TAMKANG JOURNAL OF MATHEMATICS
Volume 51, Number 1, 21-30, March 2020
doi:10.5556/j.tkjm.51.2020.2939

RATE OF CONVERGENCE OF HERMITE-FEJÉR POLYNOMIALS FOR FUNCTIONS WITH DERIVATIVES OF BOUNDED VARIATION

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Abstract. In this paper, the behavior of the Hermite-Fejér interpolation for functions with derivatives of bounded variation on $[-1,1]$ is studied by taking the interpolation over the zeros of Chebyshev polynomials of the second kind. An estimate for the rate of convergence using the zeros of the Chebyshev polynomials of the second kind is given.

1. Introduction

The Hermite-Fejér interpolation polynomials for functions of bounded variation are applicable in a variety of fields and research areas such as computer aided geometric design, computer vision, graphics, and image processing.

In 1992, Bojanic, R. and Cheng, F. H. [7] estimated the rate of convergence of the Hermite-Fejér polynomials for functions with derivatives of bounded variation using the zeros of Chebyshev polynomials of the first kind.

In this paper, the rate of convergence of the Hermite-Fejér polynomials to functions with derivatives of bounded variation using the zeros of the Chebyshev polynomials of the second kind, $U_n(x)$, as nodes of interpolation is estimated.

2. Preliminaries

The Chebyshev polynomials of the second kind, $U_n(x)$, are defined by

$$U_n(x) = \frac{\sin((n+1)\theta)}{\sin\theta}, \qquad x = \cos\theta, \quad x \in [-1,1], \quad n \geq 0.$$

The roots of $U_n(x)$ are given by $x_{kn} = \cos\theta_{kn}$, where

$$\theta_{kn} = \frac{k\pi}{(n+1)}, \quad k = 1,2,\ldots,n.$$

2010 *Mathematics Subject Classification*. .
Key words and phrases. Hermite-Fejér interpolation, Chebyshev polynomials of second kind, rate of convergence.
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Let f be a function defined on $[-1,1]$. The Hermite-Fejér interpolation polynomial, $H_n(f,x)$, of f based on the zeros x_{kn} of the Chebyshev polynomial, $U_n(x)$, is defined by

$$H_n(f,x)=\sum_{k=1}^{n} f(x_{kn})(1-x_{kn}^2)[1+2x_{kn}^2-3xx_{kn}]\frac{U_n^2(x)}{(n+1)^2(x-x_{kn})^2}. \tag{1}$$

Define a function $f_Q(x)$ as follows

$$f_Q(x)=f(-1)+\int_{-1}^{x} Q(t)\,dt, \quad x\in[-1,1], \tag{2}$$

where $Q(t)$ is a function of bounded variation on $[-1,1]$, abbreviated $BV[-1,1]$.

3. Results

If we subtract $f(x)$ from the Hermite-Fejér interpolation polynomial, $H_n(f,x)$, of f based on the zeros of the Chebyshev polynomial of the second kind, $U_n(x)$, then for any $x\in(-1,1)$ such that $x\neq x_{kn}$, $k=1,2,\dots,n$, we have from (1) and (2):

$$\begin{aligned} H_n(f_Q,x)-f_Q(x) &= \sum_{k=1}^{n}\int_{x}^{x_{kn}} Q(t)\,dt\,H_{kn}(x) \\ &= -\sum_{x_{kn}<x}\Big[\int_{x_{kn}}^{x} Q(t)\,dt\Big]H_{kn}(x)+\sum_{x_{kn}>x}\Big[\int_{x}^{x_{kn}} Q(t)\,dt\Big]H_{kn}(x), \end{aligned} \tag{3}$$

where

$$H_{kn}(x) = (1-x_{kn}^2)^2\Big[1-\frac{3x_{kn}(x-x_{kn})}{(1-x_{kn}^2)}\Big]\frac{U_n^2(x)}{(n+1)^2(x-x_{kn})^2}.$$

