SPHERICAL INDICATRICES OF INVOLUTE OF A SPACE CURVE IN EUCLIDEAN 3-SPACE

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Abstract. In this work, we studied the properties of the spherical indicatrices of involute curve of a space curve and presented some characteristic properties in the cases that involute curve and evolute curve are slant helices and helices, spherical indicatrices are slant helices and helices and we introduced new representations of spherical indicatrices.

1. Introduction

The specific curve pairs are the most popular subjects in curve and surface theory and involute-evolute pair is one of them. We can see in most textbooks various applications not only in curve theory but also in surface theory and mechanic.

In this study, the spherical indicatrices of involute of a space curve are given. In order to make a involute of a space curve and its evolute curve slant helix, the feature that spherical indicatrices curve's need to have are examined.

Let $\gamma : I \longrightarrow IR^3$ be a curve with $\gamma'(s) \neq 0$, where $\gamma'(s) = d\gamma(s)/ds$. The arc-lenght s of a curve $\gamma(s)$ is determined such that $\|\gamma'(s)\| = 1$. Let us denote $T(s) = \gamma'(s)$ and we call $T(s)$ a tangent vector of γ at $\gamma(s)$. We define the curvature of γ by $\kappa(s) = \|\gamma''(s)\|$. If $\kappa(s) \neq 0$, then the unit principal normal vector $N(s)$ of the curve at $\gamma(s)$ is given by $\gamma''(s) = \kappa(s)N(s)$. The unit vector $B(s) = T(s)\Lambda N(s)$ is called the unit binormal vector of γ at $\gamma(s)$. Then we have the Frenet-Serret formulae

$$T' = \kappa N , \ N' = -\kappa T + \tau B , \ B' = -\tau N$$

where $\tau(s)$ is the torsion of γ at $\gamma(s)$ [7, 9].

The curve γ is called evolute of $\widetilde{\gamma}$ if the tangent vectors are orthogonal at the corresponding points for each $s \in I \subset IR$. In this case, $\widetilde{\gamma}$ is called involute of the curve γ and there exists a relationship between the position vectors as

$$\widetilde{\gamma}(s^*) = \gamma(s) + \lambda T(s)$$

where λ is the distance between the curves γ and $\widetilde{\gamma}$ at the corresponding points for each s. The pair of $(\widetilde{\gamma}, \gamma)$ is called a involute-evolute pair. λ is not a constant for involute-evolute pairs [7, 9].

On the other hand, Izumiya and Takeuchi have introduced the concept of slant helix by saying that the normal lines make a constant angle with a fixed straight line. They characterize a slant helix if and only if the geodesic curvature of the principal image of the principal normal indicatrix

$$\Gamma = \frac{f'}{\kappa \left(1 + f^2\right)^{3/2}} \tag{1.1}$$

is a constant function, where $f = \frac{\tau}{\kappa}$ [5, 8] .

In this study, we denote T, N, B, κ, τ and $\widetilde{T}$, $\widetilde{N}$, $\widetilde{B}$, $\widetilde{\kappa}$, $\widetilde{\tau}$ are the Frenet equipments of γ and $\widetilde{\gamma}$, respectively. Tangent, principal normal and binormal vectors are described for the spherical curves which are called tangent, principal normal and binormal indicatrices both the curves γ and $\widetilde{\gamma}$, respectively. Throughout this study, both involute and evolute curves are regular.

2. Spherical indicatrices of involute of a curve

In this section, we introduced the spherical indicatrices of involute curve of a curve in Euclidean 3-space and gave considerable results by using the properties of the curves, similar to the previous section. Let γ be a curve with its involute curve $\widetilde{\gamma}$ then

$$\widetilde{\gamma}(s^*) = \gamma(s) + \lambda T(s) \tag{2.1}$$

where

$$\lambda = |c - s| \tag{2.2}$$

and $(c - s)$ is definitely positive. Let ϵ be the sign of $(c - s)$ such that if $c - s > 0$, $\epsilon = +1$ and if $c - s < 0$, $\epsilon = -1$. We differentiate the equation (2.1) with respect to s, we get

$$\widetilde{T}\frac{ds^*}{ds} = (1 - \epsilon) T + \kappa\epsilon (c - s) N.$$

Since T and $\widetilde{T}$ are orthogonal, there is no any component of $\widetilde{T}$ on T. Thus ϵ has to be $+1$.

