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UPPER BOUNDS FOR RICCI CURVATURES FOR SUBMANIFOLDS IN BOCHNER-KAEHLER MANIFOLDS

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Abstract. Chen established the relationship between the Ricci curvature and the squared norm of mean curvature vector for submanifolds of Riemannian space form with arbitrary codimension known as Chen-Ricci inequality. Deng improved the inequality for Lagrangian submanifolds in complex space form by using algebraic technique. In this paper, we establish the same inequalities for different submanifolds of Bochner-Kaehler manifolds. Moreover, we obtain improved Chen-Ricci inequality for Kaehlerian slant submanifolds of Bochner-Kaehler manifolds.

1. Introduction

One of the most powerful tools to find relationships between intrinsic invariants and extrinsic invariants of a submanifold is provided by Chen's invariants.The study of Chen invariants and inequalities has been an active field of research over the past two decades. Chen [8] investigated sharp relationship between the Ricci curvature and the squared norm of mean curvature for a submanifold in a Riemannian space form with arbitrary codimension. Tripathi [27] named this inequality as Chen-Ricci inequality. Matsumoto et al. [18] obtained Chen-Ricci inequality for submanifolds in complex space form. Matsumoto et al. [19] obtained the same inequality for the slant submanifolds of complex space form. After that, many research articles have been published by different geometers in this direction (see [26, 21, 24]). They obtained the similar inequalities for different submanifolds and ambient spaces in complex as well as in contact version.

Deng [10] improved the Chen-Ricci inequality for Lagrangian submanifolds in complex space form by using algebraic technique. The same author obtained the improved Chen-Ricci inequality for Quaternion space forms [11]. Mihai et al. [22] obtained the improved Chen-Ricci Inequality for Kaehlerian slant submanifolds in complex space form. Mihai [20]

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generalizes the same inequality for Lagrangian submanifolds of complex space form and Legendrian submanifolds in a Sasakian space form with semi-symmetric metric connections.

In 1949, Bochner [1] introduced a complex analogue of the Weyl conformal curvature tensor for a Kaehler manifold. This tensor is the largest irreducible component of the Riemannian curvature under the unitary group. A Kaehler metric with vanishing Bochner curvature tensor is said to be a Bochner-Kaehler metric [9]. In a seminal paper published in 2001, Bryant [2] provides an explicit local classification of Bochner-Kaehler metric and in depth study of their global geometry, generating considerable interest on this kind of manifolds (see [3, 12, 17, 23, 29]). In particular, we note that Inoue investegated [15] penrose transformation on Hermitian manifolds that are conformal to Bochner-Kaehler manifolds, using the modification of the O'Brien-Rawnsley twistor space for almost Hermitian manifolds.

There are several classes of submanifolds in Bochner-Kaehler manifolds that were investigated by many geometers: totally real submanifolds [14], anti-invariant submanifolds [28], CR-submanifolds [25] and contact hypersurfaces [13] etc.

In the first part of the paper, we obtain the Chen-Ricci inequality for submanifolds of Bochner-Kaehler manifolds and discuss the results for invariant, anti-invariant and slant submanifolds. In the second part, we improve the inequality for Kaehlerian slant submanifolds of Bochner-Kaehler manifolds using Deng's technique.

2. Preliminaries

Let $\mathcal{M}^n$ be a submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{M}}^m$. Let ∇ and $\overline{\nabla}$ be the Riemannian connections on $\mathcal{M}^n$ and $\overline{\mathcal{M}}^m$ respectively. Then the Gauss and Weingarten formulas are given by

$$\overline{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1}$$

$$\overline{\nabla}_X V = -A_V X + D_X Y, \tag{2}$$

for all X, Y tangent to $\mathcal{M}^n$ and vector field V normal to $\mathcal{M}^n$, where h, D, A_V denotes the second fundamental form, normal connection and the shape operator in the direction of V. The second fundamental form and the shape operator are related by

$$g(h(X,Y),V) = g(A_V X, Y). \tag{3}$$

Let $p \in \mathcal{M}^n$ and $\{e_1,\dots,e_n\}$ be an orthonormal basis of the tangent space $T_p\mathcal{M}^n$ and $\{e_{n+1},\dots,e_m\}$ be the orthonormal basis of $T^{\perp}\mathcal{M}^n$. We denote by H(the mean curvature vector) at p, that is

$$H(p) = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i). \tag{4}$$

Also, we set

$$h^r_{ij} = g(h(e_i, e_j), e_r), \qquad i, j \in \{1, \dots, n\}, \quad r \in \{n+1, \dots, m\}$$

and

$$\|h\|^2 = \sum_{i,j=1}^{n} (h(e_i, e_j), h(e_i, e_j)). \tag{5}$$

For any $p \in \mathcal{M}^n$ and $X \in \mathcal{M}^n$, we put $JX = PX + QX$, where PX and QX are the tangential and normal components of JX respectively.

