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ON 'USEFUL' RELATIVE INFORMATION AND J-DIVERGENCE MEASURES

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Abstract. In the present paper some new generalized measures of useful relative information have been defined and their particular cases have been studied. From these measures new useful information measures have also been derived. We have obtained useful measures of inaccuracy, dependence and J-divergence corresponding to each measure of useful relative information. In the end, an equality satisfied by useful J-divergence has been proved.

1. Introduction

Let $P = \{(p_1, p_2 \ldots, p_n), 0 \leq p_i \leq 1\}, Q = \{(q_1, q_2, \ldots, q_n), 0 \leq q_i \leq 1\}$ be two posterior and priori distributions of a random variable having utility distribution of events as $U = (u_1, u_2, \ldots, u_n)$, where $u_i > 0$ is the importance or usefulness of an event E_i. A measure of directed divergence $D(P : Q; U)$ of $(P; U)$ to $(Q; U)$ has to satisfy the following conditions: (i)$D(P : Q; U) \geq 0$, (ii)$D(P : Q; U) = 0$ iff $p_i = q_i$ for each i, (iii)$D(P : Q; U)$ is a convex or pseudo-convex function of $p_1, p_2, \ldots, p_n$ as well as of $q_1, q_2 \ldots, q_n$.

Bhaker and Hooda [1] defined and characterized 'useful' directed divergence measure given as below:

$$D(P : Q; U) = \frac{\sum_{i=1}^{n} u_i p_i \log(p_i/q_i)}{\sum_{i=1}^{n} u_i p_i} \tag{1.1}$$

This measure satisfies all three conditions (i)-(iii) provided $\sum_{i=1}^n p_i u_i \geq \sum_{i=1}^n q_i u_i$. We do come across some practical problems where we are to minimize $D(P : Q; U)$ with constraints $\sum_{i=1}^n p_i u_i \geq \sum_{i=1}^n q_i u_i$ e.g.

A manufacturer makes n different types of articles in proportions $q_1, q_2, \ldots, q_n$ having profit of $u_1, u_2, \ldots, u_n$ respectively. On the basis of availability of labour and raw material he plans to change the proportions of articles to $p_1, p_2, \ldots, p_n$; however, close to

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$q_1, q_2, \ldots, q_n$ such that the average profit may not decrease i.e. $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$. To obtain $p_i's$ we minimize

$$D(P:Q;U) = \frac{\sum_{i=1}^{n} u_i p_i \log(p_i/q_i)}{\sum_{i=1}^{n} u_i p_i} \qquad \text{subject to } \sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i.$$

It may be noted that $D(P:Q;U)$ is not a metric as it does not satisfy symmetric and triangle inequality. However, $D(P:Q;U)$ satisfies convexity property (iii) which we need to minimize $D(P:Q;U)$ as function of P or Q subject to $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$. This convexity property ensures that in each case, a local minimum will be global.

2. Some Requisite Results

(a) For all probability distributions P and Q having attached with utility distribution U,

$$\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} \gtreqless 1 \qquad \text{according to } \alpha \gtreqless 1, \qquad \text{provided } \sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i.$$

Proof. By Holder's inequality, for $\alpha > 1$.

$$\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha} \geq \Big[\sum_{i=1}^{n} (u_i^{\alpha} p_i^{\alpha})^{1/\alpha}\Big]^{\alpha} \Big[\sum_{i=1}^{n} (u_i^{1-\alpha} q_i^{1-\alpha})^{1/1-\alpha}\Big]^{1-\alpha}$$

$$\Rightarrow \sum_{i=1}^{n} u_i q_i^{\alpha} q_i^{1-\alpha} \geq \Big[\sum_{i=1}^{n} u_i p_i\Big]^{\alpha} \Big[\sum_{i=1}^{n} u_i q_i\Big]^{1-\alpha} \geq \sum_{i=1}^{n} u_i p_i, \qquad \text{since } \sum_{i=1}^{n} u_i p_i \geq \sum_{i=1}^{n} u_i q_i$$

$$\Rightarrow \frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} \geq 1.$$

For $\alpha = 1$, $\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} = 1$ for all probability distributions.

Also if $\alpha \neq 1$ and $p_i = q_i$ for each i, i.e. $P = Q$, we have

$$\frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} = 1 \qquad \text{and in other cases} \quad \frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} \neq 1.$$

(b) $(\alpha - 1)^{-1} \frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}$ is a convex function of P and Q.