Theorem 1. *Let $\widetilde{\gamma}$ be involute of a space curve, then we have Frenet formula:*

$$\widetilde{T}' = \widetilde{\kappa}\widetilde{N}, \quad \widetilde{N}' = -\widetilde{\kappa}\widetilde{T} + \widetilde{\tau}\widetilde{B}, \quad \widetilde{B}' = -\widetilde{\tau}\widetilde{N}$$

where

$$\widetilde{T} = N, \quad \widetilde{N} = \frac{-1}{\sqrt{1 + f^2}}T + \frac{f}{\sqrt{1 + f^2}}B, \quad \widetilde{B} = \frac{f}{\sqrt{1 + f^2}}T + \frac{1}{\sqrt{1 + f^2}}B$$

with the parametrization

$$\frac{ds}{ds^*} = \frac{1}{\kappa\,(c-s)} \tag{2.3}$$

and the curvature and torsion of $\widetilde{\gamma}(s^*)$ are

$$\widetilde{\kappa} = \frac{\sqrt{1+f^2}}{c-s}, \quad \widetilde{\tau} = \frac{f'}{\kappa\,(c-s)\,(1+f^2)}. \tag{2.4}$$

The geodesic curvature of the the principal image of the principal normal indicatrix of involute curve is

$$\widetilde{\Gamma} = \frac{\left\{(1+f^2)\left(f''\kappa - f'\kappa'\right) - 3\kappa f\left(f'\right)^2\right\}(1+f^2)^{\frac{3}{2}}}{\left(\kappa^2\left(1+f^2\right)^3 + \left(f'\right)^2\right)^{\frac{3}{2}}} \tag{2.5}$$

From (2.4), it is obvious that involute of γ is a planar curve if and only if γ is a generalized helix. For further usage we denote $\frac{\widetilde{\tau}}{\widetilde{\kappa}}$ as $\widetilde{f}$. By using (1.1) and (2.4) we obtained the relation

$$\widetilde{f} = \frac{f'}{\kappa\left(1+f^2\right)^{3/2}} = \Gamma$$

and so we have

$$\widetilde{f} = \Gamma. \tag{2.6}$$

Thus we have the following theorem.

Theorem 2. *Let $\widetilde{\gamma}$ be involute of a space curve γ then $\widetilde{\gamma}$ is a generalized helix if and only if its evolute is a slant helix.*

We obtained the relation between $\widetilde{\Gamma}$ and Γ by using (1.1) and (2.5) as

$$\widetilde{\Gamma} = \frac{\kappa^2\left(1+f^2\right)^4\Gamma'}{\left(\kappa^2\left(1+f^2\right)^3 + \left(f'\right)^2\right)^{\frac{3}{2}}} \tag{2.7}$$

The spherical image of tangent indicatrix of $\widetilde{\gamma}$ is

$$\widetilde{\gamma}_t(s_t^*) = \widetilde{T}(s^*)$$

with the natural parameter s_t^*.

Theorem 3. *If the Frenet frame of the tangent indicatrix $\widetilde{\gamma}_t = \widetilde{T}$ of involute of $\gamma(s)$ is $\{\widetilde{T}_t, \widetilde{N}_t, \widetilde{B}_t\}$, we have Frenet formula:*

$$\widetilde{T}_t' = \widetilde{\kappa}_t\widetilde{N}_t, \quad \widetilde{N}_t' = -\widetilde{\kappa}_t\widetilde{T}_t + \widetilde{\tau}_t\widetilde{B}_t, \quad \widetilde{B}_t' = -\widetilde{\tau}_t\widetilde{N}_t$$

where

$$\widetilde{T}_t = \widetilde{N} \ , \ \widetilde{N}_t = \frac{-\widetilde{T} + \widetilde{f}\widetilde{B}}{\sqrt{1+\widetilde{f}^2}} \ , \ \widetilde{B}_t = \frac{\widetilde{f}\widetilde{T} + \widetilde{B}}{\sqrt{1+\widetilde{f}^2}} \tag{2.8}$$

with the parametrization

$$\frac{ds^*}{ds_t^*} = \frac{1}{\widetilde{\kappa}} \tag{2.9}$$

and the curvature and torsion of $\widetilde{\gamma}_t$ are

$$\widetilde{\kappa}_t = \sqrt{1+\widetilde{f}^2}, \ \widetilde{\tau}_t = \frac{\widetilde{f}'}{\widetilde{\kappa}\left(1+\widetilde{f}^2\right)}. \tag{2.10}$$