Define $Q_x(t)$ in the following way:

$$Q_x(t)=\begin{cases} Q(t)-Q(x-)\text{ , for } t<x \\ 0\text{ , for } t=x \\ Q(t)-Q(x+)\text{ , for } t>x \end{cases}$$

then (3) can be expressed as

$$\begin{aligned} H_n(f_Q,x)-f_Q(x) &= -\sum_{x_{kn}<x}\Big[\int_{x_{kn}}^{x} Q_x(t)\,dt\Big]H_{kn}(x)+\sum_{x_{kn}>x}\Big[\int_{x}^{x_{kn}} Q_x(t)\,dt\Big]H_{kn}(x) \\ &\quad -Q(x-)\sum_{x_{kn}<x}(x-x_{kn})H_{kn}(x)+Q(x+)\sum_{x_{kn}>x}(x_{kn}-x)H_{kn}(x). \end{aligned} \tag{4}$$

Since

$$\begin{aligned} &Q(x+)\sum_{x_{kn}>x}(x_{kn}-x)H_{kn}(x)-Q(x-)\sum_{x_{kn}<x}(x-x_{kn})H_{kn}(x) \\ &\quad=\frac{Q(x+)-Q(x-)}{2}\sum_{k=1}^{n}|x_{kn}-x|H_{kn}(x)+\frac{Q(x+)+Q(x-)}{2}\sum_{k=1}^{n}(x_{kn}-x)H_{kn}(x) \end{aligned}$$

and the two summations on the right-hand side of the equation are the Hermite-Fejér interpolation polynomials of $f_x(t) = |t-x|$ and $g_x(t) = t-x$, $-1 \le t \le 1$, then we can further rewrite equation (4) as follows

$$H_n(f,x) - f(x) = \frac{\delta_x}{2} H_n(f_x,x) + \frac{\lambda_x}{2} H_n(g_x,x) + P_n(f,x), \tag{5}$$

where

$$\delta_x = \quad Q(x+) - Q(x-); \quad \lambda_x = \quad Q(x+) + Q(x-) \tag{6}$$

and

$$P_n(f,x) = -\sum_{x_{kn}<x} \Big[\int_{x_{kn}}^{x} Q_x(t)dt \Big] H_{kn}(x) + \sum_{x_{kn}>x} \Big[\int_{x}^{x_{kn}} Q_x(t)dt \Big] H_{kn}(x). \tag{7}$$

Therefore, evaluation of the rate of convergence of $H_n(f,x)$ to $f(x)$ is estimated and the details are given in three parts.

Firstly, the basic formula for the rate of convergence of Hermite-Fejér polynomial $H_n(f,x)$ for $f(x)$ stated with complete proof obtained by Bojanic, R. and Cheng, F. H. [6]. Secondly, the estimation of $H_n(f_x,x)$ and $H_n(g_x,x)$ are given in Theorem 1. Finally, the estimation of $P_n(f,x)$, is given in the result of Theorem 2.

Theorem 1. *Let $x \in (-1,1)$ and $x \ne x_{kn}$ for any $k = 1,\dots,n$, then*

$$\Big| \sum_{k=1}^{n} |x_{kn} - x| H_{kn}(x) - \frac{2U_n^2(x)(\sqrt{1-x^2})^{\frac{1}{2}} \log(n)}{\pi(n+1)} \Big| \le \frac{nC|U_n^2(x)|}{(n+1)^2}, \tag{8}$$

$$\Big| \sum_{k=1}^{n} (x_{kn} - x) H_{kn}(x) \Big| \le \frac{nC|U_n^2(x)|}{(n+1)^2}. \tag{9}$$

