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EXISTENCE OF SOLUTIONS FOR A CLASS OF $p(x)$-CURL SYSTEMS ARISING IN ELECTROMAGNETISM WITHOUT (A-R) TYPE CONDITIONS

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Abstract. In this paper, we study the existence and multiplicity of solutions for a class of $p(x)$-curl systems arising in electromagnetism. Under suitable conditions on the nonlinearities which do not satisfy Ambrosetti-Rabinowitz (A-R) type conditions, we obtain some existence and multiplicity results for the problem by using the mountain pass theorem and fountain theorem. Our main results in this paper complement and extend some earlier ones concerning the $p(x)$-curl operator in [4, 15].

1. Introduction

Motivated by the contributions on $p(x)$-curl operator in recent papers of Xiang et al. [15] and Bahrouni et al. [4], in this paper we study the existence and multiplicity of solutions for a class of stationary $p(x)$-curl systems arising in electromagnetism. Let Ω be a bounded simply connected domain of $\mathbb{R}^3$ with a $C^{1,1}$ boundary denoted by $\partial\Omega$. In what follows, vector functions and spaces of vector functions will be denoted by boldface symbols. We will use $\boldsymbol{n}$ to denote the outward unitary normal vector to $\partial\Omega$ and ∂_x to denote the partial derivative of function with respect to the variable x. Let $\boldsymbol{u} = (u_1, u_2, u_3)$ be a vector function on Ω. In order to introduce our problem precisely, we first give some notations. The divergence of $\boldsymbol{u}$ is denoted by

$$\nabla \cdot \boldsymbol{u} = \partial_{x_1} u_1 + \partial_{x_2} u_2 + \partial_{x_3} u_3$$

and the curl of $\boldsymbol{u}$ is defined by

$$\nabla \times \boldsymbol{u} = \left(\partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_2} u_3 - \partial_{x_3} u_2, \partial_{x_1} u_2 - \partial_{x_2} u_1\right).$$

Then $\nabla \times \boldsymbol{u}$ and $\nabla \cdot \boldsymbol{u}$ satisfy the following identity

$$-\Delta \boldsymbol{u} = \nabla \times (\nabla \times \boldsymbol{u}) - \nabla(\nabla \cdot \boldsymbol{u}),$$

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where $\Delta\boldsymbol{u} = (\Delta u_1, \Delta u_2, \Delta u_3)$ and $\Delta u_i = \nabla\cdot(\nabla u_i)$, $i = 1,2,3$.

In this paper, we are interested in the existence of solutions for the following stationary $p(x)$-curl systems

$$\begin{cases} -\nabla\times\left(|\nabla\times\boldsymbol{u}|^{p(x)-2}\nabla\boldsymbol{u}\right) = \boldsymbol{f}(x,\boldsymbol{u}), & \nabla\boldsymbol{u} = 0 \text{ in } \Omega \\ |\nabla\times\boldsymbol{u}|^{p(x)-2}\nabla\times\boldsymbol{u}\times\boldsymbol{n} = 0, & \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p \in C_+(\overline{\Omega})$ such that

$$\frac{6}{5} < p^- := \min_{x\in\overline{\Omega}} p(x) \le p^+ := \max_{x\in\overline{\Omega}} p(x) < 3 \tag{1.2}$$

and satisfies the logarithmic continuity, that is, there exists a function $w : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ such that

$$\forall x, y \in \overline{\Omega}, \quad |x-y| < 1, \quad |p(x)-p(y)| \le w(|x-y|), \quad \lim_{\tau\to 0^+} w(\tau)\log\frac{1}{\tau} = C < +\infty. \tag{1.3}$$

In [15], Xiang et al. considered problem (1.1) in the case when the nonlinear term $\boldsymbol{f}$ satisfies the conditions of Ambrosetti-Rabinowitz (A-R) type, that is, there exists $\mu > p^+$ such that

$$0 < \mu F(x,\boldsymbol{t}) \le \boldsymbol{f}(x,\boldsymbol{t})\cdot\boldsymbol{t} \tag{1.4}$$

for all $x \in \Omega$ and $\boldsymbol{t} \in \mathbb{R}^3\setminus\{0\}$. Using the mountain pass theorem [1] and the minimum principle, they obtained existence results for both the sublinear and superlinear cases. Condition (1.4) is a tool to study superlinear problems, it is a natural and useful condition to ensure the mountain pass geometry and the Palais-Smale (P-S) condition. Our goal is to consider the stationary $p(x)$-curl system (1.1) without Ambrosetti-Rabinowitz (A-R) type conditions. The main tools are essentially based on the mountain pass theorem [1] and fountain theorem [14]. Our situation here is different from those presented by Bahrouni et al. [4] in which the authors studied problem (1.1) by using the three critical points theorem due to Ricceri [13]. The study of the existence of solutions for $p(x)$-curl systems is a new and interesting topic. To the best of our knowledge, the only results involving the $p(x)$-curl operators can be found in [3, 4, 15]. For more information on the meaning of system (1.1) from physical point of view, we refer the readers to some results on p-curl systems and their applications [2, 9].

