TAMKANG JOURNAL OF MATHEMATICS
Volume 33, Number 2, Summer 2002

NEW BOUNDS FOR SIMPSON'S INEQUALITY

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Abstract. Some new bounds for Simpson's inequality are derived. These bounds are better than some recently obtained bounds.

1. Introduction

In recent years many authors have written about Simpson's inequality, for example see [1]-[9] and [13]. Simpson's inequality gives an error bound for the well-known Simpson's quadrature rule:

$$E(f)=\int_a^b f(t)dt-\frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]. \tag{1.1}$$

There are few known ways to express the term $E(f)$. Different variants of $E(f)$ give different estimations of the error. In this paper we give a new approach to the subject. This new approach is based on a generalization of pre-Grüss inequality which is obtained in [14]. It gives better results.

In [5] we can find the next result.

If we assume that $f^{(n-1)}$ is an absolutely continuous function on $[a,b]$ such that $f^{(n)}\in L_2(a,b)$ $(n=1,2,3)$ then we have the Simpson's inequality (for $n\in\{1,2,3\}$),

$$|E(f)|\le C_n(b-a)\sigma(f^{(n)};a,b), \tag{1.2}$$

where $E(f)$ is given by (1.1),

$$C_1=\frac{1}{6},\quad C_2=\frac{1}{12\sqrt{30}},\quad C_3=\frac{1}{48\sqrt{105}}, \tag{1.3}$$

$$\sigma(f^{(n)};a,b)=\left[\frac{1}{b-a}\left\|f^{(n)}\right\|_2^2-\left(\left[f^{(n-1)};a,b\right]\right)^2\right]^{1/2}, \tag{1.4}$$

and

$$\left[f^{(n)};a,b\right]=\frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}, \tag{1.5}$$

Received May 02, 2001.
2000 *Mathematics Subject Classification*. 26D10, 41A55.
Key words and phrases. Pre-Grüss inequality, generalization, Simpson's inequality.

$$\left\| f^{(n)} \right\|_2^2 = \int_a^b f^{(n)}(t)^2 dt. \tag{1.6}$$

The above result is an improvement of a result obtained in [13].

In this paper we give a new expression for the term $E(f)$ and use the generalization of pre-Grüss inequality to derive some better estimations of the error for Simpson's quadrature rule. In fact, we further improve (1.2).

2. Main Results

We define the Chebyshev functional:

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{(b-a)^2}\int_a^b f(t)dt \int_a^b g(t)dt \tag{2.1}$$

and the functional

$$\begin{aligned} S_\Psi(f,g) = &\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt\int_a^b g(t)dt \\ &- \int_a^b f(t)\Psi_0(t)dt \int_a^b g(t)\Psi_0(t)dt, \end{aligned} \tag{2.2}$$

where $f, g, \Psi \in L_2(a,b)$, $\Psi_0(t) = \Psi(t)/\left\|\Psi\right\|_2$. We suppose that

$$\int_a^b \Psi(t)dt = 0. \tag{2.3}$$

Further, in [11] we can find the pre-Grüss inequality

$$T(f,g)^2 \le T(f,f)T(g,g) \tag{2.4}$$

and the Grüss inequality

$$|T(f,g)| \le \frac{(\Delta-\delta)(\Gamma-\gamma)}{4}, \tag{2.5}$$

where $\delta \le f(t) \le \Delta$ and $\gamma \le g(t) \le \Gamma$, $t \in [a,b]$. Specially, we have

$$|T(f,f)| \le \frac{(\Delta-\delta)^2}{4}. \tag{2.6}$$

Theorem 1. *If $g, h, \Psi \in L_2(a,b)$ and (2.3) holds then we have*

$$|S_\Psi(g,h)| \le S_\Psi(g,g)^{1/2} S_\Psi(h,h)^{1/2}. \tag{2.7}$$

Proof. We can write

$$S_\Psi(g,h) = \int_a^b g(t)\left[h(t) - \frac{1}{b-a}\int_a^b h(s)ds - \int_a^b h(s)\Psi_0(s)ds\ \Psi_0(t)\right]dt. \tag{2.8}$$

We also have

$$\int_a^b \left[h(t) - \frac{1}{b-a}\int_a^b h(s)ds - \int_a^b h(s)\Psi_0(s)ds\ \Psi_0(t)\right]dt = 0 \tag{2.9}$$

and

$$\int_a^b \Psi_0(t)\left[h(t) - \frac{1}{b-a}\int_a^b h(s)ds - \int_a^b h(s)\Psi_0(s)ds\ \Psi_0(t)\right]dt = 0, \tag{2.10}$$

since (2.3) holds.