Theorem 2. *Let f be a function in $DBV[-1,1]$ and Q in $BV[-1,1]$, and suppose that (2) is satisfied, then for any $x \in (-1,1)$ such that $x \ne x_{kn}$, $k = 1,\dots,n$, we have*

$$\Big| H_n(f,x) - f(x) - \frac{\delta_x U_n^2(x)(\sqrt{1-x^2})^{\frac{1}{2}} \log(n)}{\pi(n+1)} \Big|$$
$$\le C\frac{(|\lambda_x| + |\delta_x|)}{2} \frac{n|U_n(x)|}{(n+1)^2} + \frac{\pi|U_n(x)|}{(n+1)} V_{x-\pi|U_n(x)|/2(n+1)}^{x+\pi|U_n(x)|/2(n+1)}(Q_x) + \frac{12U_n^2(x)}{(n+1)} \sum_{k=1}^{n} V_{x-\pi/k}^{x+\pi/k} \frac{(Q_x)}{k}. \tag{10}$$

If f is continuous at x, i.e., $\delta_x = 0$ and $\lambda_x = 2f'(x)$, then (10) can be simplified as follows:

$$\Big| H_n(f,x) - f(x) \Big| \le \frac{Cnf'(x)|U_n(x)|}{(n+1)^2} + \frac{\pi|U_n|}{(n+1)} V_{x-\pi|U_n|/2(n+1)}^{x+\pi|U_n|/2(n+1)} f'(x) + \frac{12U_n^2}{(n+1)} \sum_{k=1}^{n} V_{x-\pi/k}^{x+\pi/k} \frac{(f')}{k}. \tag{11}$$

Consider the Hermite-Fejér polynomials of the function $f(x) = x^2$ at $x = 0$ for even n. Since $U_n(0) = 1$, then for an even integer n, we have

$$\Big| H_n(f,0) - f(0) \Big| = \frac{\pi}{(n+1)} V_{-\pi/2(n+1)}^{\pi/2(n+1)}(Q_0) + \frac{12}{(n+1)} \sum_{k=1}^{n} \frac{V_{-\pi/k}^{\pi/k}(Q_0)}{k}. \tag{12}$$

Since $Q(t) = 2t$ and $Q_0(t) = Q(t)$, we have

$$\left|H_n(f,0) - f(0)\right| \leq \frac{2\pi^2}{(n+1)^2} + \frac{12}{(n+1)} \sum_{k=1}^{n} \frac{4\pi}{k^2} < \frac{c}{(n+1)} \tag{13}$$

for some $c > 0$. Hence for the function $f(x) = x^2$ when n is an even integer we have

$$\frac{1}{(n+1)} \leq \left|H_n(f,0) - f(0)\right| \leq \frac{c}{(n+1)}$$

for some constant $c \geq 0$. Therefore (10) can not be improved asymptotically.

Proof of Theorem 1. To prove (8), observe that

$$\left|H_n(f_x,x) - \sum_{k=1}^{n} |x_{kn} - x|(1-x_{kn}^2)(1-x^2)\frac{U_n^2(x)}{(n+1)^2(x-x_{kn})^2}\right| \leq \frac{nC|x|U_n^2(x)}{(n+1)^2}. \tag{14}$$

Therefore, it is sufficient to study the asymptotic behavior of the second term on the left-hand side of (14) only. Let $x = \cos(\theta)$, $\quad 0 < \theta < \pi$, $\quad x_{kn} = \cos(\theta_{kn})$, $\quad \theta_{kn} = \frac{k\pi}{(n+1)}$, k=1,2,...,n, and define

$$\sigma_\theta(\alpha) = \begin{cases} 1, & \text{for } 0 < \alpha < \theta \\ -1, & \text{for } \theta < \alpha < \pi. \end{cases}$$

Then

$$\sum_{k=1}^{n} |x_{kn} - x|(1-x_{kn}^2)(1-x^2)\frac{U_n^2}{(n+1)^2(x-x_{kn})^2} = \frac{U_n^2}{(n+1)^2}\sum_{k=1}^{n}\sigma_\theta(\theta_{kn})\frac{\sin^2(\theta)sin^2(\theta_{kn})}{(\cos\theta_{kn} - \cos\theta)}. \tag{15}$$

