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ON ABSOLUTE GENERALIZED NÖRLUND SUMMABILITY OF ORTHOGONAL SERIES

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Abstract. In this paper, we shall prove a general theorem which contains two theorems on the absolute Nörlund summability and the absolute Riesz summability of orthogonal series.

1. Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with sequence of partial sums $\{s_n\}$. Let p denote the sequence $\{p_n\}$. Given two sequences p and q, the convolution $(p*q)$ is defined by

$$(p*q)_n = \sum_{k=0}^{n} p_{n-k}q_k = \sum_{k=0}^{n} p_k q_{n-k}.$$

When $(p*q)_n \neq 0$ for all n, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_p^{p,q} = \frac{1}{(p*q)_n} \sum_{k=0}^{n} p_{n-k} q_k s_k. \tag{1}$$

If $\lim_{n\to\infty} t_n^{p,q}$ exists and is equal to s, then the series $\sum_{n=0}^{\infty} a_n$ or the sequence $\{s_n\}$ is said to be summable (N, p_n, q_n) to the value s and we write

$$\sum_{n=0}^{\infty} a_n = s(N, p_n, q_n) \quad \text{or} \quad s_n \to s(N, p_n, q_n)$$

(see Borwein [1]).

If the series $\sum_{n=0}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$ converges, then the series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|N, p_n, q_n|$ and we write $\sum_{n=0}^{\infty} a_n \in |N, p_n, q_n|$ (see Tanaka [5]).

The method $|N, p_n, q_n|$ reduces to the absolute Nörlund method $|N, p_n|$ if $q_n = 1$ for all n and to the absolute Riesz method $|\bar{N}, q_n|$ if $p_n = 1$ for all n. We know that $|N, p_n|$ mean or $|\bar{N}, q_n|$ includes as special case the absolute Cesàro mean and the absolute harmonic mean or the logarithmic mean, respectively. Finally A denotes a positive absolute constant not the same.

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2. Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the internal (a,b). We suppose that $f(x)$ belongs to $L^2(a,b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x).$$

We also write

$$R_n = (p * q)_n, R_n^j = \sum_{k=j}^{n} p_{n-k} q_k \quad \text{and} \quad R_n^{n+1} = 0.$$

Then we have

$$R_n^0 = R_n.$$

Further we put

$$P_n = (p * 1)_n = \sum_{k=0}^{n} p_k \quad \text{and} \quad Q_n = (1 * q)_n = \sum_{k=0}^{n} q_k.$$

Now we shall prove a general theorem on the absolute generalized Nörlund summability of the orthogonal series.

Theorem 1. *If the series*

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \left(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |a_j|^2 \right\}^{1/2}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

is summable $|N, p_n, q_n|$ *almost everywhere.*

Proof. Let $t_n^{p,q}(x)$ be the n-the (N, p_n, q_n) mean of the series $\sum_{n=0}^{\infty} a_n \varphi_n(x)$. Then we have

$$\begin{aligned} t_n^{p,q}(x) &= \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k s_k(x) \\ &= \frac{1}{R_n} \sum_{k=0}^{n} p_{n-k} q_k \sum_{j=0}^{k} a_j \varphi_j(x) \\ &= \frac{1}{R_n} \sum_{j=0}^{n} a_j \varphi_j(x) \sum_{k=j}^{n} p_{n-k} q_k \\ &= \frac{1}{R_n} \sum_{j=0}^{n} R_n^j a_j \varphi_j(x), \end{aligned}$$

where $s_n(x) = \sum_{k=0}^{n} a_k\varphi_k(x)$.

Thus we obtain

$$\begin{aligned} t_n^{p,q}(x) - t_{n-1}^{p,q}(x) &= \frac{1}{R_n}\sum_{j=0}^{n} R_n^j a_j\varphi_j(x) - \frac{1}{R_{n-1}}\sum_{j=0}^{n-1} R_{n-1}^j a_j\varphi_j(x) \\ &= \frac{1}{R_n}\sum_{j=1}^{n} R_n^j a_j\varphi_j(x) - \frac{1}{R_{n-1}}\sum_{j=1}^{n-1} R_{n-1}^j a_j\varphi_j(x) \\ &= \frac{1}{R_n}\sum_{j=1}^{n} R_n^j a_j\varphi_j(x) - \frac{1}{R_{n-1}}\sum_{j=1}^{n} R_{n-1}^j a_j\varphi_j(x) \\ &= \sum_{j=1}^{n}\Big(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}}\Big)a_j\varphi_j(x). \end{aligned}$$

Using the Schwarz's inequality and the orthogonality, we obtain

$$\begin{aligned} \int_a^b |\Delta t_n^{p,q}(x)|dx &\le (b-a)^{1/2}\Big\{\int_a^b |\Delta t_n^{p,q}(x)|^2 dx\Big\}^{1/2} \\ &= (b-a)^{1/2}\Big\{\sum_{j=1}^{n}\Big(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}}\Big)^2|a_j|^2\Big\}^{1/2}, \end{aligned}$$

and then

$$\sum_{n=1}^{\infty}\int_a^b |\Delta t_n^{p,q}(x)|dx \le (b-a)^{1/2}\sum_{n=1}^{\infty}\Big\{\sum_{j=1}^{n}\Big(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}}\Big)^2|a_j|^2\Big\}^{1/2}$$

which is convergent by the assumption and from the Beppo-Leni Lemma we complete the proof.

