TWO NEW DISCRETE OU-IANG TYPE INEQUALITIES

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Abstract. Two discrete Ou-Iang type inequalities are established which are supplementary to some of the existing results.

In [1], several explicit bounding sequences are found for nonnegative sequences of the form $\{x_n\}_{n=\alpha}^{\infty}$ which satisfy

$$x_n^p \leq \gamma_n + \sum_{m=\alpha}^{n-1} f_m x_m^p + \sum_{m=\alpha}^{n-1} g_m x_m^q, \ n \geq \alpha, \tag{1}$$

where α is an integer, $p, q \geq 0$ and $\{f_n\}_{n=\alpha}^{\infty}, \{g_n\}_{n=\alpha}^{\infty}, \{\gamma_n\}_{n=\alpha}^{\infty}$ are nonnegative sequences.

Here we are interested in obtaining additional explicit bounding sequences for nonnegative sequences of the form $\{x_n\}_{n=\alpha}^{\infty}$ which satisfy the above or the following functional relation

$$\varphi(x_n) \leq \gamma_n + \sum_{m=\alpha}^{n-1} f_m \phi(x_m) + \sum_{m=\alpha}^{n-1} g_m \psi(x_m), \ n \geq \alpha, \tag{2}$$

where α is an integer, $\{f_n\}_{n=\alpha}^{\infty}, \{g_n\}_{n=\alpha}^{\infty}, \{\gamma_n\}_{n=\alpha}^{\infty}$ are nonnegative sequences, and $\{\gamma_n\}_{n=\alpha}^{\infty}$ is nondecreasing.

There are many reasons for studying (2). As an example, let $\{x_n\}_{n=0}^{\infty}$ be the population of a species in time periods $n = 0, 1, \ldots$. Let φ measures the intellectual ability of the species and ψ measures the living standard. It is reasonable that the increase in intellectual ability in two consecutive time periods, that is $\varphi(x_{n+1}) - \varphi(x_n)$, is proportional to $\varphi(x_n)$ and also to $\psi(x_n)$. Let ε_n stands for nonnegative extrinsic factors such as knowledge from an outside civilization, then assuming a superposition effect, we then have

$$\varphi(x_{n+1}) - \varphi(x_n) = f\varphi(x_n) + g\psi(x_n) + \varepsilon_n, \ n = 0, 1, 2, \ldots, \tag{3}$$

where f and g are positive porportionality constants. Summing the above equation, we then obtain

$$\varphi(x_n) = \sum_{m=0}^{n-1} f\varphi(x_m) + \sum_{m=0}^{n-1} g\psi(x_m) + \gamma_n,$$

where $\gamma_n = \sum_{m=0}^{n-1} \varepsilon_m$.

In case ϕ is trivial and φ is the identity function, the following two results are known (see e.g. [2, Theorem 15.7] and [2, Lemma 15.6]).

Lemma 1. *Suppose ϕ is trivial and φ is the identity function. Suppose $\{\gamma_n\}_{n=\alpha}^{\infty}$ is a positive sequence, and ψ is a positive, nondecreasing and continuous function defined on $(0, \infty)$. Let*

$$G(u) = \int_{u_0}^{u} \frac{dt}{\psi(t)}, \ u, u_0 > 0.$$

If $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative solution of (2), then

$$x_n \le G^{-1}\left(G(\gamma_n) + \sum_{m=\alpha}^{n-1} g_m \right)$$

for $n = \alpha, \alpha+1, \ldots, \beta$, provided that

$$G(\gamma_n) + \sum_{m=\alpha}^{n-1} g_m \in G(0, \infty), \ n = \alpha, \alpha+1, \ldots, \beta.$$

We remark that when $n = \alpha$, the sum appearing in the above result is taken to be zero. This practice is in line with the general convention that empty sums are taken to be zero, and empty products to be one.