We denote by

$$\|P\|^2 = \sum_{i,j=1}^{n} g^2(Pe_i, e_j),$$

For a Riemannian manifold $\mathcal{M}^n$, we denote by $K(\pi)$ the sectional curvature of $\mathcal{M}^n$ associated with a plane section $\pi \subset T_P\mathcal{M}^n, p \in \mathcal{M}^n$. For an orthonormal basis $\{e_1, e_2, \dots, e_n\}$ of the tangent space $T_p\mathcal{M}^n$, the scalar curvature ρ is defined by

$$\rho = \sum_{i<j} K_{ij},$$

where K_{ij} denotes the sectional curvature of the 2-plane section spanned by e_i and e_j.

We recall that for a submanifold $\mathcal{M}^n$ in a Riemannian manifold, the relative null space of $\mathcal{M}^n$ at a point $p \in \mathcal{M}^n$ is defined by

$$\mathcal{N}_p = \{X \in T_p\mathcal{M}^n | h(X, Y) = 0, \forall Y \in T_p\mathcal{M}^n\}.$$

Let R be the curvature tensor of $\mathcal{M}^n$, then the Gauss equation is given by

$$\overline{R}(X, Y, Z, W) = R(X, Y, Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)),$$

for any vector fields X, Y, Z, W tangent to $\mathcal{M}^n$.

The curvature tensor of $\overline{\mathcal{M}}^m$ is given by [25]

$$\begin{aligned}\overline{R}(X, Y, Z, W) = {} & L(Y, Z)g(X, W) - L(X, Z)g(Y, W) + L(X, W)g(Y, Z)\\ & -L(Y, W)g(X, Z) + M(X, W)g(JX, W) - M(X, Z)g(JY, W)\\ & +M(X, W)g(JY, Z) - M(Y, W)g(JX, Z)\\ & -2M(X, Y)g(JZ, W) - 2M(Z, W)g(JX, Y),\end{aligned} \tag{6}$$

where

$$L(Y, Z) = \frac{1}{2n+4}\overline{\mathcal{R}ic}(Y, Z) - \frac{\overline{\rho}}{2(2n+2)(2n+4)}g(Y, Z), \tag{7}$$

$$M(Y, Z) = -L(Y, JZ), \tag{8}$$

$$L(Y, Z) = L(Z, Y), \qquad L(Y, Z) = L(JY, JZ), \qquad L(Y, JZ) = -L(JY, Z), \tag{9}$$

$\overline{\mathcal{R}ic}$ and $\overline{\rho}$ are the Ricci tensor and scalar curvature of $\overline{\mathcal{M}}^m$.

Definition 2.1. The Kaehler manifold $\overline{\mathcal{M}}^m$ is said to be Bochner-Kaehler if its Bochner curvature tensor vanishes. These spaces are also known as Bochner-flat manifolds.

Definition 2.2. A Riemannian manifold $\mathcal{M}^n$ is said to be Einstein manifold if the Ricci tensor is proportional to the metric tensor, that is, $\mathcal{R}ic(X,Y) = \lambda g(X,Y)$ for some constant λ.

Definition 2.3. A submanifold $\mathcal{M}^n$ of a Bochner-Kaehler manifold $\overline{\mathcal{M}}^m$ is said to be a slant submanifold if for any $p \in \mathcal{M}^n$ and any non zero vector $X \in T_p\mathcal{M}^n$, the angle between JX and the tangent space $T_p\mathcal{M}^n$ is constant.

Invariant and anti-invariant submanifolds are the slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively and when $0 < \theta < \frac{\pi}{2}$, then slant submanifold is called proper slant submanifold.

A proper slant submanifold is said to be Kaehlerian slant if $\nabla P = 0$. A Kaehlerian slant submanifold is a Kaehler manifold with respect to the induced metric and an almost complex structure $\overline{J} = sec\theta J$. Let $\mathcal{M}^n$ be proper slant submanifold and $\{e_1, \dots, e_n\}$ be an orthonormal basis of T_pM. If $m = n$, an orthonormal basis $\{e_1^*, \dots, e_n^*\}$ of the normal space $T^\perp M^n$ is defined by

$$e_k^* = \frac{1}{sin\theta} Qe_k, \qquad k = 1, \dots, n. \tag{10}$$

For Kaehlerian slant submanifold we have [5]

$$A_{QX}Y = A_{QY}X \qquad \forall X, Y \in T_pM^n$$

or

$$h_{ij}^k = h_{ik}^j = h_{jk}^i \tag{11}$$

where A is the shape operator and

$$h_{ij}^k = g(h(e_i, e_j), e_k^*), \qquad i, j, k = 1, \dots, n. \tag{12}$$

Now, the propositions given below characterize the submanifolds with $\nabla P = 0$.

Proposition 2.1 ([5])**.** *Let $\mathcal{M}^n$ be a submanifold of an almost Hermitian manifold $\overline{\mathcal{M}}^m$. Then $\nabla P = 0$ if and only if M is locally the Riemannian product $M_1 \times M_2 \times \dots \times M_k$, where each M_i is either a complex submanifold, a totally real submanifold or a Kaehlerian slant submanifold of $\overline{M}$.*

Proposition 2.2 ([5])**.** *Let $\mathcal{M}^n$ be an irreducible submanifold of an almost Hermitian manifold $\overline{\mathcal{M}}^m$. If M is neither invariant nor totally real, then M is a Kaehlerian slant submanifold if and only if the endomorphism P is parallel.*

Definition 2.4. A slant H-umbilical submanifold of a Kaehler manifold $\overline{\mathcal{M}}^n$ is a slant submanifold for which the second fundamental form takes the following forms

$$h(e_1,e_1) = \lambda e_1^*, h(e_2,e_2) = \cdots = h(e_n,e_n) = \mu e_1^*$$
$$h(e_1,e_1) = \mu e_j^*, h(e_j,e_k) = 0, 2 \leq j \neq k \leq n,$$

where $e_1^*,\ldots,e_n^*$ defined as in (10).