Proof. We prove this in the following steps:
Step 1. Let $S = \frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}$. If we differentiate S partially w.r.t. p_i taking all q_i and u_i fixed, then $\sum_{i=1}^{n} u_i q_i$ is fixed and thus $\sum_{i=1}^{n} u_i p_i \geq \sum_{i=1}^{n} u_i q_i$ is constant. Hence we can write

$$S = C \sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}, \qquad \text{where } \frac{1}{C} = \sum_{i=1}^{n} u_i p_i \geq \sum_{i=1}^{n} u_i q_i > 0$$

$$\Rightarrow \frac{\partial S}{\partial p_i} = C\alpha u_i p_i^{\alpha-1} q_i^{1-\alpha} \tag{2.1}$$

and

$$\frac{\partial^2 S}{\partial p_i^2} = \alpha(\alpha - 1) C u_i p_i^{\alpha-2} q_i^{1-\alpha}. \tag{2.2}$$

For $\alpha > 1$, (2.2) is positive and hence $\frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}$ is a convex function of P.

For $0 < \alpha < 1$, (2.2) is negative and hence $\frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}$ is a concave function of P.

Similarly, we can prove that $\frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}$ is a convex function of Q for $\alpha > 1$ and is a concave function of Q for $0 < \alpha < 1$.

Step 2. Since $(\alpha - 1)^{-1} \gtrless 0$ according to $\alpha \gtrless 1$, therefore, $(\alpha - 1)^{-1} \frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}$ is a convex function of P and Q for all $\alpha > 0$ provided $\sum_{i=1}^{n} p_i u_i \geq \sum_{i=1}^{n} q_i u_i$.

3. A Generalized Measure of 'Useful' Relative Information

We consider the function

$$D_\alpha(P : Q; U) = \frac{1}{\alpha - 1} \left[\phi\Big(\frac{\sum_{i=1}^{n} u_i p_i^\alpha q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} \Big) - \phi(1) \right],$$

where $\phi(x)$ is a monotonic increasing convex function of x, then

$$\alpha > 1 \Rightarrow \frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} \geq 1 \Rightarrow \phi\Big(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}\Big) \geq \phi(1)$$

$$\Rightarrow D_\alpha(P : Q : U) \geq 0.$$

$$0 < \alpha < 1 \Rightarrow \frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} \leq 1 \Rightarrow \phi\Big(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}\Big) \leq \phi(1)$$

$$\Rightarrow D_\alpha(P : Q : U) \geq 0$$

As $\alpha \to 1, D_\alpha(P : Q; U) \to \phi'(1)\dfrac{\sum_{i=1}^{n} u_i p_i \log(p_1/q_i)}{\sum_{i=1}^{n} u_i p_i}$

Also for $p_i = q_i$ for each $i, D_\alpha(P : Q; U) = 0$ and

$$D_\alpha(P : Q; U) = 0 \Rightarrow \phi\Big(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}\Big) = \phi(1) \Rightarrow \frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} = 1$$

$$\Rightarrow \text{Either } \alpha = 1 \text{ or } p_i = q_i \text{ for each } i.$$

Since an increasing convex function of a convex function is a convex function and $\phi(x)$ is a monotonic increasing convex function and $\alpha > 1$, therefore, $D_\alpha(P : Q; U)$ is a convex function of P and Q.

Similarly, if $\phi(x)$ is a concave function and $0 < \alpha < 1$, then by the facts that an increasing concave function of a concave function is a concave and the negative of a concave function is a convex function, $D_\alpha(P : Q; U)$ is a convex function of P and Q.

Thus, we can use $D_\alpha(P : Q; U)$ as a measure of 'useful' relative information if $\alpha > 1$ and $\phi(x)$ is any monotonic increasing twice differentiable convex function of x or if $0 < \alpha < 1$ and $\phi(x)$ is any monotonic decreasing twice differentiable concave function of x.

4. Special Cases

(i)
$$D_{\alpha,j}(P:Q;U)=\frac{1}{\alpha-1}\Big[\Big(\frac{\sum_{i=1}^{n}u_ip_i^{\alpha}q_i^{1-\alpha}}{\sum_{i=1}^{n}u_ip_i}\Big)^{j}-1\Big], \tag{4.1}$$

where $\alpha>1$ and $j\geq 1$ or $0<\alpha<1$ and $0<j\leq 1$.