2. Preliminaries

In order to state and prove the result of the paper in the next section, we recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the books [7, 10, 12] and the papers [4, 8, 11, 15]. Set

$$C_+(\overline{\Omega}) := \{h;\ h \in C(\overline{\Omega}), h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x \in \overline{\Omega}} h(x) \text{ and } h^- = \inf_{x \in \overline{\Omega}} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u : \text{a measurable real-valued function such that} \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

We recall the so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} := \inf \left\{ \lambda > 0;\ \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \le 1 \right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \le p^+ < +\infty$ and continuous functions are dense if $p^+ < +\infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < +\infty$ and p_1, p_2 are variable exponents so that $p_1(x) \le p_2(x)$ a.e. in Ω then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequalities

$$\left| \int_\Omega uv\, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)} \le 2 |u|_{p(x)} |v|_{p'(x)}$$

hold true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

If $u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$ then the following relations hold

$$|u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{2.1}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{2.2}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \iff \rho_{p(x)}(u_n - u) \to 0. \tag{2.3}$$

If $p \in C_+(\overline{\Omega})$ the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$, consisting of functions $u \in L^{p(x)}(\Omega)$ whose distributional gradient ∇u exists almost everywhere and belongs to $[L^{p(x)}(\Omega)]^N$, endowed with the norm

$$\|u\| := \inf \left\{ \lambda > 0;\ \int_\Omega \left[\left| \frac{\nabla u(x)}{\lambda} \right|^{p(x)} + \left| \frac{u(x)}{\lambda} \right|^{p(x)} \right] dx \le 1 \right\}$$

or

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)},$$

is a separable and reflexive Banach space. The space of smooth functions are in general not dense in $W^{1,p(x)}(\Omega)$, but if the exponent $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is,

$$|p(x) - p(y)| \le -\frac{M}{\log(|x-y|)}, \quad \forall x, y \in \Omega, \quad |x-y| \le \frac{1}{2},$$

then the smooth functions are dense in $W^{1,p(x)}(\Omega)$. The space $\left(W^{1,p(x)}(\Omega), \|.\|\right)$ is a separable and Banach space.

Proposition 2.1 (see [8, Theorem 2.3])**.** *If $p, q \in C(\overline{\Omega})$, $1 < p^- \le p^+ < 3$ and $1 \le q(x) < p^*(x) = \frac{3p(x)}{3-p(x)}$ for all $x \in \overline{\Omega}$ then the embedding*

$$W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

is continuous and compact.

Let $\boldsymbol{L}^{p(x)}(\Omega) = [L^{p(x)}(\Omega)]^3$, $\boldsymbol{W}^{1,p(x)}(\Omega) = [W^{1,p(x)}(\Omega)]^3$ and define

$$\boldsymbol{W}^{p(x)} = \left\{\boldsymbol{u} \in \boldsymbol{L}^{p(x)}(\Omega) : \nabla \times \boldsymbol{u} \in \boldsymbol{L}^{p(x)}(\Omega), \nabla \cdot \boldsymbol{u} = 0, \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\right\},$$

where $\boldsymbol{n}$ denotes the outward unitary normal vector to $\partial\Omega$. Equip $\boldsymbol{W}^{p(x)}(\Omega)$ with the norm

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} = \|\boldsymbol{u}\|_{\boldsymbol{L}^{p(x)}(\Omega)} + \|\nabla \times \boldsymbol{u}\|_{\boldsymbol{L}^{p(x)}(\Omega)}.$$

If $p^- > 1$, $\boldsymbol{W}^{p(x)}(\Omega)$ is a closed subspace of $\boldsymbol{W}_{\boldsymbol{n}}^{p(x)}(\Omega)$, where

$$\boldsymbol{W}_{\boldsymbol{n}}^{p(x)}(\Omega) = \left\{\boldsymbol{u} \in \boldsymbol{W}^{1,p(x)}(\Omega) : \boldsymbol{u} \cdot \boldsymbol{n}|_{\partial\Omega} = 0\right\}.$$

Proposition 2.2 (see [3, Theorem 2.1])**.** *If $p \in C_+(\overline{\Omega})$ satisfies $1 < p^- \le p^+ < +\infty$ and (1.3), then the embedding $\boldsymbol{W}^{1,p(x)}(\Omega)$ is a closed subspace of $\boldsymbol{W}_{\boldsymbol{n}}^{1,p(x)}(\Omega)$. Moreover, if $p^- > \frac{6}{5}$ then $\|\nabla \times .\|_{\boldsymbol{L}^{p(x)}(\Omega)}$ is a norm in $\boldsymbol{W}_{\boldsymbol{n}}^{1,p(x)}(\Omega)$ and there exists $C = C(N, p^-, p^+) > 0$ such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p(x)}(\Omega)} \le C\|\nabla \times \boldsymbol{u}\|_{\boldsymbol{L}^{s(x)}(\Omega)}.$$

Remark 2.3. By Proposition 2.1 and Proposition 2.2, the embedding $\boldsymbol{W}^{1,p(x)}(\Omega) \hookrightarrow \boldsymbol{L}^{q(x)}(\Omega)$ is compact, with $1 < p^- \le p^+ < 3$, $q \in C(\overline{\Omega})$ and $1 \le q(x) < p^*(x) = \frac{3p(x)}{3-p(x)}$ for all $x \in \overline{\Omega}$. Moreover, $\left(\boldsymbol{W}^{1,p(x)}(\Omega), \|.\|_{\boldsymbol{W}^{1,p(x)}(\Omega)}\right)$ is a reflexive and Banach space.