3. Ricci curvature and squared norm of mean curvature

In this section, we prove some inequalities of Ricci curvatures for submanifolds of Bochner-Kaehler manifolds.

Theorem 3.1. *Let $\mathcal{M}^n$ be a submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{M}}^m$, then*

(i) *for each unit vector $X \in T_p\mathcal{M}$, we have*

$$\mathcal{R}ic(X) \leq \frac{1}{4}n^2\|H\|^2 + \frac{4n^3-12n^2-2n+10-(3n^2-9n+3)\|P\|^2}{2(2n+2)(2n+4)}\overline{\rho}$$
$$\frac{6}{2n+4}\sum_{2\leq i<j\leq n}\overline{\mathcal{R}}ic(e_i,Je_j)g(e_i,Je_j) - \frac{3}{2n+4}\sum_{i,j}\overline{\mathcal{R}}ic(e_i,Je_j)g(e_i,Je_j).$$

(ii) *If $H(p)=0$, the unit tangent vector X at p satisfies the equality if and only if $X \in \mathcal{N}_p$.*

(iii) *The equality case holds identically for all unit tangent vectors at p if and only if either p is totally geodesic point or $n=2$ and p is totally umbilical point.*

Proof. (i) Let $X \in T_p\mathcal{M}$ be a unit tangent vector at p. We choose orthonormal basis $\{e_1,e_2,\ldots,e_n,e_{n+1},\ldots,e_m\}$ such that $\{e_1,e_2,\ldots,e_n\}$ are tangent to $\mathcal{M}$ at p with $e_1 = X$, then from Gauss equation we have

$$\begin{aligned}R(X,Y,Z,W) = {} & L(Y,Z)g(X,W) - L(X,Z)g(Y,W) + L(X,W)g(Y,Z)\\ & -L(Y,W)g(X,Z) + M(Y,Z)g(JX,W) - M(X,Z)g(JY,W)\\ & +M(X,W)g(JY,Z) - M(Y,W)g(JX,Z) - 2M(X,Y)(JZ,W)\\ & -2M(Z,W)g(JX,Y) + g(h(X,W),h(Y,Z)) - g(h(X,Z),h(Y,W)),\end{aligned}$$

for any X, Y, Z, W ∈ TM. Therefore we may write

$$\begin{aligned}\sum_{i,j}R(e_i,e_j,e_j,e_i) = \sum_{i,j}\Big(& L(e_j,e_j)g(e_i,e_i) - L(e_i,e_j)g(e_j,e_i) + L(e_i,e_i)g(e_j,e_j)\\ & -L(e_j,e_i)g(e_i,e_j) + M(e_j,e_j)g(Je_i,e_i) - M(e_i,e_j)g(Je_j,e_i)\\ & +M(e_i,e_i)g(Je_j,e_j) - M(e_j,e_i)g(Je_i,e_j) - 2M(e_i,e_j)(Je_j,e_i)\end{aligned}$$

$$
\begin{aligned}
& -2M(e_j,e_i)g(Je_i,e_j)+g(h(e_i,e_i),h(e_j,e_j))-g(h(e_i,e_j),h(e_j,e_i)),\Big) \\
= & \sum_{i,j}\Big(L(e_j,e_j)g(e_i,e_i)-L(e_i,e_j)g(e_j,e_i)+L(e_i,e_i)g(e_j,e_j) \\
& -L(e_j,e_i)g(e_i,e_j)-L(e_j,Je_j)g(Je_i,e_i)+L(e_i,Je_j)g(Je_j,e_i) \\
& +L(e_i,Je_i)g(Je_j,e_j)+L(e_j,Je_i)g(Je_i,e_j)+2L(e_i,Je_j)(Je_j,e_i) \\
& +2L(e_j,Je_i)g(Je_i,e_j)+g(h(e_i,e_i),h(e_j,e_j))-g(h(e_i,e_j),h(e_j,e_i))\Big).
\end{aligned}
$$

Using(4), (5) and (9), we have

$$
\begin{aligned}
\sum_{i,j}R(e_i,e_j,e_j,e_i) = & 2n\sum_i L(e_i,e_i)-2\sum_{i,j}L(e_i,e_j)g(e_i,e_j)+6\sum_{i,j}L(e_i,Je_j)g(e_i,Je_j) \\
& +n^2\|H\|^2-\|h\|^2.
\end{aligned}
$$

Last equation simplifies to,

$$
2\rho=2(n-1)\sum_i L(e_i,e_i)+6\sum_{i,j}L(e_i,Je_j)g(e_i,Je_j)+n^2\|H\|^2-\|h\|^2 \tag{13}
$$