(ii) When $j=1$, (4.1) gives a measure similar to Hooda's [2] 'useful' relative information measure of degree α. If $u_i=1$, for each i, then (4.1) gives Harvda and Charvat measure of directed-divergence.

(iii)
$$\underset{\alpha\to 1}{\mathrm{Lt}}\, D_{\alpha,j}(P:Q;U)=\frac{\sum_{i=1}^{n}u_ip_i\log(p_i/q_i)}{\sum_{i=1}^{n}u_ip_i}, \tag{4.2}$$

which is j-multiple useful relative information measure characterized and studied by Bhaker and Hooda [1].

Further if $u_i=1$ for each i, (4.2) gives $j\sum_{i=1}^{n}p_i\log(p_i/q_i)$, which is J-multiple of Kullback and Leibler [3] measure of directed divergence.

(iv)
$$D_{\alpha}(P:Q;U)=\frac{1}{\alpha-1}\log\frac{\sum_{i=1}^{n}u_ip_i^{\alpha}q_i^{1-\alpha}}{\sum_{i=1}^{n}u_ip_i}, \tag{4.3}$$

which is Bhaker and Hooda's measure of useful relative information of order α. If $\alpha>1$, (4.3) is pseudo-convex function of P and Q. If $0<\alpha<1$, (4.3) is convex function of P and Q. In case utilities are ignored or $u_i=1$ for each i, then (4.3) reduces to

$$D_{\alpha}(P:Q)=\frac{1}{\alpha-1}\log\sum_{i=1}^{n}p_i^{\alpha}q_i^{1-\alpha},$$

which is Renyi's measure [4] of directed divergence.

(v)
$$D_{\alpha,\beta,j,j'}(P:Q:U)=\frac{1}{\alpha-\beta}\left\{\Big(\frac{\sum_{i=1}^{n}u_ip_i^{\alpha}q_i^{1-\alpha}}{\sum_{i=1}^{n}u_ip_i}\Big)^{j}-\Big(\frac{\sum_{i=1}^{n}u_ip_i^{\beta}q_i^{1-\alpha}}{\sum_{i=1}^{n}u_ip_i}\Big)^{j'}\right\}, \tag{4.4}$$

where $\alpha>1, j\geq 1, \beta\geq 1, 0\leq j'\leq 1$ or $\alpha>1, 0\leq j\leq 1, \beta>1, j'>1$.

In case $j = j' = 1$, (4.4) reduces to

$$D_{\alpha,\beta}(P:Q:U) = \frac{1}{\alpha-\beta}\Big(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i} - \frac{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}\Big)$$

$$= \frac{1}{\alpha-\beta}\frac{\sum_{i=1}^{n} u_i q_i^{1-\alpha}(p_i^{\alpha} - p_i^{\beta})}{\sum_{i=1}^{n} u_i p_i} \tag{4.5}$$

which is useful relative information measure of type (α, β). When utilities are ignored or $u_i = 1$ for each i, (4.5) reduced to type (α, β) measure of directed divergence.

(vi) $$D_{\alpha,\beta}(P:Q;U) = [\alpha-\beta]^{-1}\Big\{\Big[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{1-\alpha}}\Big] - 1\Big\} \tag{4.6}$$

If $\alpha > 1, \beta < 1$ or $\alpha < 1, \beta > 1$, (4.6) is Pseudo-convex function of both P and Q.

(vii) $$D_{\alpha,\beta,a}(P:Q:U) = \frac{1}{\alpha-\beta}\Big\{\exp_a\Big(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}\Big) - \exp_a\Big(\frac{\sum_{i=1}^{n} u_i p_i^{\beta} q_i^{1-\alpha}}{\sum_{i=1}^{n} u_i p_i}\Big)\Big\}, \tag{4.7}$$

where $a > 1$ and $\alpha > 1, \beta < 1$ or $\alpha < 1, \beta > 1$.

In view of convexicity of (4.7), it can be considered as 'useful' relative information measure and consequently, we can find more special cases. If we put $\phi(x) = x \log x$, we get the limiting cases of these measures as $\alpha \to 1$ and get functions of Bhaker and Hooda's [1] measures of 'useful' information. If we put $\phi(x) = x\log x - a^{-1}(1+ax)\log(1+ax) + a^{-1}(1+a)\log(1+a)$ we get a function of 'useful' information measure and if utilities are ignored or $u_i = 1$ for each i, we get a function of Kapur's [3] measures of directed divergence.

