X, $Y \in \mathbb{R}^8$, where $\sigma := \sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the metallic number $(p, q \in N^*)$ and $\overline{\sigma} = p - \sigma$. Since

$$JZ_1 = \sigma\cos\alpha\frac{\partial}{\partial x_1} + \overline{\sigma}\cos\beta\frac{\partial}{\partial x_3} + \sigma\sin\alpha\frac{\partial}{\partial x_5} + \overline{\sigma}\sin\beta\frac{\partial}{\partial x_7},$$

$$JZ_2 = \sigma\cos\alpha\frac{\partial}{\partial x_2} + \overline{\sigma}\cos\beta\frac{\partial}{\partial x_4} + \sigma\sin\alpha\frac{\partial}{\partial x_6} + \overline{\sigma}\sin\beta\frac{\partial}{\partial x_8},$$
$$JZ_3 = \sigma Z_3, \qquad JZ_4 = \overline{\sigma} Z_4,$$

we remark that $\langle JZ_1, Z_1\rangle = \langle JZ_2, Z_2\rangle = \sigma + \overline{\sigma} = p$, $\|Z_1\|^2 = \|Z_2\|^2 = 2$.

Let us consider $D_1 = span\{Z_1, Z_2\}$ the slant distribution with the slant angle $\theta = \arccos\frac{\sigma+\overline{\sigma}}{\sqrt{2(\sigma^2+\overline{\sigma}^2)}}$ and $D_2 = span\{Z_3, Z_4\}$ an invariant distribution with respect to J. The distributions D_1 and D_2 satisfy the conditions from Definition 3 and they are completely integrable. Let M_θ and M_T be the integral manifolds of D_1 and D_2, respectively.

Therefore, $M := M_\theta \times_{\sqrt{f_1^2+f_2^2}} M_T$, with the Riemannian metric tensor

$$g := 2(df_1^2 + df_2^2) + (f_1^2 + f_2^2)(d\alpha^2 + d\beta^2) = g_{M_\theta} + (f_1^2 + f_2^2)g_{M_T}$$

is a warped product semi-slant submanifold in the metallic Riemannian manifold $(\mathbb{R}^8, \langle\cdot,\cdot\rangle, J)$.

In particular, for $p = q = 1$ and $\phi := \sigma_{1,1} = \frac{1+\sqrt{5}}{2}$ the Golden number, the immersion $i : M \to \mathbb{R}^8$ is given by

$$i(f_1, f_2, \alpha, \beta) := (f_1\cos\alpha, f_2\cos\alpha, f_1\cos\beta, f_2\cos\beta, f_1\sin\alpha, f_2\sin\alpha, f_1\sin\beta, f_2\sin\beta),$$

and the Golden structure $J : \mathbb{R}^8 \to \mathbb{R}^8$ is defined by

$$J(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) := (\phi X_1, \phi X_2, \overline{\phi} X_3, \overline{\phi} X_4, \phi X_5, \phi X_6, \overline{\phi} X_7, \overline{\phi} X_8),$$

where $\overline{\phi} = 1 - \phi$. If M_θ is the integral manifold of the slant distribution $D_1 := span\{Z_1, Z_2\}$ with the slant angle $\theta = \arccos\frac{1}{\sqrt{6}}$ and M_T is the integral manifold of the invariant distribution $D_2 = span\{Z_3, Z_4\}$ with respect to J, then $M := M_\theta \times_{\sqrt{f_1^2+f_2^2}} M_T$, with the metric

$$g := 2(df_1^2 + df_2^2) + (f_1^2 + f_2^2)(d\alpha^2 + d\beta^2) = g_{M_\theta} + (f_1^2 + f_2^2)g_{M_T}$$

is a warped product semi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^8, \langle\cdot,\cdot\rangle, J)$.

Proposition 10. *Let $M := M_\perp \times_f M_\theta$ (or $M := M_\theta \times_f M_\perp$) be a warped product hemi-slant submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ (i.e. $M_\perp$ is anti-invariant and M_θ is a proper slant submanifold in $\overline{M}$ with the slant angle $\theta \in (0, \frac{\pi}{2})$). Then M is a non proper warped product submanifold in $\overline{M}$ (i.e. the warping function f is constant on the connected components of $M_\perp$) if and only if*

$$A_{NZ}X = A_{NX}Z, \tag{5.14}$$

for any $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\theta)$ (or $X \in \Gamma(TM_\theta)$ and $Z \in \Gamma(TM_\perp)$, respectively).