Combining (7) and (13), we have

$$
\begin{aligned}
2\rho = & \frac{2(n-1)}{2n+4}\sum_i \overline{\mathscr{R}}ic(e_i,e_i)-\frac{2(n-1)\overline{\rho}}{2(2n+2)(2n+4)}\sum_i g(e_i,e_i) \\
& +\frac{6}{2n+4}\sum_{i,j}\overline{\mathscr{R}}ic(e_i,Je_j)g(e_i,Je_j)-\sum_{i,j}\frac{6\overline{\rho}}{2(2n+2)(2n+4)}g(e_i,Je_j)g(e_i,Je_j) \\
& +n^2\|H\|^2-\|h\|^2.
\end{aligned}
$$

This implies that

$$
n^2\|H\|^2=2\rho-\Big(\frac{6n^2+2n-8-6\|P\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho}+\|h\|^2-\frac{6}{2n+4}\sum_{i,j}\overline{\mathscr{R}}ic(e_i,Je_j)g(e_i,Je_j).
$$

From which we have,

$$
n^2\|H\|^2=2\rho-\Big(\frac{6n^2+2n-8-6\|P\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho}+\sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(h_{ij}^r)^2-\frac{6}{2n+4}\sum_{i,j}\overline{\mathscr{R}}ic(e_i,Je_j)g(e_i,Je_j).
$$

that is

$$
\begin{aligned}
n^2\|H\|^2 = & 2\rho-\Big(\frac{6n^2+2n-8-6\|P\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho}+\sum_{r=n+1}^{m}\Big[(h_{11}^r)^2+(h_{22}^r)^2+\cdots+(h_{nn}^r)^2+2\sum_{i<j}(h_{ij}^r)^2\Big] \\
& -\frac{6}{2n+4}\sum_{i,j}\overline{\mathscr{R}}ic(e_i,Je_j)g(e_i,Je_j).
\end{aligned}
$$

Which simplifies to

$$n^2\|H\|^2 = 2\rho - \Big(\frac{6n^2+2n-8-6\|P\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} + \sum_{r=n+1}^{m}\big[(h_{11}^r)^2+(h_{22}^r)^2+\cdots+(h_{nn}^r)^2\big]$$
$$+2\sum_{r=n+1}^{m}\sum_{i<j}(h_{ij}^r)^2 - \frac{6}{2n+4}\sum_{i,j}\overline{\mathscr{R}ic}(e_i,Je_j)g(e_i,Je_j).$$

or

$$n^2\|H\|^2 = 2\rho - \Big(\frac{6n^2+2n-8-6\|P\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} + \sum_{r=n+1}^{m}\Big[(h_{11}^r)^2+(h_{22}^r+\cdots+h_{nn}^r)^2 - 2\sum_{2\le i<j\le n}h_{ii}^r h_{jj}^r\Big]$$
$$+2\sum_{r=n+1}^{m}\sum_{i<j}(h_{ij}^r)^2 - \frac{6}{2n+4}\sum_{i,j}\overline{\mathscr{R}ic}(e_i,Je_j)g(e_i,Je_j).$$

From which we derive that,

$$n^2\|H\|^2 = 2\rho - \Big(\frac{6n^2+2n-8-6\|P\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho} + \frac{1}{2}\sum_{r=n+1}^{m}\Big[(h_{11}^r+h_{22}^r+\cdots+h_{nn}^r)^2$$
$$+(h_{11}^r-h_{22}^r-\cdots-h_{nn}^r)^2\Big] - 2\sum_{r=n+1}^{m}\sum_{2\le i<j\le n}h_{ii}^r h_{jj}^r + 2\sum_{r=n+1}^{m}\sum_{i<j}(h_{ij}^r)^2$$
$$-\frac{6}{2n+4}\sum_{i,j}\overline{\mathscr{R}ic}(e_i,Je_j)g(e_i,Je_j).$$

or

$$n^2\|H\|^2 = 2\rho - \Big(\frac{6n^2+2n-8-6\|P\|^2}{2(2n+2)(2n+4)}\Big)\overline{\rho}$$
$$+\frac{1}{2}\sum_{r=n+1}^{m}(h_{11}^r+h_{22}^r+\cdots+h_{nn}^r)^2 + \frac{1}{2}\sum_{r=n+1}^{m}(h_{11}^r-h_{22}^r-\cdots-h_{nn}^r)^2$$
$$+2\sum_{r=n+1}^{m}\sum_{j=1}^{n}(h_{1j}^r)^2 - 2\Big[\sum_{r=n+1}^{m}\sum_{2\le i<j\le n}h_{ii}^r h_{jj}^r-(h_{ij}^r)^2\Big]$$
$$-\frac{6}{2n+4}\sum_{i,j}\overline{\mathscr{R}ic}(e_i,Je_j)g(e_i,Je_j). \tag{14}$$

