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BBDF-α FOR SOLVING STIFF ORDINARY DIFFERENTIAL EQUATIONS WITH OSCILLATING SOLUTIONS

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Abstract. In this paper, the block backward differentiation α formulas (BBDF-α) is derived for solving first order stiff ordinary differential equations with oscillating solutions. The consistency and zero stability conditions are investigated to prove the convergence of the method. The stability region in the entire negative half plane shows that the derived method is A-stable for certain values of α. The implementation of the method using Newton iteration is discussed. Several numerical experiments are conducted to demonstrate the performance of the method in terms of accuracy and computational time.

1. Introduction

Ordinary differential equations (ODEs) are used frequently throughout mathematics, physics and engineering to describe how physical quantities change. The first order stiff ODEs can be found in the form

$$y'(x) = f\left(x, y\right), a \leq x \leq b, \tag{1.1}$$

where $y\left(a\right) = y_0$ is the initial condition. Lambert [1] defined equation (1.1) possesses some stiffness if the following conditions are satisfied.

a) $\mathrm{Re}\left(\lambda_t\right) < 0, t = 1, 2, \ldots, m.$

b) $\max_t \left|\mathrm{Re}\left(\lambda_t\right)\right| > \min_t \left|\mathrm{Re}\left(\lambda_t\right)\right|$ where λ_t are the eigenvalues of the Jacobian matrix, $J = \left(\frac{\partial f}{\partial y}\right)$ of the first order ODEs in equation (1.1).

According to Sunday et al. [2], a nontrivial solution (function) of ODEs is called oscillating if it does not tend either to a finite limit or to infinity. The equation (1.1) is called oscillating if it has at least one oscillating solution. Stiff ODEs with oscillating solutions frequently arise in areas such as viscous drag in mechanical systems and resistance in electronic oscillators.

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Since most of the stiff ODEs in real-life problems cannot be solved analytically, the established numerical method must be applied extensively. Several researchers such as Franco et al. [3], Tahmasbi [4], Ibrahim et al. [5, 6, 7], Nasir et al. [8, 9], Sunday et al. [10] and Zawawi et al. [11, 12] developed a variety of numerical techniques to cater such problems. Even there exist a large number of numerical methods in the scientific literature, the most accurate method among them has to be considered.

Motivated by the fact that the capability of α formulas in solving (1.1) is still not verified numerically, thus the aim of this research is to develop a new family of block method, namely block backward differentiation alpha formulas (BBDF-α) for solving stiff initial value problems (IVPs) with oscillating solutions. The advantages of this method are that it can compute the approximated solutions at two-point simultaneously, while increases the order of accuracy by selecting the suitable value of parameter α. The formula developed here will be more outstanding in terms of stability and accuracy compared to the conventional block backward differentiation formulas (BBDF) by Ibrahim et al. [6, 7]. The detailed formulation is described in the following section.

2. Formulation of the method

The method is derived using constant step size to compute the approximated solutions at y_{n+1} and y_{n+2} simultaneously at every step using polynomial $P_k(x)$ of degree k in terms of Lagrange polynomial which is defined as follows:

$$P_k(x) = \sum_{j=0}^{k} L_{k,j}(x) f\left(x_{n+1-j}\right), \tag{2.1}$$

where

$$L_{k,j}(x) = \prod_{\substack{i=0 \\ i\neq j}}^{k} \frac{(x - x_{n+1-i})}{\left(x_{n+1-j} - x_{n+1-i}\right)},$$

for each $j = 0, 1, \ldots, k$.

By using interpolating points $\left(x_{n-1}, y_{n-1}\right), \left(x_n, y_n\right), \left(x_{n+1}, y_{n+1}\right)$ and $\left(x_{n+2}, y_{n+2}\right)$, the resulting polynomial is

$$\begin{aligned} P(x) = {} & \frac{(x - x_n)(x - x_{n+1})(x - x_{n+2})}{(x_{n-1} - x_n)(x_{n-1} - x_{n+1})(x_{n-1} - x_{n+2})} y_{n-1} \\ & + \frac{(x - x_{n-1})(x - x_{n+1})(x - x_{n+2})}{(x_n - x_{n-1})(x_n - x_{n+1})(x_n - x_{n+2})} y_n \\ & + \frac{(x - x_{n-1})(x - x_n)(x - x_{n+2})}{(x_{n+1} - x_{n-1})(x_{n+1} - x_n)(x_{n+1} - x_{n+2})} y_{n+1} \\ & + \frac{(x - x_{n-1})(x - x_n)(x - x_{n+1})}{(x_{n+2} - x_{n-1})(x_{n+2} - x_n)(x_{n+2} - x_{n+1})} y_{n+2}, \end{aligned} \tag{2.2}$$

Next, we replace $x = sh + x_{n+1}$ into (2.2) and differentiate once with respect to s. Let $s = 0$, we have

$$P'(x_{n+1}) = \frac{1}{6}y_{n-1} - y_n + \frac{1}{2}y_{n+1} + \frac{1}{3}y_{n+2}. \tag{2.3}$$

We now apply the same procedure by evaluating $s = 1$ to formulate y_{n+2}, will produce the following equation:

$$P'(x_{n+2}) = -\frac{1}{3}y_{n-1} + \frac{3}{2}y_n - 3y_{n+1} + \frac{11}{6}y_{n+2}. \tag{2.4}$$