The geodesic curvature of the principal image of the principal normal indicatrix of $\widetilde{\gamma}_t$ is

$$\widetilde{\Gamma}_t = \frac{\left\{\left(1+\widetilde{f}^2\right)\left(\widetilde{f}''\widetilde{\kappa} - \widetilde{f}'\widetilde{\kappa}'\right) - 3\widetilde{\kappa}\widetilde{f}\left(\widetilde{f}'\right)^2\right\}\left(1+\widetilde{f}^2\right)^{\frac{3}{2}}}{\left(\widetilde{\kappa}^2\left(1+\widetilde{f}^2\right)^3 + \left(\widetilde{f}'\right)^2\right)^{\frac{3}{2}}}. \tag{2.11}$$

Form (2.10) we have the following theorem.

Theorem 4. *Let $\widetilde{\gamma}$ be involute of a space curve γ then spherical image of the tangent indicatrix of $\widetilde{\gamma}$ is a circle on unit sphere if and only if evolute of $\widetilde{\gamma}$ is a generalized helix.*

By using (2.6) and theorem 4, we can state the following corollary.

Corollary 1. *Spherical image of the tangent indicatrix of $\widetilde{\gamma}$ is a circle on unit sphere if and only if evolute of $\widetilde{\gamma}$ is a slant helix.*

From (2.10) and (2.11) we obtained

$$\frac{\widetilde{\tau}_t}{\widetilde{\kappa}_t} = \frac{\widetilde{f}'}{\widetilde{\kappa}\left(1+\widetilde{f}^2\right)^{1/2}} = \widetilde{\Gamma} \tag{2.12}$$

and the relation between $\widetilde{\Gamma}$ and $\widetilde{\Gamma}_t$ is

$$\widetilde{\Gamma}_t = \frac{\widetilde{\kappa}^2\left(1+\widetilde{f}^2\right)^4 \widetilde{\Gamma}'}{\left(\widetilde{\kappa}^2\left(1+\widetilde{f}^2\right)^3 + \left(\widetilde{f}'\right)^2\right)^{\frac{3}{2}}} \tag{2.13}$$

so we have the following relation too.

$$\frac{\widetilde{\tau}_t}{\widetilde{\kappa}_t} = \frac{\kappa^2\left(1+f^2\right)^4 \Gamma'}{\left(\kappa^2\left(1+f^2\right)^3 + \left(f'\right)^2\right)^{\frac{3}{2}}}$$

can obtained by using (2.7) and (2.12). Hence we can give the following theorem.

Theorem 5. *Let $\widetilde{\gamma}$ be involute of a space curve γ then followings are true.*

   i. *If γ is a slant helix then spherical image of the tangent indicatrix of its involute is a generalized helix,*

   ii. *if $\widetilde{\gamma}$ is a slant helix then its spherical image of the tangent indicatrix is a spherical helix.*

The spherical image of principal normal indicatrix of involute of the curve $\gamma(s)$ is

$$\widetilde{\gamma}_n\left(s_n^*\right) = \widetilde{N}(s^*)$$

with the natural parameter s_n^*.