Also, from Gauss equation, we have

$$K_{ij} = 2L(e_i,e_i)+6L(e_i,e_i)g(e_i,Je_j) + \sum_{r=n+1}^{m}\Big[h_{ii}^r h_{jj}^r-(h_{ij}^r)^2\Big]$$
$$= \frac{2}{2n+4}\overline{\mathscr{R}ic}(e_i,e_i) - \frac{2\overline{\rho}}{2(2n+2)(2n+4)}g(e_i,e_i) + \frac{6}{2n+4}\overline{\mathscr{R}ic}(e_i,Je_j)g(e_i,Je_j)$$
$$-\frac{6\overline{\rho}}{2(2n+2)(2n+4)}g(e_i,Je_j)g(e_i,Je_j) + \sum_{r=n+1}^{m}\Big[h_{ii}^r h_{jj}^r-(h_{ij}^r)^2\Big]$$
$$= \frac{8n+6-6\|p\|^2}{2(2n+2)(2n+4)}\overline{\rho} + \frac{6}{2n+4}\overline{\mathscr{R}ic}(e_i,Je_j)g(e_i,Je_j) + \sum_{r=n+1}^{m}\Big[h_{ii}^r h_{jj}^r-(h_{ij}^r)^2\Big]$$

and consequently

$$\sum_{2\le i<j\le n}K_{ij} = \frac{4n^3-9n^2-n+6-(3n^2-9n+6)\|p\|^2}{2(2n+2)(2n+4)}\overline{\rho}$$

$$+\frac{6}{2n+4}\sum_{2\le i<j\le n}\overline{\mathscr{R}}ic(e_i,Je_j)g(e_i,Je_j)+\sum_{r=n+1}^{m}\sum_{2\le i<j\le n}\left[h^r_{ii}h^r_{jj}-(h^r_{ij})^2\right]. \quad (15)$$

Incorporating (15) in (14), we get

$$\frac{1}{2}n^2\|H\|^2\ge 2\mathscr{R}ic(X)-\frac{6n^2+2n-8-6\|P\|^2}{2(2n+2)(2n+4)}\overline{\rho}+2\frac{4n^3-9n^2-n+6-(3n^2-9n+6)\|P\|^2}{2(2n+2)(2n+4)}\overline{\rho}$$
$$\frac{12}{2n+4}\sum_{2\le i<j\le n}\overline{\mathscr{R}}ic(e_i,Je_j)g(e_i,Je_j)-\frac{6}{2n+4}\sum_{i,j}\overline{\mathscr{R}}ic(e_i,Je_j)g(e_i,Je_j)$$

or,

$$\mathscr{R}ic(X)\le\frac{1}{4}n^2\|H\|^2+\frac{4n^3-12n^2-2n+10-(3n^2+9n-3)\|P\|^2}{2(2n+2)(2n+4)}\overline{\rho}$$
$$\frac{6}{2n+4}\sum_{2\le i<j\le n}\overline{\mathscr{R}}ic(e_i,Je_j)g(e_i,Je_j)-\frac{3}{2n+4}\sum_{i,j}\overline{\mathscr{R}}ic(e_i,Je_j)g(e_i,Je_j)$$

(ii) Suppose $H(p)=0$, equality holds if and only if

$$\begin{cases}h^r_{12}=\cdots=h^r_{1n}=0,\\ h^r_{11}=h^r_{22}+\cdots+h^r_{nn}, r\in\{n+1,\dots,m\}\end{cases}$$

Then $h^r_{1j}=0\forall j\in\{1,2,\dots,n\}, r\in\{n+1,\dots,m\}$, i.e. $X\in\mathscr{N}$.

(iii) The equality case holds for all unit vectors at p if and only if

$$\begin{cases}h^r_{12}=0, i\neq j, r\in\{n+1,\dots,m\},\\ h^r_{11}+\cdots+h^r_{nn}-2h^r_{ij}=0, i\in\{1,2,\dots,n\}, r\in\{n+1,\dots,m\}\end{cases}$$

We distinguish two cases:

(a) $n\neq 2$, then p is a totally geodesic point

(b) n=2, it follows that p is a totally umbilical point.

The converse is trivial. □

The following proposition follows from the theorem 3.1, if the submanifold $\mathscr{M}^n$ is Einstein.

Proposition 3.2. *Let $\mathscr{M}^n$ be a submanifold of a Bochner-Kaehler manifold $\overline{\mathscr{M}}^m$ which is Einstein, then*

(i) *For each unit vector $X\in T_p\mathscr{M}$, we have*

$$\mathscr{R}ic(X)\le\frac{1}{4}n^2\|H\|^2+\frac{4n^3-12n^2-2n+10-(3n^2-9n+3)\|P\|^2}{2(2n+2)(2n+4)}\overline{\rho}+\frac{3}{2n+2}\lambda\|P\|^2.$$

(ii) *If $H(p)=0$, the unit tangent vector X at p satisfies the equality if and only if $X\in\mathscr{N}_p$.*

(iii) *The equality case holds identically for all unit tangent vectors at p if and only if either p is totally geodesic point or $n = 2$ and p is totally umbilical point.*

If $\mathscr{M}^n$ is a slant submanifold of $\mathscr{M}^m$, we have the following theorem.