It follows from (2.8)-(2.10) that

$$\begin{aligned} S_\Psi(g,h) = \int_a^b &\left[g(t) - \frac{1}{b-a}\int_a^b g(s)ds - \int_a^b g(s)\Psi_0(s)ds\ \Psi_0(t)\right] \\ &\times \left[h(t) - \frac{1}{b-a}\int_a^b h(s)ds - \int_a^b h(s)\Psi_0(s)ds\ \Psi_0(t)\right]dt. \end{aligned} \tag{2.11}$$

Then, using (2.11),

$$S_\Psi(g,g) = \int_a^b \left[g(t) - \frac{1}{b-a}\int_a^b g(s)ds - \int_a^b g(s)\Psi_0(s)ds\ \Psi_0(t)\right]^2 dt \geq 0. \tag{2.12}$$

In a similar way we get $S_\Psi(h,h) \geq 0$. Now, using the Cauchy inequality and (2.11) we get

$$\begin{aligned} |S_\Psi(g,h)| &\leq \left\{\int_a^b \left[g(t) - \frac{1}{b-a}\int_a^b g(s)ds - \int_a^b g(s)\Psi_0(s)ds\ \Psi_0(t)\right]^2 dt\right\}^{1/2} \\ &\quad\times \left\{\int_a^b \left[h(t) - \frac{1}{b-a}\int_a^b h(s)ds - \int_a^b h(s)\Psi_0(s)ds\ \Psi_0(t)\right]^2 dt\right\}^{1/2} \\ &= S_\Psi(g,g)^{1/2} S_\Psi(h,h)^{1/2}. \end{aligned}$$

This completes the proof.

Remark 1. A more general result can be found in [14]. The mentioned result can be applied in this paper as it is described in [14]. In fact, we here consider only the case $n = 1$ of the mentioned result.

We also have

$$S_\Psi(f,g)=(b-a)T(f,g)-\int_a^b f(t)\Psi_0(t)dt\int_a^b g(t)\Psi_0(t)dt. \tag{2.13}$$

Hence, $S_\Psi(f,g)$ is a generalization of $T(f,g)$. From (2.13) we easily find that

$$S_\Psi(f,f)\le(b-a)T(f,f). \tag{2.14}$$

Theorem 2. *Let $I\subset R$ be a closed interval and $a,b\in Int\ I$, $a<b$.If $f:I\to R$ is an absolutely continuous function with $f'\in L_2(a,b)$ then we have*

$$\left|\frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]-\int_a^b f(t)dt\right|\le\frac{(b-a)^{3/2}}{6}K_1, \tag{2.15}$$

where

$$K_1=\left[\sigma^2(f';a,b)(b-a)-\left(\int_a^b f'(t)\Psi_0(t)dt\right)^2\right]^{1/2} \tag{2.16}$$

and $\Psi(t)=t-\frac{a+b}{2}$ while σ is defined by (1.4).

Proof. We define

$$p_1(t)=\begin{cases} t-a, & t\in\left[a,\frac{a+b}{2}\right]\\ t-b, & t\in\left(\frac{a+b}{2},b\right].\end{cases} \tag{2.17}$$

