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ON A NEW CLASS OF SEQUENCES RELATED TO THE SPACE ℓ^p

BINOD CHANDRA TRIPATHY AND MAUSUMI SEN*

Abstract. In this article we introduce a new class of sequences named as $m(\phi,p)$. This generalizes the earlier sequence space $m(\phi)$ due to Sargent [2]. We study some of the properties of this space.

1. Introduction

The space $m(\phi)$ was introduced by Sargent [2]. He studied some properties of the space $m(\phi)$. Later on it was studied from sequence space point of view and some matrix classes with one member as $m(\phi)$ were characterized by Rath and Tripathy [1], Tripathy [3] and others. In this article we generalize this space by introducing the space $m(\phi,p)$.

Throughout the article $w, \ell^p, \ell^l, \ell^\infty$ denote the spaces of all *p-absolutely summable*, *absolutely summable* and *bounded* sequences respectively. Also N and C denote the set of all natural numbers and complex numbers respectively.

2. Definitions and Background

Let $x = (x_n)$ be a sequence, then $S(x)$ denotes the set of all permutation of the elements of (x_n), i.e. $S(x) = \{(x_{\pi(n)}) : \pi(n)$ is a permutation on $N\}$. A sequence space E is said to be *symmetric* if $S(x) \subseteq E$ for all $x \in E$.

A sequence space E is said to be *solid* if $(y_n) \in E$ whenever $(x_n) \in E$ and $|y_n| \leq |x_n|$ for all $n \in N$.

A BK-space is a Banach space in which the co-ordinate maps are continuous, i.e. if $(x_k^{(n)})_k \in E$, then

$\|(x_k^{(n)}) - (x_k)\| \to 0$ as $n \to \infty \Rightarrow |x_k^{(n)} - x_k| \to 0$ as $n \to \infty$, for each fixed k.

$\wp_s$ denotes the set of all subsets of N those do not contain more than s elements. Throughout the paper $\{\phi_n\}$ denotes a non-decreasing sequence of positive numbers such that $n\phi_{n+1} \leq (n+1)\phi_n$ for all $n \in N$.

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The space $m(\phi)$ is defined as

$$m(\phi) = \Big\{(x_k) \in w : \|x\|_{m(\phi)} = \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \sum_{n\in\sigma} |x_n| < \infty\Big\}.$$

Generalizing the above sequence space, we now introduce the space $m(\phi, p)$ as follows: For $1 \leq p < \infty$,

$$m(\phi,p) = \Big\{(x_k) \in w : \|x\|_{m(\phi,p)} = \sup_{s\geq 1, \sigma\in\wp_s} \frac{1}{\phi_s} \Big\{\sum_{n\in\sigma} |x_n|^p\Big\}^{1/p} < \infty\Big\}.$$

From the definition it is clear that $m(\phi, 1) \equiv m(\phi)$. It is a routine work to verify that $m(\phi, p)$ is a normed linear space with respect to the above norm.

3. Main Results

Theorem 1. *The space $m(\phi, p)$ is complete.*

Proof. Let $\{x^{(n)}\}$ be a Cauchy sequence in $m(\phi, p)$. Hence

$$\sup_{s\geq 1, \sigma\in\wp_s} \Big[\frac{1}{\phi_s}\Big\{\sum_{i\in\sigma} |x_i^{(n)}|^p\Big\}^{1/p}\Big] < \infty, \qquad \text{for all } n (n = 1, 2, 3, \ldots).$$

Then for each $\varepsilon > 0$, there exists a positive integer n_0 such that

$$\|x^{(m)} - x^{(n)}\|_{m(\phi,p)} < \varepsilon \qquad \text{for all } m, n \geq n_0$$

$$\Rightarrow \sup_{s\geq 1, \sigma\in\wp_s} \Big[\frac{1}{\phi_s}\Big\{\sum_{i\in\sigma} |x_i^{(m)} - x_i^{(n)}|^p\Big\}^{1/p}\Big] < \varepsilon \qquad \text{for all } m, n \geq n_0 \tag{1}$$

$$\Rightarrow |x_i^{(m)} - x_i^{(n)}| < \varepsilon\phi_1 \qquad \text{for all } m, n \geq n_0, \text{ for all } i \in N.$$

Hence for each fixed $i (1 \leq i < \infty)$, the sequence $\{x_i^{(n)}\}_{n=1}^{\infty}$ is a Cauchy sequence in C. Since C is complete, it converges in C. Let $x_i^{(n)} \to x_i$ as $n \to \infty$.