By considering $hf_{n+1} = P'(x_{n+1})$ and $hf_{n+2} = P'(x_{n+2})$, the BBDF is given as follows:

$$\begin{aligned} y_{n+1} + \frac{1}{3}y_{n-1} - 2y_n + \frac{2}{3}y_{n+2} &= 2hf_{n+1}, \\ y_{n+2} - \frac{2}{11}y_{n-1} + \frac{9}{11}y_n - \frac{18}{11}y_{n+1} &= \frac{6}{11}hf_{n+2}. \end{aligned} \tag{2.5}$$

Subsequently, equations (2.5) are modified based on Celaya and Anza [13] using four independent parameters α, β, γ and μ. The following expressions are obtained:

$$\begin{aligned} &(1+\gamma)y_{n+1} - \gamma y_n + \frac{1}{3}y_{n-1} - 2\left((1+\mu)y_n - \mu y_{n-1}\right) + \frac{2}{3}\left((1+\beta)y_{n+2} - \beta y_{n+1}\right) \\ &\quad = 2h\left((1+\alpha)f_{n+1} - \alpha f_n\right), \\ &(1+\beta)y_{n+2} - \beta y_{n+1} - \frac{2}{11}y_{n-1} + \frac{9}{11}\left((1+\mu)y_n - \mu y_{n-1}\right) - \frac{18}{11}\left((1+\gamma)y_{n+1} - \gamma y_n\right) \\ &\quad = \frac{6}{11}h\left((1+\alpha)f_{n+2} - \alpha f_{n+1}\right). \end{aligned} \tag{2.6}$$

On arranging (2.6), we obtain two corrector formulas with various parameters as follows:

$$\begin{aligned} &\left(\frac{2}{3} + \frac{2}{3}\beta\right)y_{n+2} + \left(1 + \gamma - \frac{2}{3}\beta\right)y_{n+1} + (-\gamma - 2 - 2\mu)y_n + \left(2\mu + \frac{1}{3}\right)y_{n-1} \\ &\quad = (2+2\alpha)hf_{n+1} - 2\alpha hf_n. \\ &(1+\beta)y_{n+2} + \left(-\beta - \frac{18}{11} - \frac{18}{11}\gamma\right)y_{n+1} + \left(\frac{9}{11} + \frac{9}{11}\mu + \frac{18}{11}\gamma\right)y_n + \left(-\frac{9}{11}\mu - \frac{2}{11}\right)y_{n-1} \\ &\quad = \left(\frac{6}{11} + \frac{6}{11}\alpha\right)hf_{n+2} - \frac{6}{11}\alpha hf_{n+1}. \end{aligned} \tag{2.7}$$

Equations (2.7) can be written in the following way, which corresponds to the standard linear multistep method (LMM) given by

$$\sum_{j=0}^{3} A_j y_{n+j-1} = h\sum_{j=0}^{3} B_j f_{n+j-1}, \tag{2.8}$$

where

$$A_0 = \begin{bmatrix} 2\mu + \frac{1}{3} \\ -\frac{9}{11}\mu - \frac{2}{11} \end{bmatrix}, \quad A_1 = \begin{bmatrix} -\gamma - 2\mu - 2 \\ \frac{9}{11} + \frac{9}{11}\mu + \frac{18}{11}\gamma \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 + \gamma - \frac{2}{3}\beta \\ -\beta - \frac{18}{11} - \frac{18}{11}\gamma \end{bmatrix}, \quad A_3 = \begin{bmatrix} \frac{2}{3} + \frac{2}{3}\beta \\ 1 + \beta \end{bmatrix},$$

$$B_0=\begin{bmatrix}0\\0\end{bmatrix},\qquad B_1=\begin{bmatrix}-2\alpha\\0\end{bmatrix},\qquad B_2=\begin{bmatrix}2+2\alpha\\-\frac{6}{11}\alpha\end{bmatrix},\qquad B_3=\begin{bmatrix}0\\\frac{6}{11}+\frac{6}{11}\alpha\end{bmatrix}.$$

The order of the method must be determined to produce the formula with one independent parameter α. Based on Lambert [1], the LMM is said to be order p if the following condition is satisfied.

$$C_q=0 \text{ or } 0\le q\le p \quad\text{and}\quad C_{p+1}\neq 0.$$

where

$$\begin{aligned}C_0&=\sum_{j=0}^{k}A_j,\\ C_1&=\sum_{j=0}^{k}jA_j-\sum_{j=0}^{k}B_j,\\ C_q&=\sum_{j=0}^{k}\left(\frac{1}{q!}j^qA_j-\frac{1}{(q-1)!}j^{q-1}B_j\right),q=1,2,\dots,k.\end{aligned}\tag{2.9}$$