3. Main results

In order to state the main results of the paper, let us introduce the following hypotheses on the structure of the problem

(H_1) $F:\Omega\times\mathbb{R}^3\to\mathbb{R}$ is differentiable with respect to $\boldsymbol{t}\in\mathbb{R}^3$ such that $\boldsymbol{f}=\partial_{\boldsymbol{t}}F(x,\boldsymbol{t}):\Omega\times\mathbb{R}^3\to\mathbb{R}^3$ is a continuous function;

(H_2) There exists a constant $C>0$, $q\in C(\overline{\Omega})$ and $p(x)<q(x)<p^*(x)=\frac{3p(x)}{3-p(x)}$ for all $x\in\overline{\Omega}$ such that

$$|\boldsymbol{f}(x,\boldsymbol{t})|\le C(1+|\boldsymbol{t}|^{q(x)-1}),\quad \forall (x,\boldsymbol{t})\in\Omega\times\mathbb{R}^3;$$

(H_3) $\lim\limits_{\boldsymbol{t}\to 0}\frac{\boldsymbol{f}(x,\boldsymbol{t})}{|\boldsymbol{t}|^{p^+-1}}=0$ uniformly in $x\in\Omega$;

(H_4) $\lim\limits_{|\boldsymbol{t}|\to+\infty}\frac{F(x,\boldsymbol{t})}{|\boldsymbol{t}|^{p^+}}=+\infty$ uniformly in $x\in\Omega$;

(H_5) There exists $\theta\ge 1$ such that $\theta\mathscr{F}(x,\boldsymbol{t})\ge\mathscr{F}(x,\tau\boldsymbol{t})$ for all $(x,\boldsymbol{t})\in\Omega\times\mathbb{R}^3$ and all $\tau\in[0,1]$, where $\mathscr{F}(x,\boldsymbol{t})=\boldsymbol{f}(x,\boldsymbol{t})\cdot\boldsymbol{t}-p^+F(x,\boldsymbol{t})$.

Definition 3.4. We say that $\boldsymbol{u}\in\boldsymbol{W}^{p(x)}(\Omega)$ is a weak solution for problem (1.1) if

$$\int_\Omega|\nabla\times\boldsymbol{u}|^{p(x)-2}\nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{v}\,dx-\int_\Omega\boldsymbol{f}(x,\boldsymbol{u})\cdot\boldsymbol{v}\,dx=0$$

for all $\boldsymbol{v}\in\boldsymbol{W}^{p(x)}(\Omega)$.

Let us consider the functional $J:\boldsymbol{W}^{p(x)}(\Omega)\to\mathbb{R}$ defined by

$$J(\boldsymbol{u})=\int_\Omega\frac{1}{p(x)}|\nabla\times\boldsymbol{u}|^{p(x)}\,dx-\int_\Omega F(x,\boldsymbol{u})\,dx. \tag{3.1}$$

Using condition (H_2) and Remark 2.3, with the same arguments as those used in [15] we can show that $J\in C^1(\boldsymbol{W}^{p(x)}(\Omega),\mathbb{R})$ and its derivative is given by

$$J(\boldsymbol{u})(\boldsymbol{v})=\int_\Omega|\nabla\times\boldsymbol{u}|^{p(x)-2}\nabla\times\boldsymbol{u}\cdot\nabla\times\boldsymbol{v}\,dx-\int_\Omega\boldsymbol{f}(x,\boldsymbol{u})\cdot\boldsymbol{v}\,dx$$

for all $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{W}^{p(x)}(\Omega)$. Hence, we can find weak solutions of problem (1.1) as the critical points of the functional J in the space $\boldsymbol{W}^{p(x)}(\Omega)$.

The main results of this paper is given by the following theorems.

Theorem 3.5. *Assume that the conditions* (1.2), (1.3) *and the hypotheses* (H_1)–(H_5) *hold, then problem (1.1) has at least one non-trivial weak solution.*

Theorem 3.6. *Assume that the conditions* (1.2), (1.3) *and the hypotheses* (H_1)–(H_5) *hold. Moreover, we assume that*

(H_6) $\boldsymbol{f}(x,-\boldsymbol{t})=-\boldsymbol{f}(x,\boldsymbol{t})$ *for all* $x\in\Omega$ *and all* $\boldsymbol{t}\in\mathbb{R}^3$.

Then problem (1.1) has a sequence of weak solutions $\{\pm\boldsymbol{u}_k\}$ *such that* $J(\pm\boldsymbol{u}_k)\to+\infty$ *as* $k\to\infty$.

Througout this section, we alway assume that the conditions (1.2), (1.3), (H_1)-(H_5) hold. We also denote by c_i a general positive real number whose value may change from line to line.

Lemma 3.7. *There exist some constants $\rho, \alpha > 0$ such that $J(\boldsymbol{u}) \geq \alpha$ for all $\boldsymbol{u} \in \boldsymbol{W}^{p(x)}(\Omega)$ with $\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} = \rho$.*

Proof. From (H_2) and (H_3), for any $\epsilon > 0$, there exists a positive constant $c(\epsilon)$ depending on ϵ such that

$$|F(x,\boldsymbol{t})| \leq \epsilon|\boldsymbol{t}|^{p^+} + c(\epsilon)|\boldsymbol{t}|^{q(x)}, \quad \forall (x,\boldsymbol{t}) \in \overline{\Omega} \times \mathbb{R}^3. \tag{3.2}$$

Since the embeddings $\boldsymbol{W}^{p(x)}(\Omega) \hookrightarrow \boldsymbol{L}^{p^+}(\Omega)$ and $\boldsymbol{W}^{p(x)}(\Omega) \hookrightarrow \boldsymbol{L}^{q(x)}(\Omega)$ are continuous and compact, there exist constants $c_1, c_2 > 0$ such that

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^{p^+}(\Omega)} \leq c_1\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)}, \quad \|\boldsymbol{u}\|_{\boldsymbol{L}^{q(x)}(\Omega)} \leq c_2\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)}, \quad \forall \boldsymbol{u} \in \boldsymbol{W}^{p(x)}(\Omega). \tag{3.3}$$