Theorem 6. *If the Frenet frame of the binormal indicatrix $\widetilde{\gamma}_b = \widetilde{B}$ of involute of the curve $\gamma(s)$ is $\{\widetilde{T}_b, \widetilde{N}_b, \widetilde{B}_b\}$, we have Frenet formula:*

$$\widetilde{T}_b' = \widetilde{\kappa}_b \widetilde{N}_b, \quad \widetilde{N}_b' = -\widetilde{\kappa}_b \widetilde{T}_b + \widetilde{\tau}_b \widetilde{B}_b, \quad \widetilde{B}_b' = -\widetilde{\tau}_b \widetilde{N}_b$$

where

$$\widetilde{T}_b = -\widetilde{N} \; , \; \widetilde{N}_b = \frac{-\widetilde{T} + \widetilde{f}\widetilde{B}}{\sqrt{1 + \widetilde{f}^2}} \; , \; \widetilde{B}_b = \frac{-\widetilde{f}\widetilde{T} - \widetilde{B}}{\sqrt{1 + \widetilde{f}^2}} \tag{2.14}$$

with the parametrization

$$\frac{ds^*}{ds_b^*} = \frac{1}{\widetilde{\tau}} \tag{2.15}$$

and the curvature and torsion of $\widetilde{\gamma}_b$ are

$$\widetilde{\kappa}_b = \frac{\sqrt{1 + \widetilde{f}^2}}{\widetilde{f}} \; , \; \widetilde{\tau}_b = \frac{-\widetilde{f}'}{\widetilde{\kappa}\,\widetilde{f}\left(1 + \widetilde{f}^2\right)}. \tag{2.16}$$

The geodesic curvature of the principal image of the principal normal indicatrix of $\widetilde{\gamma}_b$ is

$$\widetilde{\Gamma}_b = \frac{\left(1 + \widetilde{f}^2\right)^{\frac{5}{2}}\left(-\widetilde{f}\left(\widetilde{f}'\right)^2 \widetilde{\tau} - \widetilde{\tau}^2 \widetilde{f}^2\left(\frac{\widetilde{f}'}{\widetilde{\tau}}\right)'\right) + 3\widetilde{\tau}\widetilde{f}^3\left(\widetilde{f}'\right)^2\left(1 + \widetilde{f}^2\right)^{\frac{3}{2}}}{\left(\widetilde{\tau}^2\left(1 + \widetilde{f}^2\right)^3 + \widetilde{f}^2\left(\widetilde{f}'\right)^2\right)^{\frac{3}{2}}}. \tag{2.17}$$

From (2.16) we have the following theorem.

Theorem 7. *Let γ be a space curve and $\widetilde{\gamma}$ be its involute with nonzero torsion then spherical image of binormal indicatrix of $\widetilde{\gamma}$ is a circle on unit sphere if and only if $\widetilde{\gamma}$ is a generalized helix.*

The relation

$$\frac{\widetilde{\tau}_b}{\widetilde{\kappa}_b} = \frac{-\widetilde{f}'}{\widetilde{\kappa}\left(1 + \widetilde{f}^2\right)^{3/2}} = -\widetilde{\Gamma} \tag{2.18}$$

We can easily obtaine from (2.16) and it gives us the following theorem.

Theorem 8. *Let non-planar $\widetilde{\gamma}$ be involute of a space curve γ then its spherical image of binormal indicatrix is spherical helix if and only if evolute of γ is a slant helix.*

From (2.15), (2.17) and (2.18) we can obtaine the relation between $\widetilde{\Gamma}$ and $\widetilde{\Gamma}_b$ which is

$$\widetilde{\Gamma}_b = \frac{-\widetilde{\Gamma}'}{\widetilde{\kappa}\sqrt{1+\widetilde{f}^2}\left(1+\widetilde{\Gamma}^2\right)^{3/2}}. \tag{2.19}$$

Thus we have the following theorem.

Theorem 9. *Spherical image of binormal indicatrix $\widetilde{\gamma}_b$ is a spherical slant helix if and only if $\widetilde{\gamma}$ is a slant helix.*

By using (2.3), (2.4), (2.6) and (2.16) we get the relation

$$\widetilde{\tau}_b = \frac{-\Gamma'}{\kappa\Gamma\sqrt{1+f^2}\left(1+\Gamma^2\right)}$$

and we have the following theorem.