5. Measures of 'Useful' Information

Let $C = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)$ be the uniform distribution. Then

$$\text{(a)}\ D(P:C;U) = \frac{\sum_{i=1}^{n} u_i p_i \log np_i}{\sum_{i=1}^{n} u_i p_i} \geq 0, \qquad \text{since } nP_i \geq 1.$$

$$\text{(b)}\ D(P:C;U) = 0, \qquad \text{if } P = C.$$

$$\text{(c)}\ D(P:C;U) \qquad \text{is a convex function of } P.$$

We see that $D(P:Q;U) = \log n + \frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i} = D(C,U) - D(P;U)$, where $D(P;U)$ is 'useful' information measure characterized by Bhaker and Hooda [1]. So to minimize $D(P:C;U)$ we maximize $D(P;U)$.

(d) $D(C;U) \geq D(P;U)$

(e) $D(C;U) = D(P;U)$, if $P = C$.

(f) $D(P;U)$ is a concave function of P. Thus $D(P;U)$ is 'useful' information measure of P and U corresponding to 'useful' directed divergence measure $D(P:Q;U)$.

Next we define

$$H_{\alpha,\phi}(P;U) = \frac{1}{1-\alpha}\Big[\phi\Big(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} n^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i}\Big) - \phi(n^{\alpha-1})\Big] \tag{5.1}$$

so that

$$H_{\alpha,\phi}(C;U) = \frac{1}{1-\alpha}[\phi(1) - \phi(n^{\alpha-1})] \tag{5.2}$$

and

$$\underset{\alpha\to 1}{\text{Lt}}\, H_{1,\phi}(C;U) = \log n \tag{5.3}$$

If $\phi(x) = x^j, (j \geq 1)$, we have

$$H_{\alpha,j}(P:U) = \frac{1}{1-\alpha}\Big[\Big(\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} n^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i}\Big)^j - (n^{\alpha-1})^j\Big]$$

$$= \frac{(n^{\alpha-1})^j}{1-\alpha}\Big[\Big(\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}\Big)^j - 1\Big] \tag{5.4}$$

In particular for $j = 1$, we get

$$H_{\alpha,1}(P:U) = \frac{(n^{\alpha-1})}{1-\alpha}\Big[\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i} - 1\Big] \tag{5.5}$$

When $\alpha \to 1$, the measure (5.1), (5.4) and (5.5) respecitively reduce to

$$H_{1,\phi}(P;U) = -\phi'(1)\Big(\frac{\sum_{i=1}^n u_i p_i \log p_i}{\sum_{i=1}^n u_i p_i}\Big) \tag{5.6}$$

$$H_{1,j}(P;U) = -j\frac{\sum_{i=1}^n u_i p_i \log p_i}{\sum_{i=1}^n u_i p_i} \tag{5.7}$$

$$H_{1,1}(P;U) = -\frac{\sum_{i=1}^n u_i p_i \log p_i}{\sum_{i=1}^n u_i p_i} \tag{5.8}$$

If we take $\phi(x) = \log x$ in (5.1), we get

$$H_{\alpha,\phi}(P;U) = \frac{1}{1-\alpha}\Big[\log\Big(\frac{\sum_{i=1}^n u_i p_i^\alpha n^{\alpha-1}}{\sum_{i=1}^n u_i p_i}\Big) - \log(n^{\alpha-1})\Big]$$

$$= \frac{1}{1-\alpha}\log\frac{\sum_{i=1}^n u_i p_i^\alpha}{\sum_{i=1}^n u_i p_i}, \tag{5.9}$$

which is Bhaker and Hooda's [1] measure of 'useful' information of order α. This gives Renyi's entropy in case the utilites are ignored or $u_i = 1$ for each i.

if we take $\phi(x) = x \log x$, we get

$$H_{\alpha,\phi}(P:U) = \frac{1}{1-\alpha}\Big[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha} n^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i} \log \frac{\sum_{i=1}^{n} u_i p_i^{\alpha} n^{\alpha-1}}{\sum_{i=1}^{n} u_i p_i} - n^{\alpha-1}\log n^{\alpha-1}\Big]$$

$$= \frac{1}{1-\alpha}\Big[\frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i} \log \frac{\sum_{i=1}^{n} u_i p_i^{\alpha}}{\sum_{i=1}^{n} u_i p_i}\Big] \tag{5.10}$$

Incase $\alpha \to 1$, (5.10) reduces to $H_{1,\phi}(P;U) = -\frac{\sum_{i=1}^{n} u_i p_i \log p_i}{\sum_{i=1}^{n} u_i p_i}$, which is 'useful' information measures studied by Bhaker and Hooda [1].