Theorem 3.3. *Let $\mathscr{M}^n$ be a slant submanifold of a Bochner-Kaehler manifold $\overline{\mathscr{M}}^m$, then*

(i) *for each unit vector $X \in T_p\mathscr{M}$, we have*

$$\mathscr{R}ic(X) \leq \frac{1}{4}n^2\|H\|^2 + \frac{4n^3 - 12n^2 - 2n + 10 - (3n^2 - 9n + 3)\cos^2\theta}{2(2n+2)(2n+4)}\overline{\rho} + \frac{6\cos\theta}{2n+4}\sum_{2\leq i<j\leq n}\overline{\mathscr{R}}ic(e_i, Je_j) - \frac{3}{2n+4}\cos\theta\sum_{i,j}\overline{\mathscr{R}}ic(e_i, Je_j).$$

(ii) *If $H(p) = 0$, the unit tangent vector X at p satisfies the equality if and only if $X \in \mathscr{N}_p$.*

(iii) *The equality case holds identically for all unit tangent vectors at p if and only if either p is totally geodesic point or $n = 2$ and p is totally umbilical point.*

Following corollaries can be deduced from the last theorem.

Corollary 3.4. *Let $\mathscr{M}^n$ be an anti-invariant submanifold of a Bochner-Kaehler manifold $\overline{\mathscr{M}}^m$, then*

(i) *for each unit vector $X \in T_p\mathscr{M}$, we have*

$$\mathscr{R}ic(X) \leq \frac{1}{4}n^2\|H\|^2 + \frac{4n^3 - 12n^2 - 2n + 10}{2(2n+2)(2n+4)}\overline{\rho}.$$

(ii) *If $H(p) = 0$, the unit tangent vector X at p satisfies the equality if and only if $X \in \mathscr{N}_p$.*

(iii) *The equality case holds identically for all unit tangent vectors at p if and only if either p is totally geodesic point or $n = 2$ and p is totally umbilical point.*

Corollary 3.5. *Let $\mathscr{M}^n$ be a invariant submanifold of a Bochner-Kaehler manifold $\overline{\mathscr{M}}^m$, then*

(i) *for each unit vector $X \in T_p\mathscr{M}$, we have*

$$\mathscr{R}ic(X) \leq \frac{1}{4}n^2\|H\|^2 + \frac{4n^3 - 12n^2 - 2n + 10 - (3n^2 - 9n + 3)}{2(2n+2)(2n+4)}\overline{\rho} + \frac{6}{2n+4}\sum_{2\leq i<j\leq n}\overline{\mathscr{R}}ic(e_i, Je_j) - \frac{3}{2n+4}\sum_{i,j}\overline{\mathscr{R}}ic(e_i, Je_j).$$

(ii) *If $H(p) = 0$, the unit tangent vector X at p satisfies the equality if and only if $X \in \mathscr{N}_p$.*

(iii) *The equality case holds identically for all unit tangent vectors at p if and only if either p is totally geodesic point or $n = 2$ and p is totally umbilical point.*

4. Improved Chen-Ricci inequality

In 2009, Deng improved the Chen-Ricci inequality for Lagrangian submanifolds in complex space forms using the algebraic technique. In this section we will improve the Chen-Ricci inequality for Bochner-Kaehler manifolds. The following lemmas will be helpful for the proof of the main result.

Lemma 4.1 ([10]). *Let $(x_1, x_2, \ldots, x_n)$ be a point in $\mathbf{R}^n$. If $x_1 + x_2 + \cdots + x_n = na$, we have*

$$x_2^1 + x_2^2 + \cdots + x_n^2 \geq na^2.$$

The equality sign holds if and only if $x_1 = x_2 = \cdots = x_n = a$.

Lemma 4.2 ([10]). *Let $f_1(x_1, x_2, \ldots, x_n)$ be a function in $\mathbf{R}^n$ defined by*

$$f_1(x_1, x_2, \ldots, x_n) = x_1 \sum_{j=2}^{n} x_j - \sum_{j=2}^{n} x_j^2.$$

If $x_1 + x_2 + \cdots + x_n = 2na$, we have

$$f_1(x_1, x_2, \ldots, x_n) \leq \frac{(n-1)}{4n}(x_1 + x_2 + \ldots + x_n)^2.$$

The equality sign holds if and only if $\frac{1}{n+1} x_1 = x_2 = \cdots = x_n = a$.

Lemma 4.3 ([10]). *Let $f_2(x_1, x_2, \ldots, x_n)$ be a function in $\mathbf{R}^n$ defined by*

$$f_2(x_1, x_2, \ldots, x_n) = x_1 \sum_{j=2}^{n} x_j - x_1^2.$$

If $x_1 + x_2 + \cdots + x_n = 4a$, we have

$$f_2(x_1, x_2, \ldots, x_n) \leq \frac{1}{8}(x_1 + x_2 + \ldots + x_n)^2.$$

The equality sign holds if and only if $x_1 = a$ and $x_2 + \cdots + x_n = 3a$.

Now, we prove the main result of this section

Theorem 4.4. *Let $\mathcal{M}^n$ be a Kaehlerian slant submanifold of a Bochner-Kaehler manifold $\overline{\mathcal{M}}^{2n}$. Let X be a unit tangent vector in the tangent space $T_x\mathcal{M}^n$ at a point x in $\mathcal{M}$. Then we have*

$$\begin{aligned} \mathcal{R}ic(X) \leq \frac{(n-2)}{2n+4}\overline{\mathcal{R}ic}(X) + \frac{2(n+3)}{2(2n+2)(2n+4)}\overline{\rho} - \frac{6}{2n+4}\sum_{j=2}^{n} \overline{\mathcal{R}ic}(Je_1, e_j)\cos\theta \\ + \frac{6}{2(2n+2)(2n+4)}\overline{\rho}\cos^2\theta + \frac{n(n-1)}{4}\|H\|^2 \end{aligned} \tag{16}$$

where H is the mean curvature vector of $\mathcal{M}^n$ in $\overline{\mathcal{M}}^{2n}$ and $\mathcal{R}ic(X)$ is the Ricci curvature of $\mathcal{M}^n$ at X. $\overline{\mathcal{R}ic}(X)$ and $\overline{\rho}$ are the Ricci curvature and scalar curvature of $\overline{\mathcal{M}}^{2n}$.