Let $0 < \epsilon < \frac{1}{2p^+ c_1^{p^+}}$, where c_1 is given by (3.3). From (3.2) and (3.3), for all $\boldsymbol{u} \in \boldsymbol{W}^{p(x)}(\Omega)$ with $\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} < 1$, using Remark 2.3 we have

$$\begin{aligned}
J(\boldsymbol{u}) &= \int_\Omega \frac{1}{p(x)}|\nabla \times \boldsymbol{u}|^{p(x)}\,dx - \int_\Omega F(x,\boldsymbol{u})\,dx \\
&\geq \frac{1}{p^+}\|\boldsymbol{u}\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - \epsilon\int_\Omega |\boldsymbol{u}|^{p^+}\,dx - c(\epsilon)\int_\Omega |\boldsymbol{u}|^{q(x)}\,dx \\
&\geq \frac{1}{p^+}\|\boldsymbol{u}\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - \epsilon c_1^{p^+}\|\boldsymbol{u}\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - c(\epsilon)c_2^{q^-}\|\boldsymbol{u}\|^{q^-}_{\boldsymbol{W}^{p(x)}(\Omega)} \\
&\geq \left(\frac{1}{2p^+} - c(\epsilon)c_2^{q^-}\|\boldsymbol{u}\|^{q^- - p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}\right)\|\boldsymbol{u}\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}.
\end{aligned}$$

From the above information, we can choose $\alpha > 0$ and $\rho > 0$ so that $J(\boldsymbol{u}) \geq \alpha > 0$ for all $\boldsymbol{u} \in \boldsymbol{W}^{p(x)}(\Omega)$ with $\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} = \rho$. □

Lemma 3.8. *There exists a function $\boldsymbol{e} \in \boldsymbol{W}^{p(x)}(\Omega)$ with $\|\boldsymbol{e}\|_{\boldsymbol{W}^{p(x)}(\Omega)} > \rho$ such that $J(\boldsymbol{e}) < 0$, where ρ is given by Lemma 3.7.*

Proof. From (H_4), it follows that for any $M > 0$ there exists a constant $c_M = c(M) > 0$ depending on M, such that

$$F(x,\boldsymbol{t}) \geq M|\boldsymbol{t}|^{p^+} - c_M, \quad \text{for a.e. } x \in \Omega, \quad \forall \boldsymbol{t} \in \mathbb{R}^3. \tag{3.4}$$

Take $\phi \in [C_0^\infty(\Omega)]^3$ with $\phi > 0$, from (3.4) and the definition of J, we get

$$\begin{aligned}
J(\tau\phi) &= \int_\Omega \frac{1}{p(x)}|\nabla \times (\tau\phi)|^{p(x)}\,dx - \int_\Omega F(x,\tau\phi)\,dx \\
&\leq \frac{\tau^{p^+}}{p^-}\|\phi\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - M\int_\Omega |\tau\phi|^{p^+}\,dx + c_M|\Omega|
\end{aligned}$$

$$\leq \tau^{p^+}\left(\frac{1}{p^-}\|\phi\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - M\int_\Omega |\phi|^{p^+}\,dx\right) + c_M|\Omega|,$$

where $\tau > 1$ is large enough and $|\Omega|$ denotes the Lebesgue measure of Ω. From this, if M is large enough such that

$$\frac{1}{p^-}\|\phi\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - M\int_\Omega |\phi|^{p^+}\,dx < 0,$$

then we have $\lim_{\tau\to+\infty} J(\tau\phi) = -\infty$. Therefore, there exists $\boldsymbol{e} \in \boldsymbol{W}^{p(x)}(\Omega)$ with $\|\boldsymbol{e}\|_{\boldsymbol{W}^{p(x)}(\Omega)} > \rho$ such that $J(\boldsymbol{e}) < 0$, where ρ is given by Lemma 3.7. □

Lemma 3.9. *The functional J satisfies the Cerami condition.*

Proof. For all $c \in \mathbb{R}$, let $\{\boldsymbol{u}_n\} \subset \boldsymbol{W}^{p(x)}(\Omega)$ be such that

$$J(\boldsymbol{u}_n) \to c, \quad \left(1 + \|\boldsymbol{u}_n\|_{\boldsymbol{W}^{p(x)}(\Omega)}\right) J'(\boldsymbol{u}_n) \to 0 \text{ as } n\to\infty. \tag{3.5}$$

Then we have

$$J(\boldsymbol{u}_n) = c + o_n(1), \quad J'(\boldsymbol{u}_n)(\boldsymbol{u}_n) = o_n(1). \tag{3.6}$$

where $o_n(1) \to 0$ as $n\to\infty$.

We will show that $\{\boldsymbol{u}_n\}$ is bounded in $\boldsymbol{W}^{p(x)}(\Omega)$ by contradiction. Indeed, if $\{\boldsymbol{u}_n\}$ is unbounded, then up to a subsequence, we may assume that $\|\boldsymbol{u}_n\|_{\boldsymbol{W}^{p(x)}(\Omega)} \to +\infty$ as $n\to\infty$. Put $\boldsymbol{w}_n = \frac{\boldsymbol{u}_n}{\|\boldsymbol{u}_n\|_{\boldsymbol{W}^{p(x)}(\Omega)}}$ for all $n \in \mathbb{N}$. Clearly, $\|\boldsymbol{w}_n\|_{\boldsymbol{W}^{p(x)}(\Omega)} = 1$ for all $n \in \mathbb{N}$, then there exists $\boldsymbol{w} \in \boldsymbol{W}^{p(x)}(\Omega)$ such that, up to a subsequence, still denoted by $\{\boldsymbol{w}_n\}$, we have $\boldsymbol{w}_n$ converges weakly to some function $\boldsymbol{w}$ in $\boldsymbol{W}^{p(x)}(\Omega)$ and

$$\boldsymbol{w}_n(x) \to \boldsymbol{w}(x) \text{ a.e. in } \Omega,\ n\to\infty, \tag{3.7}$$

$$\boldsymbol{w}_n \to \boldsymbol{w} \text{ strongly in } \boldsymbol{L}^{q(x)}(\Omega),\ n\to\infty, \tag{3.8}$$

$$\boldsymbol{w}_n \to \boldsymbol{w} \text{ strongly in } \boldsymbol{L}^{p^+}(\Omega),\ n\to\infty. \tag{3.9}$$