Since $(\cos\theta_{kn} - \cos\theta) = (\theta - \theta_{kn})\sin(\theta'_{kn})$ for some θ'_{kn} between θ and θ_{kn}, it follows that

$$\begin{aligned}
&\left|\sum_{k=1}^{n}\sigma_\theta(\theta_{kn})\frac{U_n^2\sin^2(\theta)\sin^2(\theta_{kn})}{(\cos\theta_{kn}-\cos\theta)(n+1)^2} - \sum_{k=1}^{n}\sigma_\theta(\theta_{kn})\frac{U_n^2\sin(\theta)\sin^2(\theta_{kn})}{(\theta-\theta_{kn})(n+1)^2}\right| \\
&\leq \frac{U_n^2\sin(\theta)}{(n+1)^2}\sum_{k=1}^{n}\left|sin^2(\theta_{kn})\left(\frac{\sin(\theta)}{(\cos\theta_{kn}-\cos\theta)} - \frac{1}{(\theta-\theta_{kn})}\right)\right| \\
&\leq \frac{U_n^2\sin(\theta)}{(n+1)^2}\sum_{k=1}^{n}\left|\frac{\sin(\theta)}{(\cos\theta_{kn}-\cos\theta)} - \frac{1}{(\theta-\theta_{kn})}\right| \\
&\leq \frac{U_n^2\sin(\theta)}{(n+1)^2}\sum_{k=1}^{n}\left|\frac{sin(\theta)}{\sin\theta'_{kn}(\theta-\theta_{kn})} - \frac{1}{(\theta-\theta_{kn})}\right| \\
&\leq \frac{U_n^2\sin(\theta)}{(n+1)^2}\sum_{k=1}^{n}\left|\frac{1}{(\theta-\theta_{kn})}\left(\frac{\sin\theta-\sin\theta'_{kn}}{\sin\theta'_{kn}}\right)\right| \\
&\leq \frac{U_n^2\sin(\theta)}{(n+1)^2}\sum_{k=1}^{n}\left|\frac{1}{(\theta-\theta_{kn})}\frac{(\theta-\theta'_{kn})}{\sin\theta'_{kn}}\right| \\
&\leq \frac{U_n^2\sin(\theta)}{(n+1)^2}\sum_{k=1}^{n}\left|\frac{1}{\sin(\theta'_{kn})}\right|.
\end{aligned}$$

Furthermore, we have

$$\Big|\sin(\frac{\theta}{2})\Big| \geq \Big|\frac{\theta}{\pi}\Big|.$$

If $|\theta| \leq \pi$, then we have

$$\begin{aligned}
\sin\theta'_{kn} &= \Big|\frac{(\cos\theta - \cos\theta_{kn})}{(\theta-\theta_{kn})}\Big| \\
&= \Big|\frac{2\sin(\frac{\theta+\theta_{kn}}{2})\sin(\frac{\theta-\theta_{kn}}{2})}{(\theta-\theta_{kn})}\Big| \\
&\geq \Big|\frac{2\sin(\frac{\theta+\theta_{kn}}{2})(\frac{\theta-\theta_{kn}}{\pi})}{(\theta-\theta_{kn})}\Big| \\
&= \frac{2}{\pi}\Big|\sin(\frac{\theta+\theta_{kn}}{2})\Big| \geq \frac{2}{\pi}M(\theta),
\end{aligned}$$

where

$$M(\theta) = min\Big(\sin\frac{\theta}{2}, \sin\frac{(\pi-\theta)}{2}\Big).$$

Therefore,

$$\Big|\sum_{k=1}^{n}\sigma_\theta(\theta_{kn})\frac{U_n^2\sin^2(\theta)\sin^2(\theta_{kn})}{(\cos\theta_{kn}-\cos\theta)(n+1)^2} - \sum_{k=1}^{n}\sigma_\theta(\theta_{kn})\frac{U_n^2\sin\theta\sin^2(\theta_{kn})}{(\theta-\theta_{kn})(n+1)^2}\Big| \leq \frac{nCU_n^2}{(n+1)^2}. \tag{16}$$