It is not difficult to verify that

$$\int_a^b p_1(t)dt=0 \tag{2.18}$$

and

$$\int_a^b \Psi(t)dt=0. \tag{2.19}$$

We also have

$$\|p_1\|_2^2=\int_a^b p_1(t)^2dt=\frac{(b-a)^3}{12}, \tag{2.20}$$

$$\|\Psi\|_2^2=\int_a^b \Psi(t)^2dt=\frac{(b-a)^3}{12} \tag{2.21}$$

such that

$$\Psi_0(t)=\frac{\Psi(t)}{\|\Psi\|_2}=\frac{\sqrt{12}}{(b-a)^{3/2}}(t-\frac{a+b}{2}) \tag{2.22}$$

and

$$\int_a^b p_1(x,t)\Psi(t)dt=-\frac{(b-a)^3}{24}. \tag{2.23}$$

Integrating by parts, we have

$$\int_a^b p_1(t)f'(t)dt = \int_a^{\frac{a+b}{2}} (t-a)f'(t)dt + \int_{\frac{a+b}{2}}^b (t-b)f'(t)dt$$
$$= f(\frac{a+b}{2})(b-a) - \int_a^b f(t)dt \tag{2.24}$$

and

$$\int_a^b \Psi(t)f'(t)dt = \int_a^b (t - \frac{a+b}{2})f'(t)dt$$
$$= \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)dt. \tag{2.25}$$

From (2.18) and (2.21)-(2.25) we have

$$\int_a^b p_1(t)f'(t)dt - \frac{1}{b-a}\int_a^b p_1(t)dt \int_a^b f'(t)dt - -\int_a^b p_1(t)\Psi_0(t)dt \int_a^b f'(t)\Psi_0(t)dt$$
$$= f(\frac{a+b}{2})(b-a) - \int_a^b f(t)dt + \frac{1}{2}\left[\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)dt\right]$$
$$= \left[f(\frac{a+b}{2}) + \frac{f(a)+f(b)}{4}\right](b-a) - \frac{3}{2}\int_a^b f(t)dt. \tag{2.26}$$

On the other hand, we have

$$\int_a^b p_1(t)f'(t)dt - \frac{1}{b-a}\int_a^b p_1(t)dt \int_a^b f'(t)dt - \int_a^b p_1(t)\Psi_0(t)dt \int_a^b f'(t)\Psi_0(t)dt = S_\Psi(p_1, f'). \tag{2.27}$$

From (2.7), (2.26) and (2.27) it follows that

$$\left|\frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b)\right] - \int_a^b f(t)dt\right| \le \frac{2}{3}S_\Psi(f',f')^{1/2}S_\Psi(p_1,p_1)^{1/2}. \tag{2.28}$$

Using (2.18), (2.20) and (2.23) we get

$$S_\Psi(p_1,p_1) = \|p_1\|_2^2 - \frac{1}{b-a}\left(\int_a^b p_1(t)dt\right)^2 - \left(\int_a^b p_1(t)\Psi_0(t)dt\right)^2 = \frac{(b-a)^3}{16}. \tag{2.29}$$

We also have

$$S_\Psi(f',f') = \|f'\|_2^2 - \frac{1}{b-a}\left(\int_a^b f'(t)dt\right)^2 - \left(\int_a^b f'(t)\Psi_0(t)dt\right)^2$$
$$= \sigma^2(f';a,b)(b-a) - \left(\int_a^b f'(t)\Psi_0(t)dt\right)^2 = K_1^2. \tag{2.30}$$

From (2.28)-(2.30) we easily get (2.15).

Remark 2. It is obvious that (2.15) is better than the corresponding inequality in (1.2).

Theorem 3. *Let $I \subset R$ be a closed interval and $a, b \in Int\ I$, $a < b$. If $f : I \to R$ is such that f' is an absolutely continuous function with $f'' \in L_2(a, b)$ then we have*

$$\left| \frac{b-a}{6} \left[f(a) + 4f(\frac{a+b}{2}) + f(b) \right] - \int_a^b f(t)dt \right| \le \frac{(b-a)^{5/2}}{12\sqrt{30}} K_2, \tag{2.31}$$

where

$$K_2 = \left[\sigma^2(f^{''}; a, b)(b-a) - \left(\int_a^b f''(t)\Psi_0(t)dt \right)^2 \right]^{1/2}, \tag{2.32}$$

$$\Psi(t) = \begin{cases} 1, & t \in \left[a, \frac{a+b}{2}\right] \\ -1, & t \in \left(\frac{a+b}{2}, b\right] \end{cases} \tag{2.33}$$

and $\Psi_0(t) = \Psi(t)/\left\|\Psi\right\|_2$.