We define $x = (x_1, x_2, x_3, \ldots)$

We now show that (a) $x \in m(\phi, p)$ and (b) $x^{(n)} \to x$.

From (1) we get, for each fixed s

$$\sum_{i\in\sigma} |x_i^{(m)} - x_i^{(n)}|^p < \varepsilon^p \phi_s^p \qquad \text{for all } m, n \geq n_0, \ \sigma \in \wp_s.$$

Letting $n \to \infty$, we get

$$\sum_{i\in\sigma} |x_i^{(m)} - x_i|^p < \varepsilon^p \phi_s^p \qquad \text{for all } m \geq n_0, \sigma \in \wp_s$$

$$\Rightarrow \sup_{s\geq 1, \sigma\in\wp_s} \Big[\frac{1}{\phi_s}\Big\{\sum_{i\in\sigma} |x_i^{(m)} - x_i|^p\Big\}^{1/p}\Big] < \varepsilon \qquad \text{for all } m \geq n_0 \tag{2}$$

$$\Rightarrow x^{(n)} - x \in m(\phi, p) \qquad \text{for all } n \geq n_0.$$

Hence $x = x^{(n)} + (x - x^{(n)}) \in m(\phi, p)$, as $m(\phi, p)$ is a linear space.

$$\text{Also (2)} \Rightarrow \|x^{(n)} - x\|_{m(\phi,p)} < \varepsilon \qquad \text{for all } n \geq n_0$$
$$\Rightarrow x^{(n)} - x \in m(\phi, p).$$

Hence $m(\phi, p)(1 \leq p < \infty)$ is a Banach space.

Theorem 2. *The space $m(\phi, p)$ is a BK-space.*

Proof. Let $\|x^{(n)} - x\|_{m(\phi,p)} \to 0$ as $n \to \infty$.
Hence given $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$\|x^{(n)} - x\|_{m(\phi,p)} < \varepsilon \qquad \text{for all } n \geq n_0$$
$$\Rightarrow \sup_{s \geq 1, \sigma \in \wp_s} \left[\frac{1}{\phi_s} \Big\{ \sum_{k \in \sigma} |x_k^{(n)} - x_k|^p \Big\}^{1/p} \right] < \varepsilon \qquad \text{for all } n \geq n_0$$
$$\Rightarrow |x_k^{(n)} - x_k| < \varepsilon \phi_1 \qquad \text{for all } n \geq n_0, \qquad \text{for all } k.$$

So, $|x_k^{(n)} - x_k| \to 0$ as $n \to \infty$ and the proof is complete.

Using the definitions we formulate the following result.

Proposition 3. (i) *The space $m(\phi, p)$ is a symmetric space. If $x \in m(\phi, p)$ and $u \in S(x)$, then $\|u\|_{m(\phi,p)} = \|x\|_{m(\phi,p)}$.*
(ii). *The space $m(\phi, p)$ is a normal space.*

Proposition 4. $m(\phi) \subseteq m(\phi, p)$

Proof. Let $x \in m(\phi)$. Then $\|x\|_{m(\phi)} = \sup_{s \geq 1, \sigma \in \wp_s} \left\{ \frac{1}{\phi_s} \sum_{n \in \sigma} |x_n| \right\} = K(< \infty)$.
Hence for each fixed s, $\sum_{n \in \sigma} |x_n| \leq K \phi_s$, $\sigma \in \wp_s$.

$$\Rightarrow \Big\{ \sum_{n \in \sigma} |x_n|^p \Big\}^{1/p} \leq K \phi_s, \sigma \in \wp_s$$
$$\Rightarrow \sup_{s \geq 1, \sigma \in \wp_s} \left[\frac{1}{\phi_s} \Big\{ \sum_{n \in \sigma} |x_n|^p \Big\}^{1/p} \right] \leq K.$$

Thus $x \in m(\phi, p)$ and this completes the proof.

Proposition 5. $m(\phi, p) \subseteq m(\psi, p)$ *if and only if* $\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s} \right) < \infty$.