When utilities are ignored in (5.10), we have

$$H_{1,\phi}(P) = \frac{1}{1-\alpha}\Big[\sum_{i=1}^{n} p_i^{\alpha} \log \sum_{i=1}^{n} p_i^{\alpha}\Big] \tag{5.11}$$

When $\alpha \to 1$, the measure (5.11) reduces to

$$H_{1,\phi}(P) = -\sum_{i=1}^{n} p_i \log p_i, \tag{5.12}$$

which is Shannon's entropy.
We can get more general measure if we put

$$\phi(x) = x\log x - \frac{1}{\alpha_j}(1+ax)\log(1+ax) \qquad \text{in (5.4).} \tag{5.13}$$

where $a \geq 1, j > 1$ or $a < 1, j < 2$.

Thus we can obtain multi-parameter families of useful information measures by taking positive combinations of these measures of 'useful' information:

$$H_{\alpha,\phi}(P;U) = \frac{C_j}{1-\alpha j}\Big[\phi_j\Big(\frac{\sum_{i=1}^{n} p_i^{\alpha_j} n^{\alpha_j - 1}}{\sum_{i=1}^{n} u_i p_i^{j}}\Big) - \phi_j(n^{\alpha_j-1})\Big]. \tag{5.14}$$

where $C_j > 0, \alpha_j > 0, \alpha_j \neq 1$ and ϕ_j's are n different increasing convex functions.

For the independent distributions P and Q having U and V respectively as utility distributions, (5.4) gives

$$\begin{aligned}
H_{\alpha,j}(P*Q:U*V) &= \frac{n^{(\alpha-1)j}}{1-\alpha}\Big[\Big(\frac{\sum_{i=1}^{n} u_i p_i^\alpha}{\sum_{i=1}^{n} u_i p_i}\cdot\frac{\sum_{i=1}^{m} v_j q_j^\alpha}{\sum_{i=1}^{n} v_j q_j}\Big)^j-1\Big]\\
&= \frac{n^{(\alpha-1)j}}{1-\alpha}\Big[\Big(\frac{\sum_{i=1}^{n} u_i p_i^\alpha}{\sum_{i=1}^{n} u_i p_i}\Big)^j\Big(\frac{\sum_{i=1}^{m} v_j q_j^\alpha}{\sum_{i=1}^{n} v_j q_j}\Big)^j-1\Big]\\
&= \frac{n^{(\alpha-1)j}}{1-\alpha}\Big[\{\frac{1-\alpha}{n^{(\alpha-1)j}}H_{\alpha,j}(P:U)+1\}\{\frac{1-\alpha}{n^{(\alpha-1)j}}H_{\alpha,j}(Q:V)+1\}-1\Big]\\
&= \frac{1-\alpha}{n^{(\alpha-1)j}}H_{\alpha,j}(P:U)H_{\alpha,j}(Q:V)+H_{\alpha,j}(P:U)+H_{\alpha,j}(Q:V) \qquad (5.15)
\end{aligned}$$

which is a well known functional equation and proves that the measure (5.4) is non-additive.

6. Measure of 'Useful' Stochastic Dependence

Let $f(x_1, x_2, \ldots, x_n)$ be the density function of joint probability distribution of $x_1, x_2, \ldots, x_m$ having $h(y_1, y_2, \ldots, y_m)$ utility function such that y_i corresponds to x_i for each i. Let $g_1(x_1)$, $g_2(x_2)$, …, $g_m(x_m)$ be the density function of the m marginal probability distributions having respectively $h_1(y_1), h_2(y_2), \ldots, h_m(y_m)$ as marginal utility distributions. Then the 'useful' relative information of $f(x_1, x_2, \ldots, x_m)$ from $g_1(x_1)$, $g_2(x_2), \ldots,$ $g_m(x_m)$ is given by