Let $\Omega_{\neq} := \{x \in \Omega : \boldsymbol{w}(x) \neq 0\}$. If $x \in \Omega_{\neq}$ then it follows from (3.7) that $|\boldsymbol{u}_n(x)| = |\boldsymbol{w}_n(x)|\|\boldsymbol{u}_n\|_{\boldsymbol{W}^{p(x)}(\Omega)} \to +\infty$ as $n\to\infty$. Moreover, from (H_4), we have

$$\lim_{m\to\infty} \frac{F(x,\boldsymbol{u}_n(x))}{|\boldsymbol{u}_n(x)|^{p^+}}|\boldsymbol{w}_n(x)|^{p^+} = +\infty, \quad x \in \Omega_{\neq}. \tag{3.10}$$

Using again the condition (H_4), there exists $\delta > 0$ such that

$$\frac{F(x,\boldsymbol{t})}{|\boldsymbol{t}|^{p^+}} > 1 \tag{3.11}$$

for all $x \in \Omega$ and $|\boldsymbol{t}| > \delta > 0$. Since $F(x,\boldsymbol{t})$ is continuous on $\overline{\Omega} \times [|\boldsymbol{t}| \leq \delta]$, there exists a positive constant c_3 such that

$$|F(x,\boldsymbol{t})| \leq c_3 \tag{3.12}$$

for all $(x,\boldsymbol{t}) \in \overline{\Omega} \times [|\boldsymbol{t}| \le \delta]$. From (3.11) and (3.12) there exists $c_4 \in \mathbb{R}$ such that

$$F(x,\boldsymbol{t}) \ge c_4 \tag{3.13}$$

for all $(x,\boldsymbol{t}) \in \overline{\Omega} \times \mathbb{R}^3$. From (3.13), for all $x \in \Omega$ and n, we have

$$\frac{F(x,\boldsymbol{u}_n(x)) - c_4}{\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}} \ge 0$$

or

$$\frac{F(x,\boldsymbol{u}_n(x))}{|\boldsymbol{u}_n(x)|^{p^+}}|\boldsymbol{w}_n(x)|^{p^+} - \frac{c_4}{\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}} \ge 0, \quad \forall x \in \Omega, \quad \forall n. \tag{3.14}$$

By (3.6), we have

$$\begin{aligned} c &= J(\boldsymbol{u}_n) + o_n(1) \\ &\ge \frac{1}{p^+}\|\boldsymbol{u}_n\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)} - \int_\Omega F(x,\boldsymbol{u}_n)\,dx + o_n(1), \end{aligned}$$

which implies that

$$\int_\Omega F(x,\boldsymbol{u}_n)\,dx \ge \frac{1}{p^+}\|\boldsymbol{u}_n\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)} - c + o_n(1) \to +\infty \text{ as } n \to \infty. \tag{3.15}$$

We also have

$$\begin{aligned} c &= J(\boldsymbol{u}_n) + o_n(1) \\ &\le \frac{1}{p^-}\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - \int_\Omega F(x,\boldsymbol{u}_n)\,dx + o_n(1) \end{aligned}$$

and by (3.15),

$$\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} \ge p^-\int_\Omega F(x,\boldsymbol{u}_n)\,dx + p^-c - o(1) > 0 \text{ for } n \text{ large enough.} \tag{3.16}$$

Next, we will claim that $|\Omega_{\neq}| = 0$. In fact, if $|\Omega_{\neq}| \ne 0$, then by relations (3.10), (3.14), (3.16) and the Fatou lemma, we have

$$\begin{aligned} +\infty &= (+\infty)|\Omega_{\neq}| \\ &= \int_{\Omega_{\neq}} \liminf_{n\to\infty} \frac{F(x,\boldsymbol{u}_n(x))}{|\boldsymbol{u}_n(x)|^{p^+}}|\boldsymbol{w}_n(x)|^{p^+}\,dx - \int_{\Omega_{\neq}} \limsup_{n\to\infty} \frac{c_4}{\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}}\,dx \\ &= \int_{\Omega_{\neq}} \liminf_{n\to\infty} \left(\frac{F(x,\boldsymbol{u}_n(x))}{|\boldsymbol{u}_n(x)|^{p^+}}|\boldsymbol{w}_n(x)|^{p^+} - \frac{c_4}{\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}} \right) dx \\ &\le \liminf_{n\to\infty} \int_{\Omega_{\neq}} \left(\frac{F(x,\boldsymbol{u}_n(x))}{|\boldsymbol{u}_n(x)|^{p^+}}|\boldsymbol{w}_n(x)|^{p^+} - \frac{c_4}{\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}} \right) dx \end{aligned}$$

$$\begin{aligned}
&\leq \liminf_{n\to\infty} \int_\Omega \left(\frac{F(x,\boldsymbol{u}_n(x))}{|\boldsymbol{u}_n(x)|^{p^+}} |\boldsymbol{w}_n(x)|^{p^+} - \frac{c_4}{\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}} \right) dx \\
&= \liminf_{n\to\infty} \int_\Omega \frac{F(x,\boldsymbol{u}_n(x))}{\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}} \, dx - \limsup_{n\to\infty} \int_\Omega \frac{c_4}{\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}} \, dx \\
&= \liminf_{n\to\infty} \int_\Omega \frac{F(x,\boldsymbol{u}_n(x))}{\|\boldsymbol{u}_n\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)}} \, dx \\
&\leq \liminf_{n\to\infty} \frac{\int_\Omega F(x,\boldsymbol{u}_n(x))\, dx}{p^- \int_\Omega F(x,\boldsymbol{u}_n)\, dx + p^- c - o(1)}.
\end{aligned}$$

Combining this with (3.15), we obtain

$$+\infty \leq \frac{1}{p^-},$$

which is a contradiction. This shows that $|\Omega_{\neq}| = 0$ and thus $\boldsymbol{w}(x) = 0$ a.e. in Ω.