The term C_{p+1} is the error constant of (2.8). Hence, the value of $C_q=\begin{bmatrix}c_{q,1}\\c_{q,2}\end{bmatrix}$, $q=0,\dots,4$ for BBDF-α is determined as follows:

$$C_0=\sum_{j=0}^{3}A_j=\begin{bmatrix}0\\0\end{bmatrix},$$

$$C_1=\sum_{j=0}^{3}(jA_j-B_j)=\begin{bmatrix}\gamma-2\mu+\frac{2}{3}\beta\\-\frac{18}{11}\gamma+\frac{9}{11}\mu+\beta\end{bmatrix},$$

$$C_2=\sum_{j=0}^{3}\left(\frac{1}{2}j^2A_j-jB_j\right)=\begin{bmatrix}\frac{3}{2}\gamma-\mu+\frac{5}{3}\beta-2\alpha\\\frac{9}{22}\mu-\frac{27}{11}\gamma+\frac{5}{2}\beta-\frac{6}{11}\alpha\end{bmatrix},$$

$$C_3=\sum_{j=0}^{3}\left(\frac{1}{6}j^3A_j-\frac{1}{2}j^2B_j\right)=\begin{bmatrix}\frac{7}{6}\gamma-\frac{1}{3}\mu+\frac{19}{9}\beta-3\alpha\\\frac{3}{22}\mu-\frac{21}{11}\gamma+\frac{19}{6}\beta-\frac{15}{11}\alpha\end{bmatrix},$$

$$C_4=\sum_{j=0}^{3}\left(\frac{1}{24}j^4A_j-\frac{1}{6}j^3B_j\right)=\begin{bmatrix}\frac{5}{8}\gamma-\frac{1}{12}\mu+\frac{65}{36}\beta+\frac{1}{3}\alpha+\frac{17}{6}\\-\frac{45}{44}\gamma+\frac{3}{88}\mu+\frac{65}{24}\beta-\frac{31}{11}\alpha-\frac{1}{2}\end{bmatrix}.$$

By solving C_q, $q=0,\dots,3$ simultaneously using Maple software, the following expressions are obtained:

First point:

$$\mu=\frac{1}{2}\alpha,\qquad \beta=\frac{3}{2}\alpha,\qquad \gamma=0.\tag{2.10}$$

Second point:

$$\mu = \frac{1}{3}\alpha, \qquad \beta = \frac{9}{11}\alpha, \qquad \gamma = \frac{2}{3}\alpha. \tag{2.11}$$

Then we substitute (2.10) into $c_{q,1}$, $q = 0,\ldots,4$ and (2.11) into $c_{q,2}$, $q = 0,\ldots,4$ to produce $C_0 = C_1 = C_2 = C_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $C_4 = \begin{bmatrix} 3\alpha + \frac{17}{6} \\ -\frac{14}{11}\alpha - \frac{1}{2} \end{bmatrix}$. Since the error constant, $C_4 \neq 0$, it can be concluded that the derived method is order 3. By substituting (2.10) and (2.11) into (2.7), the BBDF-α is obtained as follows:

$$\begin{aligned}
y_{n+1} &= \left(\frac{-\alpha - \frac{1}{3}}{1-\alpha}\right) y_{n-1} + \left(\frac{2+\alpha}{1-\alpha}\right) y_n + \left(\frac{-\frac{2}{3} - \alpha}{1-\alpha}\right) y_{n+2} + \left(\frac{-2\alpha}{1-\alpha}\right) h f_n + \left(\frac{2+2\alpha}{1-\alpha}\right) h f_{n+1}, \\
y_{n+2} &= \left(\frac{\frac{3}{11}\alpha + \frac{2}{11}}{1+\frac{9}{11}\alpha}\right) y_{n-1} + \left(\frac{-\frac{9}{11} - \frac{15}{11}\alpha}{1+\frac{9}{11}\alpha}\right) y_n + \left(\frac{\frac{18}{11} + \frac{21}{11}\alpha}{1+\frac{9}{11}\alpha}\right) y_{n+1} + \left(\frac{-\frac{6}{11}\alpha}{1+\frac{9}{11}\alpha}\right) h f_{n+1} \\
&\quad + \left(\frac{\frac{6}{11} + \frac{6}{11}\alpha}{1+\frac{9}{11}\alpha}\right) h f_{n+2}.
\end{aligned} \tag{2.12}$$

3. Stability analysis

This section discusses the stability analysis of the method (2.12). The definitions of zero stability, absolute stability and A-stability are given by Lambert [1]. The LMM is said to be zero-stable if all the roots of the first characteristic polynomial have modulus less than or equal to unity, and those of modulus unity are simple. The stability properties of BBDF-α can be determined through the application of the standard linear test equation:

$$f = y' = \lambda y. \tag{3.1}$$

We substitute (3.1) into (2.12) to obtain

$$\begin{aligned}
&\left(\frac{2}{3} + \alpha\right) y_{n+2} + (1-\alpha)\, y_{n+1} - (2+2\alpha)\, h\lambda y_{n+1} \\
&\quad = -2\alpha h\lambda y_n - (-2-\alpha)\, y_n - \left(\alpha + \frac{1}{3}\right) y_{n-1}, \\
&\left(1 + \frac{9}{11}\alpha\right) y_{n+2} + \left(-\frac{18}{11} - \frac{21}{11}\alpha\right) y_{n+1} - \left(\frac{6}{11} + \frac{6}{11}\alpha\right) h\lambda y_{n+2} + \frac{6}{11}\alpha h\lambda y_{n+1} \\
&\quad = -\left(\frac{9}{11} + \frac{15}{11}\alpha\right) y_n - \left(-\frac{3}{11}\alpha - \frac{2}{11}\right) y_{n-1}.
\end{aligned} \tag{3.2}$$