We can obtain two the following corollaries from our theorem.

Corollary 1.([2,4]) *If the series*

$$\sum_{n=1}^{\infty}\frac{p_n}{P_nP_{n-1}}\Big\{\sum_{j=1}^{n} p_{n-j}^2\Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\Big)^2|a_j|^2\Big\}^{1/2}$$

converges, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n\varphi_n(x)$$

is summable $|N, p_n|$ *almost everywhere.*

Proof. The proof follows from our theorem and the fact that

$$\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} = \frac{P_{n-j}}{P_n} - \frac{P_{n-1-j}}{P_{n-1}}$$

$$= \frac{1}{P_n P_{n-1}}(P_{n-1}P_{n-j} - P_n P_{n-1-j})$$
$$= \frac{1}{P_n P_{n-1}}\{(P_n - p_n)P_{n-j} - P_n(P_{n-j} - p_{n-j})\}$$
$$= \frac{1}{P_n P_{n-1}}(P_n P_{n-j} - p_n P_{n-j} - P_n P_{n-j} + p_{n-j}P_n)$$
$$= \frac{p_n}{P_n P_{n-1}}\Big(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\Big)p_{n-j} \qquad \text{for all } q_n = 1.$$

Corollary 2.([3]) *If the series*

$$\sum_{n=1}^{\infty} \frac{q_n}{Q_n Q_{n-1}}\Big\{\sum_{j=1}^{n} Q_{j-1}^2 a_j^2\Big\}^{1/2}$$

convergeces, then the orthogonal series

$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

is summable $|\bar{N}, q_n|$ almost everywhere.

Proof. The proof follows from theorem 1 and the fact that

$$\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} = \frac{Q_n - Q_{j-1}}{Q_n} - \frac{Q_{n-1} - Q_{j-1}}{Q_{n-1}}$$
$$= Q_{j-1}\Big(\frac{1}{Q_n} - \frac{1}{Q_{n-1}}\Big)$$
$$= -\frac{q_n Q_{j-1}}{Q_n Q_{n-1}} \qquad \text{for all } p_n = 1.$$

For the application of these corollaries, see Okuyama [2,3,4].

Furthermore, if we put

$$w(j) = \frac{1}{j}\sum_{n=j}^{\infty} n^2\Big(\frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}}\Big)^2, \tag{2}$$

then we have the following theorem from theorem 1.

Theorem 2. *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty} |a_n|^2 \Omega(n) w(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} a_n \varphi_n(x)$ is summable $|N, p_n, q_n|$ almost everywhere, where $w(n)$ is defined by (2).*

Proof. We have by Schwarz inequality

$$\begin{aligned}
\sum_{n=1}^{\infty}\int_a^b |\Delta t_n^{p,q}(x)|dx &\le A\sum_{n=1}^{\infty}\Big\{\sum_{j=1}^{n}\Big(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\Big)^2|a_j|^2\Big\}^{1/2}\\
&= A\sum_{n=1}^{\infty}\frac{1}{n^{1/2}\Omega(n)^{1/2}}\Big\{n\Omega(n)\sum_{j=1}^{n}\Big(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\Big)^2|a_j|^2\Big\}^{1/2}\\
&\le A\Big\{\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}\Big\}^{1/2}\Big\{\sum_{n=1}^{\infty}n\Omega(n)\sum_{j=1}^{n}\Big(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\Big)^2|a_j|^2\Big\}^{1/2}\\
&\le A\Big\{\sum_{j=1}^{\infty}|a_j|^2\sum_{n=j}^{\infty}n\Omega(n)\Big(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\Big)^2\Big\}^{1/2}\\
&\le A\Big\{\sum_{j=1}^{\infty}|a_j|^2\frac{\Omega(j)}{j}\sum_{n=j}^{\infty}n^2\Big(\frac{R_n^j}{R_n}-\frac{R_{n-1}^j}{R_{n-1}}\Big)^2\Big\}^{1/2}\\
&= A\Big\{\sum_{j=1}^{\infty}|a_j|^2\Omega(j)w(j)\Big\}^{1/2} < \infty
\end{aligned}$$

by virtue of the hypotheses of Theorem 2. Thus this completes the proof of Theorem 2 from the same reason of the proof of Theorem 1.

References

[1] D. Borwein, *On product of sequences*, J. London Math. Soc. **33**(1958), 352-357.

[2] Y. Okuyama, *On the absolute Nörlund summability of orthogonal series*, Proc. Japan Acad. **54**(1978), 113-118.

[3] Y. Okuyama and T. Tsuchikura, *On the absolute Riesz summability of orthogonal series*, Analysis Math, **7**(1981), 199-208.

[4] Y. Okuyama, *Absolute summability of Fourier series and orthogonal series*, Lecture Notes in Math. No.1067(1984), Springer-Verlag.

[5] M. Tanaka, *On generalized Nörlund methods of summability*, Bull. Austral. Math. Soc. **19**(1978), 381-402.

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