Since the function $\tau \mapsto J(\tau u_m)$ is continuous in $\tau \in [0,1]$, for each n there exists $\tau_n \in [0,1]$ such that

$$J(\tau_n \boldsymbol{u}_n) := \max_{\tau\in[0,1]} J(\tau \boldsymbol{u}_n), \quad n = 1,2,\dots \tag{3.17}$$

Clearly, $\tau_n > 0$ and $J(\tau_n \boldsymbol{u}_n) \geq c > 0 = J(0) = J(0.\boldsymbol{u}_n)$. If $\tau_n < 1$ then $\frac{d}{dt} J(\tau \boldsymbol{u}_n)|_{\tau=\tau_n} = 0$ which gives $J'(\tau_n \boldsymbol{u}_n)(\tau_n \boldsymbol{u}_n) = 0$. If $\tau_n = 1$, then $J'(\boldsymbol{u}_n)(\boldsymbol{u}_n) = o(1)$. So we always have

$$J'(\tau_n \boldsymbol{u}_n)(\tau_n \boldsymbol{u}_n) = o(1). \tag{3.18}$$

Now, let us fix $k \geq 1$ so that $\|\boldsymbol{u}_k\|_{\boldsymbol{W}^{p(x)}(\Omega)} > 1$ and define the sequence $\{\overline{\boldsymbol{w}}_n\}$ by the following formula

$$\overline{\boldsymbol{w}}_n = \left(2p^+ \|\boldsymbol{u}_k\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)}\right)^{\frac{1}{p^-}} \boldsymbol{w}_n, \quad n = 1,2,\dots \tag{3.19}$$

Fix k, since $\boldsymbol{w}_n \to 0$ strongly in the spaces $\boldsymbol{L}^{q(x)}(\Omega)$ and $\boldsymbol{L}^{p^+}(\Omega)$ as $n \to \infty$, using (3.2), we deduce that

$$\left| \int_\Omega F(x, \overline{\boldsymbol{w}}_n)\, dx \right| \leq \epsilon \int_\Omega |\overline{\boldsymbol{w}}_n|^{p^+} \, dx + c(\epsilon) \int_\Omega |\overline{\boldsymbol{w}}_n|^{q(x)} \, dx \to 0 \text{ as } n \to \infty. \tag{3.20}$$

Since $\|\boldsymbol{u}_n\|_{\boldsymbol{W}^{p(x)}(\Omega)} \to +\infty$ as $n \to \infty$, we can find a constant $n_k > k$ depending on k such that

$$0 < \frac{\left(2p^+ \|\boldsymbol{u}_k\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)}\right)^{\frac{1}{p^-}}}{\|\boldsymbol{u}_n\|_{\boldsymbol{W}^{p(x)}(\Omega)}} < 1 \text{ for all } n > n_k. \tag{3.21}$$

Hence,

$$J(\tau_n \boldsymbol{u}_n) \geq J\left(\frac{\left(2p^+ \|\boldsymbol{u}_k\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)}\right)^{\frac{1}{p^-}}}{\|\boldsymbol{u}_n\|_{\boldsymbol{W}^{p(x)}(\Omega)}} u_n \right)$$

$$
\begin{aligned}
&= J(\overline{\boldsymbol{w}}_n)\\
&= \int_\Omega \frac{1}{p(x)} |\nabla \times \overline{\boldsymbol{w}}_n|^{p(x)}\,dx - \int_\Omega F(x,\overline{\boldsymbol{w}}_n)\,dx\\
&\geq \frac{1}{p^+}\int_\Omega \left(\|\boldsymbol{u}_k\|^{p(x)}_{\boldsymbol{W}^{p(x)}(\Omega)} . (2p^+)^{\frac{p(x)}{p^-}} . |\nabla \times \boldsymbol{w}_n|^{p(x)}\right) dx - \int_\Omega F(x,\overline{\boldsymbol{w}}_n)\,dx\\
&\geq 2\|\boldsymbol{u}_k\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)} - \int_\Omega F(x,\overline{\boldsymbol{w}}_n)\,dx\\
&\geq \|\boldsymbol{u}_k\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)}
\end{aligned}
$$

for any $n > n_k > k$ large enough.