By considering $\hat{h} = \lambda h$, equations (3.2) can be written in matrix form as follows:

$$\begin{bmatrix} 1-\alpha-2\hat{h}-2\alpha\hat{h} & \frac{2}{3}+\alpha \\ -\frac{18}{11} - \frac{21}{11}\alpha + \frac{6}{11}\alpha\hat{h} & 1 + \frac{9}{11}\alpha - \frac{6}{11}\hat{h} - \frac{6}{11}\alpha\hat{h} \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} -\alpha - \frac{1}{3} & 2+\alpha-2\alpha\hat{h} \\ \frac{3}{11}\alpha + \frac{2}{11} & -\frac{9}{11} - \frac{15}{11}\alpha \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix}, \tag{3.3}$$

where

$$A = \begin{bmatrix} 1-\alpha-2\hat{h}-2\alpha\hat{h} & \frac{2}{3}+\alpha \\ -\frac{18}{11}-\frac{21}{11}\alpha+\frac{6}{11}\alpha\hat{h} & 1+\frac{9}{11}\alpha-\frac{6}{11}\hat{h}-\frac{6}{11}\alpha\hat{h} \end{bmatrix},$$

$$B = \begin{bmatrix} -\alpha-\frac{1}{3} & 2+\alpha-2\alpha\hat{h} \\ \frac{3}{11}\alpha+\frac{2}{11} & -\frac{9}{11}-\frac{15}{11}\alpha \end{bmatrix}.$$

By solving $\det(At-B)=0$, the stability polynomial, $p\left(t,\hat{h},\alpha\right)$ is obtained as follows:

$$\begin{aligned} p(t,\hat{h},\alpha) =& \frac{6}{11}\alpha-2t+\frac{4}{11}\alpha\hat{h}-\frac{36}{11}t\alpha-\frac{20}{11}t\hat{h}+\frac{23}{11}t^2+\frac{30}{11}t^2\alpha-\frac{28}{11}t^2\hat{h}+\frac{12}{11}t^2\alpha^2 \\ &-\frac{24}{11}t\alpha^2+\frac{12}{11}t^2\hat{h}^2+\frac{12}{11}\alpha^2+\frac{6}{11}\alpha^2\hat{h}-\frac{8}{11}t\alpha\hat{h}-4t^2\alpha\hat{h}-\frac{18}{11}t^2\alpha^2\hat{h} \\ &+\frac{24}{11}t^2\alpha\hat{h}^2+\frac{12}{11}t^2\alpha^2\hat{h}^2+\frac{12}{11}t\alpha^2\hat{h}-\frac{12}{11}t\alpha^2\hat{h}^2-\frac{1}{11}. \end{aligned} \tag{3.4}$$

To determine the zero stability of the BBDF-α, we set $\hat{h}=0$ in (3.4) and obtain the following equation:

$$p(t,\alpha)=\frac{6}{11}\alpha-2t-\frac{36}{11}t\alpha+\frac{23}{11}t^2+\frac{30}{11}t^2\alpha+\frac{12}{11}t^2\alpha^2-\frac{24}{11}t\alpha^2+\frac{12}{11}\alpha^2-\frac{1}{11}. \tag{3.5}$$

Then, we solve (3.5) to produce the following roots:

$$t_1=1, \qquad t_2=\frac{6\alpha-1+12\alpha^2}{30\alpha+12\alpha^2+23}.$$

Since t_2 possesses α, the graph of t_2 in relation to some values of α is plotted in Figure 1. To view the graph clearly, the values of α in the range of $[-5,5]$ are selected.

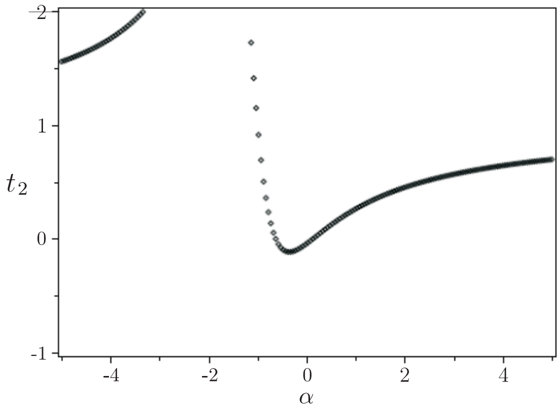

Figure 1: Graph of t_2 versus α for BBDF-α.

It can be seen that if $\alpha \le -1$, then $t_2 \ge 1$. If $\alpha > -1$, then $t_2 < 1$. Therefore, we conclude that the BBDF-α is zero-stable when $\alpha \in [-1,\infty)$ where ∞ is referred to the largest positive values which satisfy the roots of zero stability. The LMM is said to be absolutely stable in a region

Table 1: Roots of stability polynomial, t_1 and t_2 for $\alpha = 0.3, 3.0$.