Let j be an integer such that $\theta_{jn} < \theta < \theta_{j+1,n}$. It is easy to see that

$$j = \Big[\frac{(n+1)\theta}{\pi}\Big].$$

Since $(n+1)(\theta-\theta_{kn}) = \pi\Big(\frac{(n+1)\theta}{\pi} - k\Big)$, we have

$$\begin{aligned}
&\sum_{k=1}^{n}\sigma_\theta(\theta_{kn})\frac{U_n^2\sin(\theta)\sin^2(\theta_{kn})}{(\theta-\theta_{kn})(n+1)^2} \\
&\leq \frac{U_n^2\sin(\theta)}{(n+1)^2}\sum_{k=1}^{n}\sigma_\theta(\theta_{kn})\frac{1}{(\theta-\theta_{kn})} \\
&= \frac{U_n^2\sin(\theta)}{(n+1)}\sum_{k=1}^{n}\sigma_\theta(\theta_{kn})\frac{1}{(\theta-\theta_{kn})(n+1)} \\
&= \frac{U_n^2\sin(\theta)}{(n+1)}\Big[\sum_{k=1}^{j}\frac{1}{\pi[\frac{(n+1)\theta}{\pi}-k]} - \sum_{k=j+1}^{n}\frac{1}{\pi[\frac{(n+1)\theta}{\pi}-k]}\Big] \\
&= \frac{U_n^2\sin(\theta)}{(n+1)\pi}\Big[\sum_{k=0}^{j-1}\frac{1}{[\frac{(n+1)\theta}{\pi}-j+k]} - \sum_{k=0}^{n-j-1}\frac{1}{[\frac{(n+1)\theta}{\pi}-j-1-k]}\Big] \\
&= \frac{U_n^2\sin(\theta)}{(n+1)\pi}\Big[\sum_{k=0}^{j-1}\frac{1}{[\frac{(n+1)\theta}{\pi}-j+k]} + \sum_{k=0}^{n-j-1}\frac{1}{[k+1-\frac{(n+1)\theta}{\pi}+j]}\Big] \\
&= \frac{U_n^2\sin(\theta)}{(n+1)\pi}\Big[\sum_{k=0}^{j-1}\frac{1}{S((n+1)\theta)+k} + \sum_{k=0}^{n-j-1}\frac{1}{1-S((n+1)\theta)+k}\Big] =: w_1 + w_2,
\end{aligned} \tag{17}$$

where

$$S(x)=\frac{x}{\pi}-\Big[\frac{x}{\pi}\Big]. \tag{18}$$

We shall prove that w_1 and w_2 are both asymptotically equal to

$$\frac{U_n^2\sin(\theta)\log(n)}{\pi(n+1)}.$$

First, observe that

$$w_1-\frac{U_n^2\sin(\theta)\log(n)}{\pi(n+1)}=w_{1,1}+w_{1,2}, \tag{19}$$

where

$$w_{1,1}=\frac{U_n^2\sin(\theta)}{\pi(n+1)}\Big[\frac{1}{S((n+1)\theta)}\Big],\quad w_{1,2}=\frac{U_n^2\sin(\theta)}{\pi(n+1)}\Big[\sum_{k=1}^{j-1}\frac{1}{S((n+1)\theta)+k}-\log(n)\Big].$$

Since

$$\sin((n+1)\theta)=(-1)^n\sin(S((n+1)\theta)\pi)=(-1)^n\sin\theta,\quad j=\Big[\frac{(n+1)\theta}{\pi}\Big]$$

$$|w_{1,1}|\le\frac{|U_n^2|}{\pi(n+1)}. \tag{20}$$

On the other hand, it is easy to see that

$$\Big|\sum_{k=1}^{j-1}\frac{1}{S((n+1)\theta)+k}-\log(n)\Big|\le 5.$$

Hence,

$$|w_{1,2}|\le\frac{5|U_n^2|}{\pi(n+1)}. \tag{21}$$

Therefore, from (20) and (21) we have

$$\Big|w_1-\frac{U_n^2\sin(\theta)\log n}{\pi(n+1)}\Big|\le\frac{3|U_n^2|}{(n+1)}. \tag{22}$$