Proof. We define

$$p_2(t) = \begin{cases} \frac{1}{2}(t-a)(t - \frac{2a+b}{3}), & t \in \left[a, \frac{a+b}{2}\right] \\ \frac{1}{2}(t-b)(t - \frac{a+2b}{3}), & t \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{2.34}$$

It is not difficult to verify that

$$\int_a^b p_2(t)dt = 0, \tag{2.35}$$

$$\int_a^b \Psi(t)dt = 0, \tag{2.36}$$

$$\int_a^b p_2(t)\Psi(t)dt = 0. \tag{2.37}$$

Integrating by parts, we have

$$\begin{aligned} \int_a^b p_2(t)f''(t)dt &= \frac{1}{2}\int_a^{\frac{a+b}{2}} (t-a)(t - \frac{2a+b}{3})f''(t)dt + \frac{1}{2}\int_{\frac{a+b}{2}}^b (t-b)(t - \frac{a+2b}{3})f^{''}(t)dt \\ &= \frac{(b-a)^2}{24}f'(\frac{a+b}{2}) - \frac{(b-a)^2}{24}f'(\frac{a+b}{2}) - \int_a^{\frac{a+b}{2}} (t - \frac{5a+b}{6})f'(t)dt \\ &\quad - \int_{\frac{a+b}{2}}^b (t - \frac{a+5b}{6})f'(t)dt \\ &= -\frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b) \right] + \int_a^b f(t)dt. \end{aligned} \tag{2.38}$$

From (2.35), (2.37) and (2.38) it follows that

$$\int_a^b p_2(t)f''(t)dt - \frac{1}{b-a}\int_a^b p_2(t)dt\int_a^b f''(t)dt - \int_a^b p_2(t)\Psi_0(t)dt\int_a^b f''(t)\Psi_0(t)dt$$
$$= -\frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]+\int_a^b f(t)dt. \tag{2.39}$$

On the other hand, we have

$$\int_a^b p_2(t)f''(t)dt - \frac{1}{b-a}\int_a^b p_2(t)dt\int_a^b f''(t)dt - \int_a^b p_2(t)\Psi_0(t)dt\int_a^b f''(t)\Psi_0(t)dt$$
$$= S_\Psi(p_2, f'') \tag{2.40}$$

Using (2.7), (2.39) and (2.40) we have

$$\left|\frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]-\int_a^b f(t)dt\right| \le S_\Psi(p_2,p_2)^{1/2}S_\Psi(f'',f'')^{1/2}. \tag{2.41}$$

We also have

$$S_\Psi(p_2,p_2) = \|p_2\|_2^2 - \frac{1}{b-a}\left(\int_a^b p_2(t)dt\right)^2 - \left(\int_a^b p_2(t)\Psi_0(t)dt\right)^2$$
$$= \frac{(b-a)^5}{4320} \tag{2.42}$$

and

$$S_\Psi(f'',f'') = \sigma^2(f'';a,b)(b-a) - \left(\int_a^b f''(t)\Psi_0(t)dt\right)^2 = K_2^2. \tag{2.43}$$

From (2.41)-(2.43) we easily get (2.31).

Remark 3. It is obvious that (2.31) is better than the corresponding estimation in (1.2).

Corollary 1. *Let the assumptions of Theorem* 3 *be satisfied. If there exist constants* $\gamma, \Gamma \in R$ *such that* $\gamma \le f''(t) \le \Gamma$, $t \in [a,b]$ *then we have*

$$\left|\frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]-\int_a^b f(t)dt\right| \le \frac{(b-a)^3}{12\sqrt{30}}K, \tag{2.44}$$

where

$$K = \left[\frac{(\Gamma-\gamma)^2}{4} - \left(\frac{f'(a)-2f'(\frac{a+b}{2})+f'(b)}{b-a}\right)^2\right]^{1/2}. \tag{2.45}$$

Proof. The proof immediately follows from (2.31)-(2.33) and (2.1), (2.6).