α	$\hat{h}$	t_1	t_2
0.3	-5	0.06999812281	-0.129188236600
	-2	0.08146584082	-0.098279291580
	-0.5	0.36622060960	0.046304575120
	0	1.00000000000	0.056831922000
	0.5	2.68004160800	0.061296504180
	2	2.80135289800	0.066994440480
	5	0.44398549880	0.073101439110
3.0	-5	0.56256964650	-0.054953902280
	-2	0.56242867270	-0.007621569259
	-0.5	0.55937114810	0.382489317000
	0	1.00000000000	0.565610859000
	0.5	2.59326884300	0.564138564800
	2	1.31950888300	0.564549087500
	5	0.56170017910	0.237478947100

$\Re$ (real part) of the complex plane if, for all $\hat{h} \in \Re$, all roots of the stability polynomial, $p\left(t, \hat{h}\right)$ associated with the method, satisfy $|t_s| < 1, s = 1, 2, \cdots, k$. For instance, we choose $\alpha = 0.3, 3.0$ and $\hat{h} \in [-5, 5]$ to determine the absolute stability of the proposed method. By evaluating t in (3.4), the roots of stability polynomial are presented in Table 1.

From Table 1, it can be seen that the BBDF-α is absolutely stable for certain region which most at the negative real part of complex plane. If its region of absolute stability (or simply the stability region) contains the whole of the left half-plane, $\text{Re}\left(\hat{h}\right) < 0$, the method has A-stable property. In Figure 2, the graph of stability region for $\alpha = 0.3, 3.0$ is illustrated by the set of points which is determined using the boundary, $t = e^{i\theta}$, where $0 \le \theta \le 2\pi$. However, there are some values of α that are not to be considered since the boundaries of the stability region do not intersect the $\text{Re}\left(\hat{h}\right)$ axis.

Figure 2 demonstrates the comparison of stability regions between BBDF-$\alpha = 0.3, 3.0$ and BBDF by Ibrahim et al. [7]. The intervals of unstable region for BBDF-$\alpha = 3.0$, BBDF-$\alpha = 0.3$ and BBDF are $[0, 2.29]$, $[0, 3.25]$ and $[0, 4.0]$ respectively. It has to be noted that for every formulas, the stability region lies outside the closed regions, while the unstable region lies inside the circles. It can be observed that the BBDF-α has smaller unstable region compared to the BBDF. Since the stability region of BBDF-$\alpha = 0.3, 3.0$ covers the entire left half-plane, the proposed method is A-stable, thus it is suitable for solving stiff problems. On the other hand, it is a fact that the parameter α appears in the error constant, C_4 will somehow affect the magnitude of the truncation error. Therefore, it is advisable to choose the suitable value

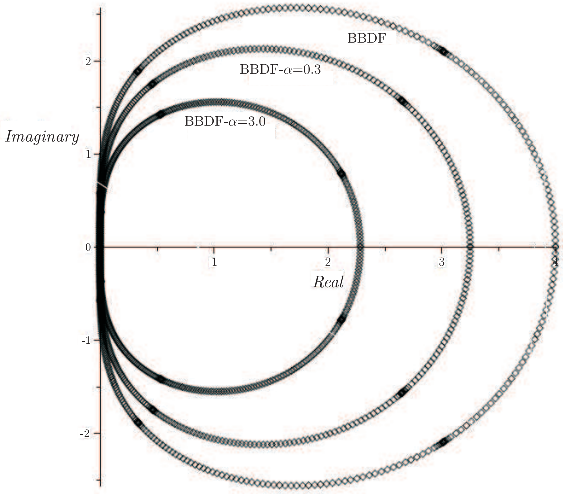

Figure 2: The graph of stability regions for BBDF and BBDF-α.

of α without sacrificing the accuracy and A-stability property.

4. Convergence properties

Lambert [1] stated that the convergence of the LMM requires consistency and zero stability. It is known that the LMM is said to be consistent if it has at least order one. Since the derived method is order 3, we can conclude that the BBDF-α is consistent. In Section 3, the zero stability of the derived method is proven. This property, together with the consistency shows that the derived method is convergent.

5. Implementation of the method

In this section, the Newton iteration is applied to implement the derived method. The corrector formulas (2.12) can be written as

$$
\begin{aligned}
F_1 &= y_{n+1} - \left(\frac{-\frac{2}{3}-\alpha}{1-\alpha}\right) y_{n+2} - \left(\frac{2+2\alpha}{1-\alpha}\right) h f_{n+1} - \mu_1,\\
F_2 &= y_{n+2} - \left(\frac{\frac{18}{11}+\frac{21}{11}\alpha}{1+\frac{9}{11}\alpha}\right) y_{n+1} - \left(\frac{\frac{6}{11}+\frac{6}{11}\alpha}{1+\frac{9}{11}\alpha}\right) h f_{n+2} - \left(\frac{-\frac{6}{11}\alpha}{1+\frac{9}{11}\alpha}\right) h f_{n+1} - \mu_2,
\end{aligned}
\tag{5.1}
$$

where μ_1 and μ_2 are the known previous values. The $y^{(i+1)}_{n+1}$ denotes the $(i+1)^{th}$ iterative value of y_{n+1} while the $y^{(i+1)}_{n+2}$ denotes the $(i+1)^{th}$ iterative value of y_{n+2}. Then we consider

$$
e^{(i+1)}_{n+1} = y^{(i+1)}_{n+1} - y^{(i)}_{n+1}, e^{(i+1)}_{n+2} = y^{(i+1)}_{n+2} - y^{(i)}_{n+2}. \tag{5.2}
$$