The evaluation of the asymptotic behavior of w_2 can be carried out in a similar way. First, observe that

$$w_2-\frac{U_n^2\sin(\theta)\log(n)}{\pi(n+1)}=w_{2,1}+w_{2,2}, \tag{23}$$

where

$$w_{2,1}=\frac{U_n^2\sin(\theta)}{\pi(n+1)}\Big[\frac{1}{1-S((n+1)\theta)}\Big],\quad w_{2,2}=\frac{U_n^2\sin(\theta)}{\pi(n+1)}\Big[\sum_{k=1}^{j-1}\frac{1}{1-S((n+1)\theta)+k}-\log(n)\Big].$$

It follows that,

$$|w_{2,1}|\le\frac{|U_n^2|}{\pi(n+1)}. \tag{24}$$

On the other hand, it is also easy to see that

$$\left|\sum_{k=1}^{n-j-1} \frac{1}{1-S((n+1)\theta)+k} - \log(n)\right| \leq 5.$$

Hence,

$$|w_{2,2}| \leq \frac{5|U_n^2|}{\pi(n+1)}.$$

Therefore, from (23) and (24), we have

$$\left|w_2 - \frac{U_n^2 \sin(\theta)\log(n)}{\pi(n+1)}\right| \leq \frac{3|U_n^2|}{(n+1)}, \tag{25}$$

and the estimate (8) follows from (14), (15), (16), (17), (18), (20) and (23). To prove (9), observe that by using a similar technique we can show that

$$\left|H_n(g_x,x) - \sum_{k=1}^{n}(x_{kn}-x)(1-x_{kn}^2)(1-x^2)\frac{U_n^2}{(n+1)^2(x-x_{kn})^2}\right| \leq \frac{nC|x||U_n^2(x)|}{(n+1)^2} \tag{26}$$

$$\sum_{k=1}^{n}(x_{kn}-x)(1-x_{kn}^2)(1-x^2)\frac{U_n^2}{(n+1)^2(x-x_{kn})^2} = \sum_{k=1}^{n}\frac{U_n^2\sin^2\theta\sin^2\theta_{kn}}{(\cos\theta_{kn}-\cos\theta)(n+1)^2} \tag{27}$$

$$\left|\sum_{k=1}^{n}\frac{U_n^2\sin^2(\theta)\sin^2(\theta_{kn})}{(\cos\theta_{kn}-\cos\theta)(n+1)^2} - \sum_{k=1}^{n}\frac{U_n^2\sin\theta\sin^2(\theta_{kn}}{(\theta-\theta_{kn})(n+1)^2}\right| \leq \frac{nCU_n^2}{(n+1)^2}, \tag{28}$$

where $x=\cos\theta, x_{kn}=\cos\theta_{kn}, \theta_{kn}=\frac{k\pi}{(n+1)}, k=1,\ldots,n$, and

$$\sum_{k=1}^{n}\frac{U_n^2\sin^2\theta_{kn}\sin^2\theta}{(\theta-\theta_{kn})} = w_1 - w_2. \tag{29}$$

Therefore (9) follows from (26), (27), (28), and (29).