Lemma 2. *Suppose ϕ is trivial and φ is the identity function. Suppose further that $\gamma_n = \gamma + \sum_{m=\alpha}^{n-1} \sigma_m$ for $n \ge \alpha$, where $\{\sigma_m\}_{m=\alpha}^{\infty}$ is a nonnegative sequence, and ψ is a positive, nondecreasing and continuous function defined on $[0, \infty)$. Let*

$$G(u) = \int_{u_0}^{u} \frac{dt}{\psi(t)}, \ u, u_0 > 0.$$

If $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative solution of (2), then

$$x_n \le G^{-1}\left(G(\gamma_n) + \sum_{m=\alpha}^{n-1} g_m \right)$$

for $n = \alpha, \alpha+1, \ldots, \beta$, provided that

$$G(\gamma_n) + \sum_{m=\alpha}^{n-1} g_m \in G(0, \infty), \ n = \alpha, \alpha+1, \ldots, \beta.$$

In case ϕ is the identity function, the following is known (see e.g. [2, Theorem 15.10]).

Lemma 3. *Suppose ϕ and φ are the identity function, and suppose ψ is a nondecreasing and continuous function on $[0, \infty)$ such that $\psi(u) > 0$ for $u > 0$. Let*

$$G(x) = \int_{\beta}^{x} \frac{dt}{\psi(t)}, \ \beta, x > 0,$$

and

$$A_n = \gamma_n + \sum_{m=\alpha}^{n-1} \gamma_m f_m \prod_{i=m+1}^{n-1} (1 + f_i), \ n \geq \alpha.$$

If $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative solution of (2), then

$$x_n \leq G^{-1} \left(G(A_n) + \sum_{m=\alpha}^{n-1} g_m \prod_{i=m+1}^{n-1} (1 + f_i) \right)$$

for $n = \alpha, \alpha + 1, \ldots, \beta$, provided

$$G(A_n) + \sum_{m=\alpha}^{n-1} g_m \prod_{i=m+1}^{n-1} (1 + f_i) \in G(0, \infty), \ n = \alpha, \alpha + 1, \ldots, \beta.$$

In case $\varphi = \phi$, we may show the following.

Theorem 1. *Suppose $\varphi = \phi$, and ϕ, ψ are continuous and nonnegative functions defined on $[0, \infty)$, ϕ is increasing and ψ is nondecreasing. Let*

$$\Omega(r) = \int_{\beta}^{r} \frac{dt}{\psi(\phi^{-1}(t))}, \ r, \beta > 0, \tag{4}$$

and

$$\Gamma_n = \gamma_n + \sum_{m=\alpha}^{n-1} \gamma_m f_m \prod_{j=m+1}^{n-1} (1 + f_j), \ n \geq \alpha.$$

If $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative solution of (2), then

$$x_n \leq \phi^{-1} \left(\Omega^{-1} \left[\Omega(\Gamma_n) + \sum_{m=\alpha}^{n-1} g_m \prod_{i=m+1}^{n-1} (1 + f_i) \right] \right) \tag{5}$$

for $n = \alpha, \alpha + 1, \ldots, \beta$, provided

$$\Omega(\Gamma_n) + \sum_{m=\alpha}^{n-1} g_m \prod_{i=m+1}^{n-1} (1 + f_i) \in \Omega(0, \infty), \ n = \alpha, \alpha + 1, \ldots, \beta.$$

Proof. Let

$$u_n = \gamma_n + \sum_{m=\alpha}^{n-1} f_m \phi(x_m) + \sum_{m=\alpha}^{n-1} g_m \psi(x_m), \ n \geq \alpha.$$

Then $\{u_n\}_{n=\alpha}^{\infty}$ is nonnegative, $u_\alpha = \gamma_\alpha$ and

$$\Delta u_n = \Delta \gamma_n + f_n \phi(x_n) + g_n \psi(x_n) \geq 0, \ n \geq \alpha.$$

Since ϕ is increasing, thus $x_n \leq \phi^{-1}(u_n)$ for $n \geq \alpha$. As a consequence,

$$\Delta u_n \leq \Delta \gamma_n + f_n u_n + g_n \psi(\phi^{-1}(u_n)), \ n \geq \alpha,$$

which implies

$$u_n \leq \gamma_n + \sum_{m=\alpha}^{n-1} f_m u_m + \sum_{m=\alpha}^{n-1} g_m \psi(\phi^{-1}(u_n)), \ n \geq \alpha.$$

According to Lemma 3,

$$u_n \leq \Omega^{-1} \left[\Omega(\Gamma_n) + \sum_{m=\alpha}^{n-1} g_m \prod_{i=m+1}^{n-1} (1 + f_i) \right] \tag{6}$$

for $n = \alpha, \alpha + 1, \ldots, \beta$, provided that

$$\Omega(\Gamma_n) + \sum_{m=\alpha}^{n-1} g_m \prod_{i=m+1}^{n-1} (1 + f_i) \in \Omega(0, \infty), \ n = \alpha, \alpha + 1, \ldots, \beta.$$

Finally, (5) follows from (6) and $\phi(x_n) \leq u_n$. The proof is complete.