Newton iteration takes the form

$$y_{n+1}^{(i+1)} = y_{n+1}^{(i)} - F_1\left(y_{n+1}^{(i)}\right)\left[F_1'\left(y_{n+1}^{(i)}\right)\right]^{-1}, y_{n+2}^{(i+1)} = y_{n+2}^{(i)} - F_2\left(y_{n+2}^{(i)}\right)\left[F_2'\left(y_{n+2}^{(i)}\right)\right]^{-1}. \tag{5.3}$$

We substitute (5.1) and (5.2) into (5.3) to produce

$$\left[1+\left(\frac{2+2\alpha}{1-\alpha}\right)h\frac{\partial f_{n+1}}{\partial y_{n+1}}\right]e_{n+1}^{(i+1)} = -y_{n+1}^{(i)} + \left(\frac{-\frac{2}{3}-\alpha}{1-\alpha}\right)y_{n+2}^{(i)} + \left(\frac{2+2\alpha}{1-\alpha}\right)hf_{n+1}^{(i)} + \mu_1, \tag{5.4}$$

$$\begin{aligned}\left[1-\left(\frac{\frac{6}{11}+\frac{6}{11}\alpha}{1+\frac{9}{11}\alpha}\right)h\frac{\partial f_{n+2}}{\partial y_{n+2}}\right]e_{n+2}^{(i+1)} &= -y_{n+2}^{(i)} + \left(\frac{\frac{18}{11}+\frac{21}{11}\alpha}{1+\frac{9}{11}\alpha}\right)y_{n+1}^{(i)} + \left(\frac{\frac{6}{11}+\frac{6}{11}\alpha}{1+\frac{9}{11}\alpha}\right)hf_{n+2}^{(i)} \\ &\quad + \left(\frac{-\frac{6}{11}\alpha}{1+\frac{9}{11}\alpha}\right)hf_{n+1}^{(i)} + \mu_2.\end{aligned} \tag{5.5}$$

Hence, equations (5.4) and (5.5) can be written in the following matrix form

$$\begin{aligned}&\begin{bmatrix} 1+\left(\frac{2+2\alpha}{1-\alpha}\right)h\frac{\partial f_{n+1}}{\partial y_{n+1}} & 0 \\ 0 & 1-\left(\frac{\frac{6}{11}+\frac{6}{11}\alpha}{1+\frac{9}{11}\alpha}\right)h\frac{\partial f_{n+2}}{\partial y_{n+2}}\end{bmatrix}\begin{bmatrix} e_{n+1}^{(i+1)} \\ e_{n+2}^{(i+1)}\end{bmatrix} \\ &= \begin{bmatrix} -1 & \frac{-\frac{2}{3}-\alpha}{1-\alpha} \\ \frac{\frac{18}{11}+\frac{21}{11}\alpha}{1+\frac{9}{11}\alpha} & -1\end{bmatrix}\begin{bmatrix} y_{n+1}^{(i)} \\ y_{n+2}^{(i)}\end{bmatrix} + h\begin{bmatrix} \frac{-2-2\alpha}{1-\alpha} & 0 \\ \frac{\frac{6}{11}\alpha}{1+\frac{9}{11}\alpha} & \frac{\frac{6}{11}+\frac{6}{11}\alpha}{1+\frac{9}{11}\alpha}\end{bmatrix}\begin{bmatrix} f_{n+1}^{(i)} \\ f_{n+2}^{(i)}\end{bmatrix} + \begin{bmatrix}\mu_1 \\ \mu_2\end{bmatrix},\end{aligned} \tag{5.6}$$

where $e_{n+1}^{(i+1)} = y_{n+1}^{(i+1)} - y_{n+1}^{(i)}$ and $e_{n+2}^{(i+1)} = y_{n+2}^{(i+1)} - y_{n+2}^{(i)}$ as the increment.

6. Numerical results

The numerical results are usually presented by comparing several methods based on computing the error only at the endpoint of interval. In this section, the values of maximum error, average error and computational time over the whole interval are presented in Tables 2-4. To test the performance of BBDF-α, the values of $\alpha = 0.3, 3.0$ are considered. The graphs of Log(MAXE) against Log(TIME) are illustrated in Figures 3-5. The notations used in tables and figures are as follow:

h: Step size
MAXE: Maximum of absolute error
AVER: Average of error
TIME: Time execution (microseconds)
BBDF: Block backward differentiation formulas in Ibrahim et al. [6]
BBDF-α: Block backward differentiation α-formulas

Three stiff problems with oscillating solutions are given as follows:
Problem 1 (Stiff):

$$y' = 100\left(\sin x - y\right), 0 \le x \le 3.$$

Initial value: $y(0) = 0$.
Eigenvalue: $\lambda = -100$.
Exact solution:

$$y(x) = \frac{\sin x - 0.01\cos x + 0.01e^{-100x}}{1.0001}.$$

Source: Ibrahim et al. [6].

Problem 2 (Stiff linear system):

$$y_1' = y_3, y_2' = y_4, y_3' = -y_1, y_4' = -1000y_2, 0 \le x \le 3.$$

Initial value: $y_1(0) = 0, y_2(0) = 0, y_3(0) = 1, y_4(0) = 0$.
Eigenvalues: $\lambda = -1, 1, 10\sqrt{10}, -10\sqrt{10}$.
Exact solution: $y_1(x) = \sin x, y_2(x) = 0, y_3(x) = \cos x, y_4(x) = 0$.
Source: Franco et al. [3].