Theorem 10. *Spherical image of binormal indicatrix $\widetilde{\gamma}_b$ is circle if and only if γ is a slant helix.*

Corollary 2. *There are the following equations between the Frenet vectors of the spherical indicatrices of an involute curve in Euclidean 3-space.*

$$\widetilde{T}_t = -\widetilde{T}_b, \ \ \widetilde{N}_t = \widetilde{T}_n = \widetilde{N}_b, \ \ \widetilde{B}_t = -\widetilde{B}_b.$$

Example 1. In [6], Kula et al. obtained the general equation of a slant helix family similar to following

$$\gamma_\mu(s) = \left(\gamma_\mu^1(s), \gamma_\mu^2(s), \gamma_\mu^3(s)\right)$$

where

$$\gamma_\mu^1(s) = \frac{(w-1)\sin[(w+1)s]}{2w(w+1)} + \frac{(w+1)\sin[(w-1)s]}{2w(w-1)}$$
$$\gamma_\mu^2(s) = \frac{(w+1)\cos[(w-1)s]}{2w(w-1)} + \frac{(w-1)\cos[(w+1)s]}{2w(w+1)}$$
$$\gamma_\mu^3(s) = -\frac{\cos(s)}{\mu w}.$$

and $w = \frac{\sqrt{1+\mu^2}}{\mu}$, $\mu \in IR - \{0\}$. By using the theorem 2, we can obtained the general equation of general helix family in Euclidean 3-space according to the non-zero constant μ as follows

$$\widetilde{\gamma}_\mu(s_*) = \left(\widetilde{\gamma}_\mu^1(s), \widetilde{\gamma}_\mu^2(s), \widetilde{\gamma}_\mu^3(s)\right)$$

where

$$\widetilde{\gamma}_{\mu}^{1}(s) = \frac{(c-s)}{2w}\{(w+1)\cos[(w-1)s] + (w-1)\cos[(w+1)s]\}$$

$$+\frac{1}{2w(w^2-1)}\left\{(w+1)^2\sin[(w-1)s] + (w-1)^2\sin[(w+1)s]\right\}$$

$$\widetilde{\gamma}_{\mu}^{2}(s) = \frac{-(c-s)}{2w}\{(w+1)\sin[(w-1)s] + (w-1)\sin[(w+1)s]\}$$

$$+\frac{1}{2w(w^2-1)}\left\{(w+1)^2\cos[(w-1)s] + (w-1)^2\cos[(w+1)s]\right\}$$

$$\widetilde{\gamma}_{\mu}^{3}(s) = \frac{(c-s)\sin(s) - \cos(s)}{\mu w}$$

and $w = \frac{\sqrt{1+\mu^2}}{\mu}$, $\mu \in IR - \{0\}$, with the parametrization

$$\frac{ds}{ds_*} = \frac{\mu w\sqrt{2}}{(c-s)\sqrt{\left(\mu^2(w^2-1)^2+1\right)(\cos(2s)+1)}}$$

and the curvature and torsion of curve $\widetilde{\gamma}_{\mu}(s_*)$ is

$$\widetilde{\kappa} = \frac{\sqrt{2}}{(c-s)\sqrt{(\cos(2s)+1)}} \quad , \quad \widetilde{\tau} = \frac{-\mu}{(c-s)\cos(s)}.$$

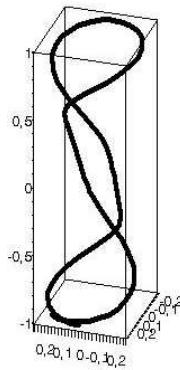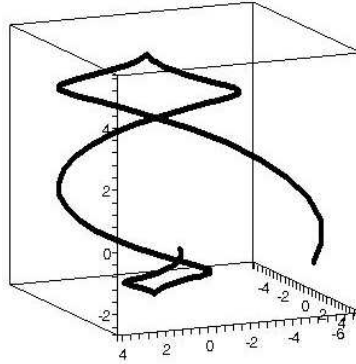
Figure 1: Slant helix.

Figure 2: General helix with $c = -1, mu = \pm 1$.

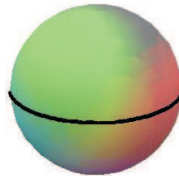
Figure 3: Tangent indicatrix of involute of a slant helix.

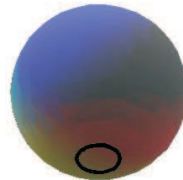Figure 4: Principal normal indicatrix of involute of a slant helix.

Figure 5: Binormal indicatrix of involute of a slant helix.

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