On the other hand, by (H_5), relation (3.18) and the fact that $\theta \geq 1$, for all $n > n_k > k$ large enough, we have

$$
\begin{aligned}
J(\tau_n \boldsymbol{u}_n) &= J(\tau_n \boldsymbol{u}_n) - \frac{1}{p^+} J'(\tau_n \boldsymbol{u}_n)(\tau_n \boldsymbol{u}_n) + o_n(1)\\
&= \int_\Omega \frac{1}{p(x)} |\nabla \times (\tau_n \boldsymbol{u}_n)|^{p(x)}\,dx - \int_\Omega F(x,\tau_n \boldsymbol{u}_n)\,dx\\
&\qquad - \frac{1}{p^+}\int_\Omega |\nabla \times (\tau_n \boldsymbol{u}_n)|^{p(x)}\,dx + \frac{1}{p^+}\int_\Omega \boldsymbol{f}(x,\tau_n \boldsymbol{u}_n)\cdot(\tau_n \boldsymbol{u}_n)\,dx + o_n(1)\\
&= \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right) |\nabla \times (\tau_n \boldsymbol{u}_n)|^{p(x)}\,dx + \frac{1}{p^+}\int_\Omega \mathscr{F}(x,\tau_n \boldsymbol{u}_n)\,dx\\
&\leq \theta \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{p^+}\right) |\nabla \times \boldsymbol{u}_n|^{p(x)}\,dx + \frac{\theta}{p^+}\int_\Omega \mathscr{F}(x,\boldsymbol{u}_n)\,dx + o_n(1)\\
&= \theta\left[\int_\Omega \frac{1}{p(x)} |\nabla \times \boldsymbol{u}_n|^{p(x)}\,dx - \int_\Omega F(x,\boldsymbol{u}_n)\,dx\right]\\
&\qquad - \frac{\theta}{p^+}\left(\int_\Omega |\nabla \times \boldsymbol{u}_n|^{p(x)}\,dx - \int_\Omega \boldsymbol{f}(x,\boldsymbol{u}_n)\cdot \boldsymbol{u}_n\,dx\right) + o_n(1)\\
&= \theta J(\boldsymbol{u}_n) - \frac{\theta}{p^+} J'(\boldsymbol{u}_n)(\boldsymbol{u}_n) + o_n(1)\\
&\to \theta c \text{ as } n \to \infty,
\end{aligned}
$$

which is a contradiction since $J(\tau_n \boldsymbol{u}_n) \to +\infty$ as $n \to \infty$. This ensures that the sequence $\{\boldsymbol{u}_n\}$ is bounded in $\boldsymbol{W}^{p(x)}(\Omega)$.

Now, since the Banach space $\boldsymbol{W}^{p(x)}(\Omega)$ is reflexive, there exists $\boldsymbol{u} \in \boldsymbol{W}^{p(x)}(\Omega)$ such that passing to a subsequence, still denoted by $\{\boldsymbol{u}_n\}$, it converges weakly to $\boldsymbol{u}$ in $\boldsymbol{W}^{p(x)}(\Omega)$ and converges strongly to u in the spaces $\boldsymbol{L}^{q(x)}(\Omega)$. Using the condition (H_2) and the Hölder inequality, we deduce that

$$
\begin{aligned}
\left|\int_\Omega \boldsymbol{f}(x,\boldsymbol{u}_n)\cdot(\boldsymbol{u}_n - \boldsymbol{u})\,dx\right| &\leq \int_\Omega |\boldsymbol{f}(x,\boldsymbol{u}_n)||\boldsymbol{u}_n - \boldsymbol{u}|\,dx\\
&\leq C\int_\Omega (1 + |\boldsymbol{u}_n|^{q(x)-1})|\boldsymbol{u}_n - \boldsymbol{u}|\,dx
\end{aligned}
$$

$$\le c_6\left(|1|_{L^{q'(x)}} + \left||u_n|^{q(x)-1}\right|_{L^{q'(x)}(\Omega)}\right)|u_n - u|_{L^{q(x)}(\Omega)}$$
$$\to 0 \text{ as } n \to \infty,$$

which yields

$$\lim_{n\to\infty}\int_\Omega f(x,u_n)\cdot(u_n - u)\,dx = 0. \tag{3.22}$$

From (3.22) and the fact that

$$\lim_{n\to\infty} J'(u_n)(u_n - u) = 0$$

we get

$$\lim_{n\to\infty}\int_\Omega |\nabla\times u_n|^{p(x)-2}\nabla\times u_n\cdot(\nabla\times u_n - \nabla\times u)\,dx = 0. \tag{3.23}$$

Now, using similar arguments as in the proof of [15, Lemma 3.3] we can show that the sequence $\{u_n\}$ converges strongly to u in $W^{p(x)}(\Omega)$ and the functional J satisfies the (C_c) condition for any $c > 0$. The proof of Lemma 3.9 is complete. □

Proof of Theorem 3.5. By Lemmas 3.7-3.9, the functional J satisfies all the assumptions of the mountain pass theorem [1]. Then we deduce a function $u \in W^{p(x)}(\Omega)$ as a non-trivial critical point of J with $J(u) = c > 0$ and thus a non-trivial weak solution of problem (1.1). □

Because $W^{p(x)}(\Omega)$ is a reflexive and separable Banach space, there exist $\{e_j\} \subset W^{p(x)}(\Omega)$ and $\{e_j^*\} \subset (W^{p(x)}(\Omega))^*$ such that

$$W^{p(x)}(\Omega) = \overline{\mathrm{span}\{e_j : j = 1,2,\ldots,\}}, \quad (W^{p(x)}(\Omega))^* = \overline{\mathrm{span}\{e_j^* : j = 1,2,\ldots,\}},$$

and

$$\left\langle e_i, e_j^*\right\rangle = \begin{cases} 1, & \text{if } i = j,\\ 0, & \text{if } i \neq j.\end{cases}$$

For convenience, we write $X_j = \mathrm{span}\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$ and $Z_k = \oplus_{j=k}^\infty X_j$. We first have the following lemma which will be used in the proof of our multiplicity result.

Lemma 3.10. *If $\alpha \in C_+(\overline{\Omega})$, $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$ denote*

$$\beta_k = \sup\left\{|u|_{L^{\alpha(x)}(\Omega)} : \|u\|_{W^{p(x)}(\Omega)} = 1,\ u \in Z_k\right\},$$

then $\lim_{k\to\infty}\beta_k = 0$.