Proof of Theorem 2. Since the evaluation of $H_n(f_x,x)$ and $H_n(g_x,x)$ is already done in Theorem 1, the only thing we have to do now is the evaluation of $P_n(f,x)$ using the result of Theorem 1. For any $x\in(-1,1)$, such that $x \neq x_{kn}$ for $k=1,2,\ldots,n$, we have

$$\begin{aligned}
\left|P_n(f,x)\right| &= \left|-\sum_{x_{kn}<x}\left[\int_{x_{kn}}^{x} Q_x(t)\,dt\right]H_{kn}(x) + \sum_{x_{kn}>x}\left[\int_{x}^{x_{kn}} Q_x(t)\,dt\right]H_{kn}(x)\right| \\
&\leq \sum_{k=1}^{n}\left|\int_{x_{kn}}^{x} V_{x-t_{kn}}^{x+t_{kn}}(Q_x)\,dt\right|\left|H_{kn}(x)\right| \\
&\leq \sum_{k=1}^{n}|x-x_{kn}|V_{x-t_{kn}}^{x+t_{kn}}(Q_x)\left|H_{kn}(x)\right|,
\end{aligned}$$

where $V_a^b(Q_x)$ is the total variation of Q_x on $[a,b]$. Let $\theta_i = \frac{i\pi}{(n+1)}$ and define

$$E_r(n,\theta)=\left\{k:\frac{r\pi}{2(n+1)}<|\theta-\theta_{kn}|\le\frac{(r+1)\pi}{2(n+1)}\right\},\ r=0,1,2,\dots,2n-1.$$

We have

$$|\theta-\theta_i|\le\frac{\pi|U_n(x)|}{2(n+1)}.$$

Thus

$$\left|P_n(f,x)\right|\le\sum_{r=0}^{2n-1}\sum_{k\in E_r(n,\theta)}|x-x_{kn}|V_{x-t_{kn}}^{x+t_{kn}}(Q_x)H_{kn}(x).$$

Since

$$t_{kn}=|x-x_{kn}|\le|\theta-\theta_{kn}|\le\frac{\pi|U_n|}{2(n+1)}$$

and $E_0(n,\theta)$ has at most two elements. If $k\in E_0(n,\theta)$ it follows that:

$$\sum_{k\in E_0(n,\theta)}|x-x_{kn}|V_{x-t_{kn}}^{x+t_{kn}}(Q_x)H_{kn}(x)\le\frac{\pi|U_n|}{(n+1)}V_{x-\pi|U_n|/2(n+1)}^{x+\pi|U_n|/2(n+1)}(Q_x). \tag{30}$$

On the other hand, since

$$\begin{aligned}|x-x_{kn}|H_{kn}(x)&=|x-x_{kn}|(1-x_{kn}^2)[1+2x_{kn}^2-3xx_{kn}]\frac{U_n^2}{(n+1)^2(x-x_{kn})^2}\\&=\frac{U_n^2\sin^2\theta_{kn}[1+2\cos^2\theta_{kn}-3\cos\theta\cos\theta_{kn}]}{(n+1)^2|\cos\theta-\cos\theta_{kn}|}\\&\le\frac{CU_n^2}{(n+1)^2\sin\theta'_{kn}|\theta-\theta_{kn}|}\\&\le\frac{CU_n^2}{(n+1)^2|\theta-\theta_{kn}|}\frac{1}{\frac{2}{\pi}M(\theta)}\\&\le\frac{C\pi U_n^2}{2(n+1)^2|\theta-\theta_{kn}|}.\end{aligned}$$

But

$$|\theta-\theta_{kn}|>\frac{r\pi}{2(n+1)},$$

$$\frac{1}{|\theta-\theta_{kn}|}<\frac{2(n+1)}{r\pi},$$

we have

$$|x-x_{kn}|H_{kn}(x)\le\frac{CU_n^2}{(n+1)r}$$

and $t_{kn}\le\frac{(r+1)\pi}{2(n+1)}$; if $k\in E_r(n,\theta)$, we have for $r=1,2,\dots,2n-1$:

$$\sum_{k\in E_r(n,\theta)}|x-x_{kn}|V_{x-t_{kn}}^{x+t_{kn}}(Q_x)H_{kn}(x)\le\frac{2CU_n^2}{(n+1)r}V_{x-(r+1)\pi/2(n+1)}^{x+(r+1)\pi/2(n+1)}(Q_x). \tag{31}$$