$$\begin{aligned}
&\frac{\iint h(y_1,\ldots,y_m)f(x_1,\ldots,x_m)\log\frac{f(x_1,x_2,\ldots,x_m)}{g_1(x_1)g_2(x_2)\cdots g_m(x_m)}dx_1\cdots dx_m dy_1\cdots dy_m}{\iint h(y_1,\ldots,y_m)f(x_1,\ldots,x_m)dx_1\cdots dx_m dy_1\cdots dy_m}\\
&= \frac{\iint h(y_1,\ldots,y_m)f(x_1,\ldots,x_m)\log f(x_1\cdots x_m)dx_1\cdots dx_m dy_1\cdots dy_m}{\iint h(y_1,\ldots,y_m)f(x_1,\ldots,x_m)dx_1\cdots dx_m dy_1\cdots dy_m}\\
&\quad -\frac{\iint h(y_1)g_1(x)\log g_1(x_1)dx_1\cdot dy_1}{\iint h_1(y_1)g_1(x_1)dx_1 dy_1}\cdots\frac{\iint h_m(y_m)g_m(x_m)\log(x_m)dx_m\cdot dy_m}{\iint h_m(y_m)g_m(x_m)dx_m\cdot dy_m}\\
&= S_1+S_2+\ldots+S_m-S=D. \qquad (6.1)
\end{aligned}$$

where S_i is the 'useful' information measure of i^{th} marginal distribution and S is the 'useful' information measure of the joint probability distribution. It may be seen that $D \geq 0$ and vanishes if and only if $x_1, x_2 \cdots, x_m$ are statistically independent. D can, therefore, be called a 'useful' measure of stochastic dependence among the variates $x_1, x_2, \ldots, x_m$.

If the variates are independent, $D = 0$, where as if these are dependent, $D > 0$. Also the greater the value of D, the greater would be the dependence among the variates.

7. Measure of Useful J-Divergence

Corresponding to $D(P : Q; U)$, we get a measure of symmetric divergence called J-divergence.

$$J(P : Q; U) = D(P : Q; U) + D(Q : P; U) = \frac{\sum_{i=1}^{n} u_i(p_i - q_i)\log(p_i/q_i)}{\sum_{i=1}^{n} u_i p_i} \tag{7.1}$$

we see that (7.1) reduces to $J(P : Q) = \sum_{i=1}^{n}(p_i - q_i)\log\left(\frac{p_i}{q_i}\right)$ in case utilities are ignored or $u_i = 1$ for each i.

Theorem 1. *$J(P : Q; U)$ is not a homogeneous function in U. However, if $U^\beta = (U_1^\beta, U_2^\beta, \dots U_n^\beta), \beta > 0$ be the β-power utility distribution, then the following inequality holds:*

$$\sum_{i=1}^{n} u_i J(P : Q; U^\beta) = \frac{\beta J(P : Q; U^\beta)}{E(P)}, \qquad \text{where } E(P) = \sum_{i=1}^{n} u_i^\beta p_i = \text{Constant} \tag{7.2}$$

Proof. Let $J(P : Q; U)$ be a homogeneous function of degree β in utility distribution. Then by Euler's theorem, we have

$$\begin{aligned} J(P : Q; \lambda U) &= \lambda^\beta J(P : Q; U) \\ &= \frac{\lambda^\beta \sum_{i=1}^{n} u_i(p_i - q_i)\log(p_i/q_i)}{\sum_{i=1}^{n} u_i p_i} \\ &= \lambda^{\beta-1} J(P : Q; \lambda U) \end{aligned} \tag{7.3}$$

It implies $\lambda^{\beta-1} = 1$ or $\beta = 1$.

Hence $J(P : Q; U)$ is not a homogeneous function in U.
Next from (7.1), we have

$$J(P : Q; U^\beta) = \frac{\sum_{i=1}^{n} u_i^\beta (p_i - q_i)\log(p_i/q_i)}{\sum_{i=1}^{n} u_i^\beta p_i} = \frac{\sum_{i=1}^{n} u_i^\beta (p_i - q_i)\log(p_i/q_i)}{E(P)} \tag{7.4}$$

Differentiating (7.4) with respect to u_i and multiplying by u_i both sides and taking summation over i, we get

$$\sum_{i=1}^{n} u_i \frac{\delta J}{\delta u_i}(P : Q; U^\beta) = \frac{\beta J(P : Q; U^\beta)}{E(P)}, \tag{7.5}$$

where $E(P)=\sum_{i=1}^{n} u_i^{\beta} p_i=$ Constant. Hence the theorem is proved.

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