Problem 3 (Stiff non-linear system):

$$\begin{aligned} y_1' &= y_3, y_2' = y_4, y_3' = -y_1 + \frac{1}{10}\left({y^2}_1 + {y^2}_2 + {y^2}_3 + {y^2}_4 - 1\right), \\ y_4' &= -1000y_2 + \frac{1}{10}\left({y^2}_1 + {y^2}_2 + {y^2}_3 + {y^2}_4 - 1\right), 0 \le x \le 3. \end{aligned}$$

Initial value: $y_1(0) = 1, y_2(0) = 0, y_3(0) = 0, y_4(0) = 0$.
Eigenvalues:$\lambda = -1, 1, 10\sqrt{10}, -10\sqrt{10}$.
Exact solution: $y_1(x) = \cos x, y_2(x) = 0, y_3(x) = -\sin x, y_4(x) = 0$.
Source: Franco et al. [3].

The numerical results demonstrate the performance of the BBDF and BBDF-α in terms of maximum of absolute error, average error and computational time. Generally, the BBDF-α gives more precise approximation as the step size decreases. For all tested problems, it can be observed that the BBDF-α outperforms the BBDF method in terms of maximum error and average error. In Table 2, although both methods show a comparable values of maximum error at $h = 10^{-2}, 10^{-3}$, the BBDF-α manages to reduce the error at $h = 10^{-4}, 10^{-5}$ significantly. Furthermore, it can be seen obviously that the BBDF-α has smaller values of average error compared to the BBDF for all step sizes. In terms of execution time, the BBDF-α needs slightly longer computational time than the BBDF for all tested problems. This is expected since the BBDF-α has parameters and more coefficients in the formulas, thus it requires extra effort to complete the computation.

Table 2: Numerical results for Problem 1.

h	Method	MAXE	AVER	TIME
10^{-2}	BBDF	7.324899e-04	1.874597e-04	145
	BBDF-$\alpha = 0.3$	1.826637e-04	2.593747e-05	160
	BBDF-$\alpha = 3.0$	1.826164e-04	4.260650e-06	252
10^{-3}	BBDF	5.671098e-04	1.781096e-05	947
	BBDF-$\alpha = 0.3$	1.208403e-04	1.834959e-06	1419
	BBDF-$\alpha = 3.0$	1.682939e-04	3.756808e-06	2000
10^{-4}	BBDF	7.183008e-05	1.964093e-06	9489
	BBDF-$\alpha = 0.3$	1.666201e-06	2.557606e-08	14088
	BBDF-$\alpha = 3.0$	3.143596e-06	5.641789e-08	16891
10^{-5}	BBDF	7.339910e-06	1.984082e-07	94538
	BBDF-$\alpha = 0.3$	1.739445e-08	2.648204e-10	139597
	BBDF-$\alpha = 3.0$	3.329428e-08	5.888808e-10	148838

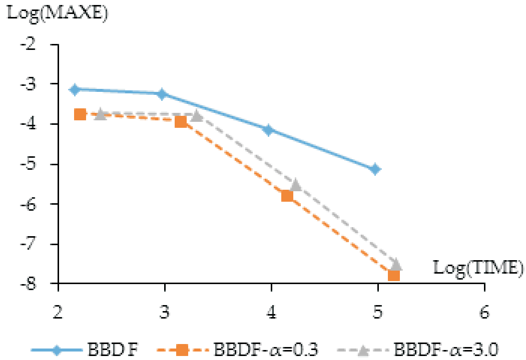

Figure 3: The graph of Log(MAXE) versus Log(TIME) for Problem 1.

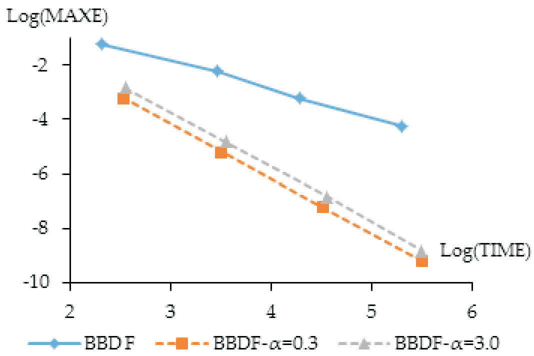

Figure 4: The graph of Log(MAXE) versus Log(TIME) for Problem 2.

Table 3: Numerical results for Problem 2.

h	Method	MAXE	AVER	TIME
10^{-2}	BBDF	5.965608e-02	3.838632e-02	209
	BBDF-$\alpha = 0.3$	6.392246e-04	4.472969e-04	333
	BBDF-$\alpha = 3.0$	1.476713e-03	9.790988e-04	354
10^{-3}	BBDF	5.943627e-03	3.875837e-03	2948
	BBDF-$\alpha = 0.3$	6.475903e-06	4.555039e-06	3179
	BBDF-$\alpha = 3.0$	1.507500e-05	1.016446e-05	3572
10^{-4}	BBDF	5.940333e-04	3.879181e-04	19113
	BBDF-$\alpha = 0.3$	6.484130e-08	4.564160e-08	31901
	BBDF-$\alpha = 3.0$	1.510489e-07	1.020270e-07	36072
10^{-5}	BBDF	5.939994e-05	3.879513e-05	199272
	BBDF-$\alpha = 0.3$	6.473784e-10	4.499082e-10	315116
	BBDF-$\alpha = 3.0$	1.516417e-09	1.022879e-09	315435