The equality sign holds for any unit tangent vector X at a point x iff either

(i) *M is a totally geodesic or*

(ii) *$n=2$ and M is an H-umbilical surface with $\lambda=3\mu$.*

Proof. Let X be a unit tangent vector at any fixed point x in $\mathscr{M}^n$. For choosing an orthonormal frame $\{e_1, e_2, \dots, e_n\}$ in $T_x\mathscr{M}^n$ such that $e_1 = X$ and $\{e_1^* = \frac{Qe_1}{\sin\theta}, e_2^* = \frac{Qe_2}{\sin\theta}, \dots, e_n^* = \frac{Qe_n}{\sin\theta}\}$ an orthonormal frame in $T_x^{\perp}\mathscr{M}^n$. Using Gauss equation we have

$$R(e_j,e_1,e_1,e_j) = \overline{R}(e_j,e_1,e_1,e_j) + g(h(e_j,e_j),h(e_1,e_1)) - g(h(e_j,e_1),h(e_1,e_j)),$$

By putting $X = W = e_j$ and $Y = Z = e_1$, $j = 2,\dots,n$ in the curvature tensor of Bochner-Kaehler manifold and using (7), we have

$$\begin{aligned}\overline{R}(e_j,e_1,e_1,e_j) = {}& \frac{(n-2)}{2n+4}\overline{\mathscr{R}ic}(X) + \frac{2(n+3)}{2(2n+2)(2n+4)}\overline{\rho} + \frac{6}{2n+4}\overline{\mathscr{R}ic}(Je_1,e_j)g(Je_1,e_j)\\ & - \frac{6}{2(2n+2)(2n+4)}\overline{\rho}g^2(Je_1,e_j)\end{aligned}$$

By summing after j=2,…,n. From above two equations, we have

$$\begin{aligned}\mathscr{R}ic(X) = {}& \frac{(n-2)}{2n+4}\overline{\mathscr{R}ic}(X) + \frac{2(n+3)}{2(2n+2)(2n+4)}\overline{\rho} + \frac{6}{2n+4}\sum_{j=2}^{n}\overline{\mathscr{R}ic}(Je_1,e_j)g(Je_1,e_j)\\ & - \frac{6}{2(2n+2)(2n+4)}\overline{\rho}\sum_{j=2}^{n}g^2(Je_1,e_j) + \sum_{r=1}^{n}\sum_{j=2}^{n}[h_{11}^r h_{jj}^r - (h_{1j}^r)^2].\end{aligned}$$

Whereby we obtain

$$\begin{aligned}&\mathscr{R}ic(X) - \frac{(n-2)}{2n+4}\overline{\mathscr{R}ic}(X) - \frac{2(n+3)}{2(2n+2)(2n+4)}\overline{\rho} - \frac{6}{2n+4}\sum_{j=2}^{n}\overline{\mathscr{R}ic}(Je_1,e_j)g(Je_1,e_j)\\ &\quad + \frac{6}{2(2n+2)(2n+4)}\overline{\rho}\sum_{j=2}^{n}g^2(Je_1,e_j) \le \sum_{r=1}^{n}\sum_{j=2}^{n}h_{11}^r h_{jj}^r - \sum_{j=2}^{n}(h_{1j}^1)^2 - \sum_{j=2}^{n}(h_{1j}^1)^2.\end{aligned}$$

Since M is a Kaehlerian slant submanifold,using (11) we have

$$\begin{aligned}&\mathscr{R}ic(X) - \frac{(n-2)}{2n+4}\overline{\mathscr{R}ic}(X) - \frac{2(n+3)}{2(2n+2)(2n+4)}\overline{\rho} - \frac{6}{2n+4}\sum_{j=2}^{n}\overline{\mathscr{R}ic}(Je_1,e_j)\cos\theta\\ &\quad + \frac{6}{2(2n+2)(2n+4)}\overline{\rho}\cos^2\theta \le \sum_{r=1}^{n}\sum_{j=2}^{n}h_{11}^r h_{jj}^r - \sum_{j=2}^{n}(h_{11}^j)^2 - \sum_{j=2}^{n}(h_{jj}^1)^2.\end{aligned}$$

Now, we suppose that

$$f_1(h_{11}^1, h_{22}^1, \dots, h_{nn}^1) = h_{11}^1\sum_{j=2}^{n}h_{jj}^1 - \sum_{j=2}^{n}(h_{jj}^1)^2$$

and

$$f_r(h_{11}^r, h_{22}^r, \dots, h_{nn}^r) = h_{11}^r \sum_{j=2}^{n} h_{jj}^r - (h_{11}^r)^2, \quad for \quad r = 2, \dots, n$$