Proof. Suppose $\sup_{s \geq 1} \left(\frac{\phi_s}{\psi_s} \right) = K(< \infty)$. Then $\phi_s \leq K \psi_s$.
Now if $(x_k) \in m(\phi, p)$, then

$$\sup_{s \geq 1, \sigma \in \wp_s} \left[\frac{1}{\phi_s} \Big\{ \sum_{n \in \sigma} |x_n|^p \Big\}^{1/p} \right] < \infty$$

$$\Rightarrow \sup_{s\geq 1,\sigma\in\wp_s}\left[\frac{1}{K\psi_s}\left\{\sum_{n\in\sigma}|x_n|^p\right\}^{1/p}\right]<\infty$$
$$\Rightarrow \|x\|_{m(\psi,p)}<\infty.$$

Hence $m(\phi,p)\subseteq m(\psi,p)$.

Conversely suppose that $m(\phi,p)\subseteq m(\psi,p)$. To show that

$$\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right)=\sup_{s\geq 1}(\eta_s)<\infty.$$

Suppose if possible $\sup_{s\geq 1}(\eta_s)=\infty$. Then there exists a subsequence (η_{s_i}) of (η_s) such that $\lim_{i\to\infty}(\eta_{s_i})=\infty$.

Then for $(x_k)\in m(\phi,p)$ we have

$$\sup_{s\geq 1,\sigma\in\wp_s}\left[\frac{1}{\psi_s}\left\{\sum_{n\in\sigma}|x_n|^p\right\}^{1/p}\right]\geq \sup_{s_i\geq 1,\sigma\in\wp_{s_i}}\left[\eta_{s_i}\frac{1}{\phi_{s_i}}\left\{\sum_{n\in\sigma}|x_n|^p\right\}^{1/p}\right]=\infty$$

$\Rightarrow (x_k)\notin m(\psi,p)$, a contradiction. This completes the proof.

We formulate the following result in view of the above result.

Corollary 6. *$m(\phi,p)=m(\psi,p)$ if and only if $\sup_{s\geq 1}(\eta_s)<\infty$ and $\sup_{s\geq 1}(\eta_s^{-1})<\infty$, where $\eta_s=\frac{\phi_s}{\psi_s}$.*

Theorem 7. *$\ell^p\subseteq m(\phi,p)\subseteq\ell^\infty$.*

Proof. Since $m(\phi,p)=\ell^p$ for $\phi_n=1$, for all $n\in N$, so the first inclusion is clear.
Next we suppose that $x\in m(\phi,p)$
Then $\sup_{s\geq 1,\sigma\in\wp_s}\left[\frac{1}{\phi_s}\left\{\sum_{n\in\sigma}|x_n|^p\right\}^{1/p}\right]=K(<\infty)\Rightarrow |x_n|\leq K\phi_1$, for all $n\in N$.
Thus $x\in\ell^\infty$, and the proof is complete.

Putting $\psi_n=1$, for all $n\in N$, in Corollary 6, we have

Proposition 8. *$m(\phi,p)=\ell^p$ if and only if $\sup_{s\geq 1}(\phi_s)<\infty$ and $\sup_{s\geq 1}(\phi_s^{-1})<\infty$.*

Using the properties of ℓ^p spaces we have the following result.

Proposition 9. *If $p<q$, then $m(\phi,p)\subset m(\phi,q)$.*

Proposition 10. *$m(\phi,p)\subset m(\psi,q)$ if $p<q$ and $\sup_{s\geq 1}\left(\frac{\phi_s}{\psi_s}\right)<\infty$.*

Corollary 11. *$m(\phi,p)=\ell^\infty$ if $\lim_{s\to\infty}(\phi_s/s)>0$.*

Proof. $m(\phi,p)=\ell^\infty$ if $p=1$ and $\phi_n=n(n=1,2,3,\ldots)$.

Hence from proposition 10 it follows that $\ell^\infty\subseteq m(\phi,p)$ if $\sup_{s\geq 1}\left(\frac{s}{\phi_s}\right)<\infty$. This completes the proof.

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Mathematical Sciences Division, Insitute of Advanced Study in Science and Technology, Khanapara, Guwahati-781 022 ASSAM, India.

E-mail: tripathybc@yahoo.com