Proof. Let $M := M_\perp \times_f M_\theta$ be a warped product hemi-slant submanifold in a locally metallic Riemannian manifold $(\overline{M}, \overline{g}, J)$ and $X \in \Gamma(TM_\perp)$ (i.e. $JX = NX$), $Z \in \Gamma(TM_\theta)$ (i.e. $JZ = TZ + NZ$).

From $\overline{\nabla} J = 0$, we have $\overline{\nabla}_Z JX = J\overline{\nabla}_Z X$ and using (3.1) and (3.6), we get:

$$-A_{NX}Z + \nabla^\perp_Z NX = J(\nabla_Z X + h(Z,X)).$$

By using Lemma 1(2), we obtain

$$-A_{NX}Z + \nabla^\perp_Z NX = X(\ln f)TZ + X(\ln f)NZ + th(Z,X) + nh(Z,X). \tag{5.15}$$

From the equality of tangential component from (5.15), we obtain

$$-A_{NX}Z = X(\ln f)TZ + th(Z,X), \tag{5.16}$$

for any $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\theta)$. From $\overline{\nabla}_X JZ = J\overline{\nabla}_X Z$ and using (3.1) and (3.6), we get:

$$\nabla_X TZ + h(X,TZ) - A_{NZ}X + \nabla^\perp_X NZ = J(\nabla_X Z + h(X,Z)).$$

By using Lemma 1(2), we obtain

$$X(\ln f)TZ + h(X,TZ) - A_{NZ}X + \nabla^\perp_X ZX = X(\ln f)TZ + \nabla_X NZ + th(Z,X) + nh(Z,X). \tag{5.17}$$

From the equality of tangential component from (5.17), we obtain

$$-A_{NZ}X = th(Z,X), \tag{5.18}$$

for any $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\theta)$. Thus, from (5.16) and (5.18) we get

$$X(\ln f)TZ = A_{NZ}X - A_{NX}Z, \tag{5.19}$$

for any $X \in \Gamma(TM_\perp)$ and $Z \in \Gamma(TM_\theta)$. Therefore, the warping function f is constant on the connected components of $M_\perp$ if and only if (5.14) holds.

For $M := M_\theta \times_f M_\perp$, we use a similar proof to obtain the conclusion. □

Remark 5. In a metallic (or Golden) Riemannian manifold $(\overline{M}, \overline{g}, J)$ there exist proper hemi-slant warped product submanifolds of the form $M := M_\perp \times_f M_\theta$ or $M := M_\theta \times_f M_\perp$, where $M_\perp$ is anti-invariant submanifold and M_θ is a proper slant submanifold in $\overline{M}$, with the slant angle $\theta \in (0, \frac{\pi}{2})$, as we can see in the next examples.

Example 5. Let $\mathbb{R}^5$ be the Euclidean space endowed with the usual Euclidean metric $\langle\cdot,\cdot\rangle$. Let $i: M \to \mathbb{R}^5$ be the immersion given by:

$$i(f,\alpha) := \left(f\sin\alpha, f\cos\alpha, f\sin\alpha, f\cos\alpha, \sqrt{\frac{p\sigma}{q}}f\right),$$

where $M := \{(f,\alpha) \mid f>0, \alpha \in (0,\frac{\pi}{2})\}$ and $\sigma := \sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the metallic number ($p,q \in N^*$).

We can find a local orthogonal frame on TM given by:

$$Z_1 = \sin\alpha\frac{\partial}{\partial x_1} + \cos\alpha\frac{\partial}{\partial x_2} + \sin\alpha\frac{\partial}{\partial x_3} + \cos\alpha\frac{\partial}{\partial x_4} + \sqrt{\frac{p\sigma}{q}}\frac{\partial}{\partial x_5},$$
$$Z_2 = f\cos\alpha\frac{\partial}{\partial x_1} - f\sin\alpha\frac{\partial}{\partial x_2} + f\cos\alpha\frac{\partial}{\partial x_3} - f\sin\alpha\frac{\partial}{\partial x_4}.$$

We define the metallic structure $J: \mathbb{R}^5 \to \mathbb{R}^5$ by:

$$J(X_1,X_2,X_3,X_4,X_5) := (\sigma X_1, \sigma X_2, \overline{\sigma}X_3, \overline{\sigma}X_4, \overline{\sigma}X_5),$$

which verifies $J^2X = pJ + qI$ and $\langle JX, Y\rangle = \langle X, JY\rangle$, for any X, $Y \in \mathbb{R}^5$. Since