Theorem 4. *Let $I \subset R$ be a closed interval and $a, b \in Int\ I$, $a < b$. If $f : I \to R$ is such that f'' is an absolutely continuous function with $f''' \in L_2(a, b)$ then we have*

$$\left| \frac{b-a}{6} \left[f(a) + 4f(\frac{a+b}{2}) + f(b) \right] - \int_a^b f(t)dt \right| \le \frac{(b-a)^{7/2}}{48\sqrt{105}} K_3, \tag{2.46}$$

where

$$K_3 = \left[\sigma^2(f^{'''}; a, b)(b-a) - \left(\int_a^b f^{'''}(t)\Psi_0(t)dt \right)^2 \right]^{1/2} \tag{2.47}$$

and

$$\Psi(t) = \begin{cases} t - \frac{7a+3b}{10}, & t \in \left[a, \frac{a+b}{2}\right] \\ t - \frac{3a+7b}{10}, & t \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{2.48}$$

Proof. We define

$$p_3(t) = \begin{cases} \frac{1}{6}(t-a)^2(t - \frac{a+b}{2}), & t \in \left[a, \frac{a+b}{2}\right] \\ \frac{1}{6}(t-b)^2(t - \frac{a+b}{2}), & t \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{2.49}$$

It is not difficult to verify that

$$\int_a^b p_3(t)dt = 0, \tag{2.50}$$

$$\int_a^b \Psi(t)dt = 0, \tag{2.51}$$

$$\int_a^b p_3(t)\Psi(t)dt = 0. \tag{2.52}$$

Integrating by parts, we have

$$\begin{aligned} \int_a^b p_3(t)f^{'''}(t)dt &= \frac{1}{6}\int_a^{\frac{a+b}{2}} (t-a)^2(t-\frac{a+b}{2})f^{'''}(t)dt + \frac{1}{6}\int_{\frac{a+b}{2}}^b (t-b)^2(t-\frac{a+b}{2})f^{'''}(t)dt \\ &= -\int_a^b p_2(t)f''(t)dt \\ &= \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b) \right] - \int_a^b f(t)dt. \end{aligned} \tag{2.53}$$

From (2.50), (2.52) and (2.53) it follows that

$$\begin{aligned} &\int_a^b p_3(t)f^{'''}(t)dt - \frac{1}{b-a}\int_a^b p_3(t)dt \int_a^b f^{'''}(t)dt - \int_a^b p_3(t)\Psi_0(t)dt \int_a^b f^{'''}(t)\Psi_0(t)dt \\ &= \frac{b-a}{6}\left[f(a) + 4f(\frac{a+b}{2}) + f(b) \right] - \int_a^b f(t)dt. \end{aligned} \tag{2.54}$$

On the other hand, we have

$$\int_a^b p_3(t)f^{'''}(t)dt - \frac{1}{b-a}\int_a^b p_3(t)dt\int_a^b f^{'''}(t)dt - \int_a^b p_3(t)\Psi_0(t)dt\int_a^b f'''(t)\Psi_0(t)dt$$
$$= S_\Psi(p_3, f^{'''}). \tag{2.55}$$

Using (2.54), (2.55) and (2.7) we get

$$\left|\frac{b-a}{6}\left[f(a)+4f(\frac{a+b}{2})+f(b)\right]-\int_a^b f(t)dt\right| \leq S_\Psi(p_3,p_3)^{1/2}S_\Psi(f^{'''},f^{'''})^{1/2}. \tag{2.56}$$

We also have

$$\begin{aligned} S_\Psi(p_3,p_3) &= \|p_3\|_2^2 - \frac{1}{b-a}\left(\int_a^b p_3(t)dt\right)^2 - \left(\int_a^b p_3(t)\Psi_0(t)dt\right)^2 \\ &= \frac{(b-a)^7}{48\cdot 48\cdot 105} \end{aligned} \tag{2.57}$$

and

$$S_\Psi(f^{'''},f^{'''}) = \sigma^2(f''';a,b)(b-a) - \left(\int_a^b f^{'''}(t)\Psi_0(t)dt\right)^2 = K_3^2. \tag{2.58}$$

From (2.56)-(2.58) we easily get (2.46).

Remark 4. It is clear that (2.46) is better than the corresponding estimation in (1.2).

Remark 5. Further improvements of the obtained results are possible. If we really need better error bounds then we can apply the procedure described in this section and a procedure described in [14]. However, some complications may occur - see [14].

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