Therefore, by (30) and (31),

$$\left|P_n(f,x)\right| \leq \frac{\pi|U_n|}{(n+1)}\left[V_{x-\pi|U_n|/2(n+1)}^{x+\pi|U_n|/2(n+1)}(Q_x)\right]+\frac{2CU_n^2}{(n+1)}\sum_{r=1}^{2n-1}\frac{1}{r}V_{x-\pi|U_n|/2(n+1)}^{x+\pi|U_n|/2(n+1)}(Q_x). \tag{32}$$

Let $Q(t) = V_{x-t}^{x+t}(Q_x)$, then

$$\begin{aligned}\sum_{r=1}^{2n-1}\frac{1}{r}V_{x-(r+1)/2(n+1)}^{x+(r+1)/2(n+1)}(Q_x) &= \sum_{r=2}^{2n}\frac{1}{r-1}Q\Big(\frac{r\pi}{2(n+1)}\Big)\\ &\leq 2\sum_{r=2}^{2n}\frac{1}{r}Q\Big(\frac{r\pi}{2(n+1)}\Big).\end{aligned} \tag{33}$$

But $Q(t)$ is non-decreasing function

$$\begin{aligned}\int_{\frac{r\pi}{2(n+1)}}^{\frac{(r+1)\pi}{2(n+1)}}\frac{Q(t)}{t}dt &\geq Q\Big(\frac{r\pi}{2(n+1)}\Big)\int_{\frac{r\pi}{2(n+1)}}^{\frac{(r+1)\pi}{2(n+1)}}\frac{dt}{t}\\ &\geq Q\Big(\frac{r\pi}{2(n+1)}\Big)\log(1+\frac{1}{r}),\end{aligned}$$

or

$$Q\Big(\frac{r\pi}{2(n+1)}\Big)\frac{1}{r} \leq \frac{3}{2}\int_{\frac{r\pi}{2(n+1)}}^{\frac{(r+1)\pi}{2(n+1)}}\frac{Q(t)}{t}dt.$$

Hence

$$\sum_{r=2}^{2n}\frac{1}{r}Q\Big(\frac{r\pi}{2(n+1)}\Big) \leq \frac{3}{2}\int_{\frac{\pi}{(n+1)}}^{\frac{(2n+1)\pi}{2(n+1)}}\frac{Q(t)}{t}dt.$$

We notice that $Q(\frac{\pi}{t})$ is non-decreasing and $Q(\frac{\pi}{t}) = Q(\pi)$, for $0 < t \leq 1$ then we have the following:

$$\begin{aligned}\int_{\frac{\pi}{(n+1)}}^{\frac{(2n+1)\pi}{2(n+1)}}\frac{Q(t)}{t}dt &= \int_{\frac{2(n+1)}{(2n+1)}}^{(n+1)}\frac{Q(\frac{\pi}{t})}{t}dt\\ &\leq \int_{\frac{2(n+1)}{2n+1}}^{1}\frac{Q(\frac{\pi}{t})}{t}dt+\int_{1}^{(n+1)}\frac{Q(\frac{\pi}{t})}{t}dt\\ &\leq \frac{Q(\pi)}{(2n+1)}+\sum_{k=1}^{n}\frac{Q(\frac{\pi}{k})}{k}\\ &\leq 2\sum_{k=1}^{n}\frac{Q(\frac{\pi}{k})}{k},\end{aligned}$$

and therefore

$$\sum_{r=2}^{2n}\frac{1}{r}Q\Big(\frac{r\pi}{2(n+1)}\Big)\leq 3\sum_{k=1}^{n}\frac{Q(\frac{\pi}{k})}{k}. \tag{34}$$

The proof of Theorem 2 follows from (32) and (34).

Acknowledgements

The authors thank the reviewers for invaluable comments to improve this paper. This research is funded by Jordan University of Science and Technology.

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