$$JZ_1 = \sigma\sin\alpha\frac{\partial}{\partial x_1} + \sigma\cos\alpha\frac{\partial}{\partial x_2} + \overline{\sigma}\sin\alpha\frac{\partial}{\partial x_3} + \overline{\sigma}\cos\alpha\frac{\partial}{\partial x_4} + \overline{\sigma}\sqrt{\frac{p\sigma}{q}}\frac{\partial}{\partial x_5},$$
$$JZ_2 = \sigma f\cos\alpha\frac{\partial}{\partial x_1} - \sigma f\sin\alpha\frac{\partial}{\partial x_2} + \overline{\sigma}f\cos\alpha\frac{\partial}{\partial x_3} - \overline{\sigma}f\sin\alpha\frac{\partial}{\partial x_4},$$

we remark that $JZ_1 \perp span\{Z_1, Z_2\} = TM$ and $\langle JZ_2, Z_2\rangle = f^2(\sigma+\overline{\sigma}) = f^2p$. We find that $\|Z_1\|^2 = 1 + \frac{\sigma^2}{q}$ and $\|Z_2\|^2 = 2f^2$.

Let us consider $D_1 = span\{Z_1\}$ the anti-invariant distribution (with respect to J) and $D_2 = span\{Z_2\}$ the slant distribution with the slant angle $\theta = \arccos\frac{\sigma+\overline{\sigma}}{\sqrt{2(\sigma^2+\overline{\sigma}^2)}}$. The distributions D_1 and D_2 satisfy the conditions from Definition 3 and they are completely integrable. Let $M_\perp$ and M_θ be the integral manifolds of D_1 and D_2, respectively.

Therefore, $M := M_\perp \times_{\sqrt{2}f} M_\theta$ with the Riemannian metric tensor

$$g := \left(1+\frac{\sigma^2}{q}\right)df^2 + 2f^2d\alpha^2 = g_{M_\perp} + 2f^2g_{M_\theta}$$

is a warped product hemi-invariant submanifold in the metallic Riemannian manifold $(\mathbb{R}^5, \langle\cdot,\cdot\rangle, J)$.

In particular, for $p = q = 1$ and $\phi := \sigma_{1,1} = \frac{1+\sqrt{5}}{2}$ the Golden number, the immersion $i: M \to \mathbb{R}^5$ is given by

$$i(f,\alpha,\beta) := (f\sin\alpha, f\cos\alpha, f\sin\beta, f\cos\beta, \sqrt{\phi}f),$$

and the Golden structure $J : \mathbb{R}^5 \to \mathbb{R}^5$ is defined by

$$J(X_1, X_2, X_3, X_4, X_5) := (\phi X_1, \phi X_2, \overline{\phi} X_3, \overline{\phi} X_4, \overline{\phi} X_5),$$

where $\overline{\phi} = 1 - \phi$. If $M_\perp$ is the integral manifold of the anti-invariant distribution $D_1 := span\{Z_1\}$ and M_θ is the integral manifold of the slant distribution $D_2 := span\{Z_2\}$ with the slant angle $\theta = \arccos \frac{1}{\sqrt{6}}$, then $M := M_\perp \times_{\sqrt{2}f} M_\theta$ with the metric

$$g := (1+\phi^2)df^2 + 2f^2 d\alpha^2 = g_{M_\perp} + 2f^2 g_{M_\theta}$$

is a warped product hemi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^5, \langle\cdot,\cdot\rangle, J)$.

Example 6. Let $\mathbb{R}^7$ be the Euclidean space endowed with the usual Euclidean metric $\langle\cdot,\cdot\rangle$. Let $i : M \to \mathbb{R}^7$ be the immersion given by:

$$i(f,\alpha) := \left(f\sin\alpha, f\cos\alpha, \frac{\sigma}{\sqrt{q}} f\sin\alpha, \frac{\sigma}{\sqrt{q}} f\cos\alpha, \frac{1}{\sqrt{2}} f, \frac{1}{\sqrt{2}} f, -f \right),$$

where $M := \{(f,\alpha) \mid f > 0, \alpha \in (0, \frac{\pi}{2})\}$ and $\sigma := \sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the metallic number $(p, q \in N^*)$.