Proposition 3.11 (see [14, Fountain theorem])**.** *Assume that $(X, \|.\|)$ is a separable Banach space, $J \in C^1(X,\mathbb{R})$ is an even functional satisfying the (PS) condition. Moreover, for each $k = 1,2,\ldots$, there exist $\rho_k > r_k > 0$ such that*

(A_1) $\inf_{\{u\in Z_k:\ \|u\|=r_k\}} J(u) \to +\infty$ *as* $k \to \infty$*;*

(A_2) $\max_{\{u\in Y_k:\ \|u\|=\rho_k\}} J(u) \le 0$.

Then J has a sequence of critical values which tends to $+\infty$.

Proof of Theorem 3.6. According to (H_6) and Lemma 3.9, J is an even functional and satisfies the (Ce) condition. We will prove Theorem 3.6 by using the fountain theorem, see Proposition 2.1. Indeed, we will show that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A_1) and (A_2) hold. Thus, the assertion of conclusion can be obtained.

(A_1): Using (3.1), for any $u \in Z_k$,

$$\begin{aligned}
J(\boldsymbol{u}) &= \int_\Omega \frac{1}{p(x)}|\nabla\times \boldsymbol{u}|^{p(x)}\,dx - \int_\Omega F(x,\boldsymbol{u})\,dx\\
&\ge \frac{1}{p^+}\|\boldsymbol{u}\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)} - c_7\int_\Omega (|\boldsymbol{u}|+|\boldsymbol{u}|^{q(x)})\,dx\\
&\ge \frac{1}{p^+}\|\boldsymbol{u}\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8|\boldsymbol{u}|^{q(\xi)}_{q(x)} - c_8\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)},\ \text{ where } \xi\in\Omega\\
&\ge \begin{cases} \frac{1}{p^+}\|\boldsymbol{u}\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8 - c_8\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} & \text{if } |\boldsymbol{u}|_{\boldsymbol{L}^{q(x)}(\Omega)}\le 1,\\ \frac{1}{p^+}\|\boldsymbol{u}\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8\beta_k^{q^+}\|u\|^{q^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} & \text{if } |\boldsymbol{u}|_{\boldsymbol{L}^{q(x)}(\Omega)}\le 1\end{cases}\\
&\ge \frac{1}{p^+}\|\boldsymbol{u}\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8\beta_k^{q^+}\|u\|^{q^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8,
\end{aligned}$$

where

$$\beta_k = \sup\left\{|\boldsymbol{u}|_{\boldsymbol{L}^{\alpha(x)}(\Omega)} :\ \|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} = 1,\ u\in Z_k\right\}. \tag{3.24}$$

Now, for any $\boldsymbol{u}\in Z_k$, $\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} = r_k = \left(c_8 q^+\beta_k^{q^+}\right)^{\frac{1}{p^- - q^+}}$, we have

$$\begin{aligned}
J(\boldsymbol{u}) &\ge \frac{1}{p^+}\|\boldsymbol{u}\|^{p^-}_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8\beta_k^{\alpha^+}\|\boldsymbol{u}\|^{q^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8\|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)} - c_8\\
&= \frac{1}{p^+}\left(c_8 q^+\beta_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}} - c_8\beta_k^{q^+}\left(c_8 q^+\beta_k^{q^+}\right)^{\frac{q^+}{p^- - q^+}}\\
&\qquad - c_8\left(c_8 q^+\beta_k^{q^+}\right)^{\frac{1}{p^- - q^+}} - c_8\\
&= \left(\frac{1}{p^+} - \frac{1}{q^+}\right)\left(c_8 q^+\beta_k^{q^+}\right)^{\frac{p^-}{p^- - q^+}} - c_8\left(c_8 q^+\beta_k^{q^+}\right)^{\frac{1}{p^- - q^+}},
\end{aligned}$$

which tends to $+\infty$ as $k\to+\infty$, because $p^+ < q^- \le q(x) < p^*(x)$ and $\beta_k\to 0$ as $k\to\infty$, see Lemma 3.10.

(A_2): By (3.1), for any $\psi\in Y_k$ with $\|\psi\|_{\boldsymbol{W}^{p(x)}(\Omega)} = 1$ and $t>1$, we have

$$\begin{aligned}
J(\tau\psi) &= \int_\Omega \frac{1}{p(x)}|\nabla\times(\tau\psi)|^{p(x)}\,dx - \int_\Omega F(x,\tau\psi)\,dx\\
&\le \frac{1}{p^-}\|\tau\psi\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - M\int_\Omega |\tau\psi|^{p^+}\,dx + c_M|\Omega|
\end{aligned}$$

$$= \tau^{p^+}\left(\frac{1}{p^-}\|\psi\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - M\int_\Omega |\psi|^{p^+}\,dx\right) + c_M|\Omega|.$$

Let us choose $M > 0$ large enough such that

$$\frac{1}{p^-}\|\psi\|^{p^+}_{\boldsymbol{W}^{p(x)}(\Omega)} - M\int_\Omega |\psi|^{p^+}\,dx < 0,$$

we then deduce that $\lim_{\tau\to+\infty} J(\tau\psi) = -\infty$. Hence, there exists $\underline{\tau} > r_k > 1$ large enough such that $J(\underline{\tau}\psi) \le 0$ and thus, if we set $\rho_k = \underline{\tau}$ we conclude that

$$b_k := \max_{\left\{\boldsymbol{u}\in Y_k:\ \|\boldsymbol{u}\|_{\boldsymbol{W}^{p(x)}(\Omega)}=\rho_k\right\}} J(\boldsymbol{u}) \le 0.$$

Conclusion of Theorem 3.6 is reached by the fountain theorem. □

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