Table 4: Numerical results for Problem 3.

h	Method	MAXE	AVER	TIME
10^{-2}	BBDF	-	-	-
	BBDF-$\alpha = 0.3$	5.159812e-04	4.336740e-04	376
	BBDF-$\alpha = 3.0$	1.082598e-03	9.759240e-04	403
10^{-3}	BBDF	4.946086e-03	3.321793e-03	2142
	BBDF-$\alpha = 0.3$	5.235607e-06	4.368993e-06	3680
	BBDF-$\alpha = 3.0$	1.105587e-05	9.612067e-06	3903
10^{-4}	BBDF	4.942338e-04	3.308826e-04	20691
	BBDF-$\alpha = 0.3$	5.243138e-08	4.378260e-08	36513
	BBDF-$\alpha = 3.0$	1.107903e-07	9.649800e-08	37882
10^{-5}	BBDF	4.941958e-05	3.309038e-05	216621
	BBDF-$\alpha = 0.3$	5.261320e-10	4.334403e-10	367761
	BBDF-$\alpha = 3.0$	1.111623e-09	9.664590e-10	364554

7. Conclusions

We have derived the block backward differentiation α formulas, namely BBDF-α for solving first order stiff ODEs possessing oscillating solutions. The stability analysis and convergence properties were investigated theoretically. The numerical results indicate that the BBDF-α has better accuracy than the BBDF method cited in the literature. The influence of independent parameter α adopted in the coefficients of BBDF has shown tremendous improvement in approximation of stiff and oscillatory IVPs. Therefore, it would be of interest

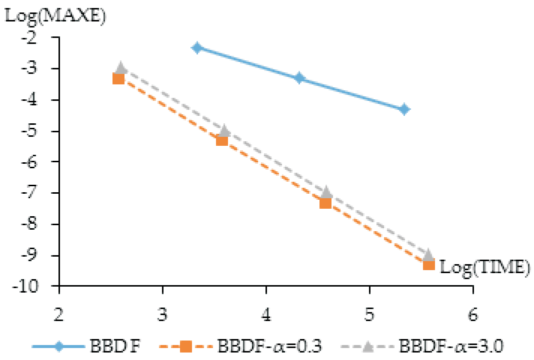

Figure 5: The graph of Log(MAXE) versus Log(TIME) for Problem 3.

to study further applications such as damped and undamped oscillatory problems in mass-spring systems or electrical circuits.

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References

[1] J. D. Lambert, Computational Methods in Ordinary Differential Equations, Wiley, London, 1973.

[2] J. Sunday, M. R. Odekunle, A. A. James and A. O. Adesanya, *Numerical solution of stiff and oscillatory differential equations using a block integrator*, Brit. J. Math. Comput. Sci., **4**(17) (2014), 2471–2481.

[3] J. M. Franco, I. Gomez and L. Randez, *SDIRK methods for stiff ODES with oscillating solutions*, J. Comput. Appl. Math., **81** (1997), 197–209.

[4] A. Tahmasbi, *Numerical solutions for stiff ordinary differential equations systems*, I. Math. Forum., **3** (2008), 703–711.

[5] Z. B. Ibrahim, Block method for multistep formulas for solving ordinary differential equations, Universiti Putra Malaysia, Ph.D thesis, 2006.

[6] Z. B. Ibrahim, M. Suleiman and K. I. Othman, *Fixed coefficients block backward differentiation formulas for the numerical solution of stiff ordinary differential equations*, Eur. J. Sci. Res., **21**(3) (2008), 508–520.

[7] Z. B. Ibrahim, M. B. Suleiman and K. I. Othman, *Implicit r point block backward differentiation formula for solving first order stiff ordinary differential equations*, Appl. Math. Comput., **186** (2007), 558–565.

[8] N. A. A. M. Nasir, Z. B. Ibrahim, K. I. Othman and M. Suleiman, *Fifth order two-point block backward differentiation formulas for solving ordinary differential equations*, Appl. Math. Sci., **5**(71) (2011), 3505–3518.

[9] N. A. A. M. Nasir, Z. B. Ibrahim, K. I. Othman and M. Suleiman, *Numerical solution of first order stiff ordinary differential equations using fifth order block backward differentiation formulas*, Sains Malays., **41** (2012), 489–492.

[10] J. Sunday, Y. Skwame and M. R. Odekunle, *A continuous block integrator for the solution of stiff and oscillatory differential equations*, IOSR J. Math., **8**(3) (2013), 75–80.

[11] I. S. M. Zawawi, Z. B. Ibrahim and K. I. Othman, *Derivation of diagonally implicit block backward differentiation formulas for solving stiff initial value problem*, Math. Probl. Eng., **2015** (2015), Article ID 179231, 13 pages.

[12] I. S. M. Zawawi, Z. B. Ibrahim, F. Ismail and Z. A. Majid, *Diagonally implicit block backward differentiation formulas for solving ordinary differential equations*, Int. J. Math. Math. Sci., **2012** (2012), Article ID 767328, 8 pages.

[13] E. A. Celaya and J. J. Anza, *BDF-α: A multistep method with numerical damping control*, Universal J. Comput. Math., **1** (2013), 96–108.

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