We can find a local orthogonal frame on TM given by:

$$\begin{aligned} Z_1 &= \sin\alpha \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_2} + \frac{\sigma}{\sqrt{q}} \sin\alpha \frac{\partial}{\partial x_3} + \frac{\sigma}{\sqrt{q}} \cos\alpha \frac{\partial}{\partial x_4} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_5} + \frac{1}{\sqrt{2}} \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7} \\ Z_2 &= f\cos\alpha \frac{\partial}{\partial x_1} - f\sin\alpha \frac{\partial}{\partial x_2} + \frac{\sigma}{\sqrt{q}} f\cos\alpha \frac{\partial}{\partial x_3} - \frac{\sigma}{\sqrt{q}} f\sin\alpha \frac{\partial}{\partial x_4}. \end{aligned}$$

We define the metallic structure $J : \mathbb{R}^7 \to \mathbb{R}^7$ by:

$$J(X_1, X_2, X_3, X_4, X_5, X_6, X_7) := (\sigma X_1, \sigma X_2, \overline{\sigma} X_3, \overline{\sigma} X_4, \sigma X_5, \sigma X_6, \overline{\sigma} X_7),$$

which verifies $J^2 X = pJ + qI$ and $\langle JX, Y\rangle = \langle X, JY\rangle$, for any X, $Y \in \mathbb{R}^5$. Since

$$\begin{aligned} JZ_1 &= \sigma\left(\sin\alpha \frac{\partial}{\partial x_1} + \cos\alpha \frac{\partial}{\partial x_2}\right) - \sqrt{q}\left(\sin\alpha \frac{\partial}{\partial x_3} + \cos\alpha \frac{\partial}{\partial x_4}\right) + \frac{\sigma}{\sqrt{2}}\left(\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_6}\right) - \overline{\sigma} \frac{\partial}{\partial x_7}, \\ JZ_2 &= \sigma\left(f\cos\alpha \frac{\partial}{\partial x_1} - f\sin\alpha \frac{\partial}{\partial x_2}\right) - \sqrt{q}\left(f\cos\alpha \frac{\partial}{\partial x_3} - f\sin\alpha \frac{\partial}{\partial x_4}\right), \end{aligned}$$

we remark that $JZ_2 \perp span\{Z_1, Z_2\}$ and $\langle JZ_1, Z_1\rangle = \sigma + \overline{\sigma}$.

We find that $\|Z_1\|^2 = 3 + \frac{\sigma^2}{q}$ and $\|Z_2\|^2 = f^2(\frac{\sigma^2}{q} + 1)$.

We denote by $D_1 := span\{Z_1\}$ the slant distribution with the slant angle θ, where $\cos\theta = \frac{\sqrt{q}(\sigma+\overline{\sigma})}{\sqrt{(\sigma^2+3q)(\sigma^2+q+2)}}$ and by $D_2 := span\{Z_2\}$ an anti-invariant distribution (with respect to J). The distributions D_1 and D_2 satisfy conditions from Definition 3.

If M_θ and $M_\perp$ are the integral manifolds of the distributions D_1 and D_2, respectively, then $M := M_\theta \times_{\sqrt{(\frac{\sigma^2}{q}+1)f}} M_\perp$ with the Riemannian metric tensor

$$g := \left(\frac{\sigma^2}{q}+3\right)df^2 + f^2\left(\frac{\sigma^2}{q}+1\right)d\alpha^2 = g_{M_\theta} + f^2\left(\frac{\sigma^2}{q}+1\right)g_{M_\perp}$$

is a warped product hemi-invariant submanifold in the metallic Riemannian manifold $(\mathbb{R}^7, \langle\cdot,\cdot\rangle, J)$.

In particular, for $p = q = 1$ and $\phi := \sigma_{1,1}$ is the Golden number, the immersion $i : M \to \mathbb{R}^7$ is given by

$$i(f,\alpha) := \left(f\sin\alpha, f\cos\alpha, \phi f\sin\alpha, \phi f\cos\alpha, \frac{1}{\sqrt{2}}f, \frac{1}{\sqrt{2}}f, -f\right),$$

and the Golden structure $J : \mathbb{R}^7 \to \mathbb{R}^7$ is defined by

$$J(X_1, X_2, X_3, X_4, X_5) := (\phi X_1, \phi X_2, \overline{\phi}X_3, \overline{\phi}X_4, \phi X_5, \phi X_6, \overline{\phi}X_7),$$

where $\overline{\phi} = 1-\phi$.

If M_θ is the integral manifold of the slant distribution $D_1 := span\{Z_1\}$ with the slant angle $\theta = \arccos\frac{1}{\phi^2+3}$ and $M_\perp$ is the integral manifold of the anti-invariant distribution $D_2 := span\{Z_2\}$, then $M := M_\theta \times_{\sqrt{\phi^2+1}f} M_\perp$ with the metric

$$g := (\phi^2+3)df^2 + f^2(\phi^2+1)d\alpha^2 = g_{M_\theta} + f^2(\phi^2+1)g_{M_\perp}$$

is a warped product hemi-slant submanifold in the Golden Riemannian manifold $(\mathbb{R}^7, \langle\cdot,\cdot\rangle, J)$.

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