It is known that

$$nH^1 = h_{11}^1 + h_{22}^1 + \dots h_{nn}^1.$$

Now by using lemma(4.2), we have

$$f_1(h_{11}^1, h_{22}^1, \dots, h_{nn}^1) \leq \frac{n-1}{4n}(nH^1)^2$$

Also by using lemma(4.3), we get

$$f_r(h_{11}^r, h_{22}^r, \dots, h_{nn}^r) \leq \frac{1}{8}(nH^r)^2 \leq \frac{n-1}{4n}(nH^r)^2$$

Thus, we have

$$\mathcal{R}ic(X) \leq \frac{(n-2)}{2n+4}\overline{\mathcal{R}ic}(X) + \frac{2(n+3)}{2(2n+2)(2n+4)}\overline{\rho} + \frac{6}{2n+4}\sum_{j=2}^{n}\overline{\mathcal{R}ic}(Je_1, e_j)\cos\theta$$
$$-\frac{6}{2(2n+2)(2n+4)}\overline{\rho}\cos^2\theta + \frac{n(n-1)}{4}\|H\|^2$$

Now, consider the equality case, for $n \geq 3$, we choose Qe_1 parallel to the mean curvature vector H. Then, we have $H^r = 0$, for $r \geq 2$.

From lemma(4.3), we get

$$h_{1j}^1 = h_{11}^j = \frac{nH^j}{4} = 0, \qquad for \quad j \geq 2$$

and

$$h_{jk}^1 = 0, \qquad for \quad j, k \geq 2, \quad j \neq k.$$

Also from lemma (4.2), we have

$$h_{11}^1 = (n+1)a, \qquad h_{jj}^1 = a, \qquad for \quad j \geq 2 \quad \text{with} a = \frac{H^1}{2}$$

Similarly computing $\overline{\mathcal{R}ic}(e_2)$ as we compute $\mathcal{R}ic(X) = \overline{\mathcal{R}ic}(e_1)$, in the inequality 16, from the equality we get

$$h_{2j}^r = h_{jr}^2 = 0, \quad for \quad r \neq 2, \quad j \neq 2, \quad r \neq j$$

Using the lemma(4.2) and the equality, we get

$$\frac{h_{11}^2}{n+1} = h_{22}^2 = \cdots = h_{nn}^2 = \frac{H^2}{2} = 0$$

Since the equality holds for all unit vector fields, thus the argument is also true for matrices (h^r_{jk}). So $h^2_{2j} = h^j_{22} = \frac{H^j}{2} = 0$ for all $j \geq 3$.

Thus the matrix (h^2_{jk}) has only two non-zero possible entries $h^2_{12} = h^2_{21} = h^1_{22} = \frac{H^1}{2}$, similarly the matrix (h^r_{jk}) has also only two non-zero possible entries $h^r_{1r} = h^r_{r1} = h^1_{rr} = \frac{H^1}{2}$ for $r \geq 3$.

Now for computing $\overline{\mathscr{R}ic}(e_2)$, put $X = Z = e_2$ and $Y = W = e_j$ in the Gauss equation, we have

$$\overline{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - \left(\frac{H^1}{2}\right)^2, \forall j \geq 3$$

By putting $X = Z = e_2$ and $Y = W = e_1$ in the Gauss equation, we get

$$\overline{R}(e_2, e_j, e_2, e_j) = R(e_2, e_j, e_2, e_j) - (n+1)\left(\frac{H^1}{2}\right)^2 + -\left(\frac{H^1}{2}\right)^2$$

on combining the above two relations, we get

$$\mathscr{R}ic(X) - \frac{(n-2)}{2n+4}\overline{\mathscr{R}ic}(X) - \frac{2(n+3)}{2(2n+2)(2n+4)}\overline{\rho} + \frac{6}{2n+4}\sum_{j=2}^{n}\overline{\mathscr{R}ic}(Je_1, e_j)\cos\theta$$
$$- \frac{6}{2(2n+2)(2n+4)}\overline{\rho}\cos^2\theta = 2(n-1)\left(\frac{H^1}{2}\right)^2$$

on the other hand, the equality case of (16)implies that

$$\mathscr{R}ic(X) - \frac{(n-2)}{2n+4}\overline{\mathscr{R}ic}(X) - \frac{2(n+3)}{2(2n+2)(2n+4)}\overline{\rho} + \frac{6}{2n+4}\sum_{j=2}^{n}\overline{\mathscr{R}ic}(Je_1, e_j)\cos\theta$$
$$- \frac{6}{2(2n+2)(2n+4)}\overline{\rho}\cos^2\theta = n(n-1)\left(\frac{H^1}{2}\right)^2$$

We know that $n \neq 1, 2$, thus from last two equations, we find that $H^1 = 0$. Thus (h^r_{jk}) are all zero i.e M is totally geodesic submanifold in $\overline{\mathscr{M}}^{2n}$.

For $n = 2$, if M^2 is not totally geodesic, we have

$$h(e_1, e_1) = \lambda e^*_1, h(e_2, e_2) = \mu e^*_1, h(e_1, e_2) = \mu e^*_2$$

with $\lambda = 3\mu$ and such a surface is called H-umbilical surface. □

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