As a corollary, let $\phi(u) = u^p$ and $\psi(u) = u^q$ where p, q are distinct positive numbers. The function Ω defined by (4) is then equal to

$$\Omega(r) = \int_1^r \frac{dt}{t^{q/p}} = \frac{p}{p-q} \left(r^{(p-q)/p} - 1 \right), \ r > 0,$$

and

$$\Omega^{-1}(t) = \left(\frac{p-q}{p} t + 1 \right)^{p/(p-q)}$$

where

$$\Omega(0, \infty) = \mathrm{dom}\Omega^{-1} = \begin{cases} (p/(q-p), \infty) & p > q \\ (-\infty, p/(q-p)) & p < q \end{cases}.$$

Therefore, the conclusion of Theorem 1 is now

$$x_n \leq \left\{ \Gamma_n^{(p-q)/p} + \frac{p-q}{p} \sum_{m=\alpha}^{n-1} g_m \prod_{i=m+1}^{n-1} (1 + f_i) \right\}^{1/(p-q)}$$

for $n = \alpha, \alpha + 1, \ldots, \beta$, provided that

$$\Gamma_n^{(p-q)/p} + \frac{p-q}{p} \sum_{m=\alpha}^{n-1} g_m \prod_{i=m+1}^{n-1} (1 + f_i) \in \mathrm{dom}\Omega^{-1}, \ n = \alpha, \alpha + 1, \ldots, \beta.$$

We remark that the above conclusion is different from the result stated in [1, Theorem 1] for equation (1), and therefore Theorem 1 is not a direct generalization but a complementary result of Theorem 1 in [1].

Theorem 2. *Suppose φ is the identity function. Suppose $\gamma_\alpha > 0$. Suppose further that ϕ, ψ are positive, continuous and nondecreasing on $[0, \infty)$. Let*

$$\Psi(u) = \int_{u_0}^{u} \frac{dt}{\psi(t)}, \, u, u_0 > 0,$$

$$\Phi(u) = \int_{u_0}^{u} \frac{dt}{\phi(t)}, \, u, u_0 > 0,$$

and $\Phi(u + v) \geq \Phi(u) + \Phi(v)$ for $u, v > 0$, and $\lim_{u \to +\infty} \Phi(u) = +\infty$. If $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative solution of (2), then

$$x_n \leq \Psi^{-1}\left(\Psi\left(\gamma_n + \Phi^{-1}\left(\sum_{m=\alpha}^{n-1} f_m\right)\right) + \sum_{m=\alpha}^{n-1} g_m\right)$$

for $n = \alpha, \alpha + 1, \dots, \beta$, provided that

$$\Psi\left(\gamma_n + \Phi^{-1}\left(\sum_{m=\alpha}^{n-1} f_m\right)\right) + \sum_{m=\alpha}^{n-1} g_m \in \Psi(0, \infty), \, n = \alpha, \alpha + 1, \dots, \beta.$$

Proof. Let

$$u_n = \gamma_n + \sum_{m=\alpha}^{n-1} f_m \phi(x_m) + \sum_{m=\alpha}^{n-1} g_m \psi(x_m), \, n \geq \alpha.$$

Then $\{u_n\}_{n=\alpha}^{\infty}$ is a nonnegative sequence, $u_\alpha = \gamma_\alpha$ and

$$\Delta u_n = \Delta \gamma_n + f_n \phi(x_n) + g_n \psi(x_n) \geq 0, \, n \geq \alpha.$$

Since $x_n \leq u_n$ for $n \geq \alpha$,

$$\Delta u_n \leq \Delta \gamma_n + f_n \phi(u_n) + g_n \psi(u_n), \, n \geq \alpha,$$

which implies

$$u_n \leq \left(\gamma_\alpha + \sum_{m=\alpha}^{n-1} (g_m \psi(u_n) + \Delta \gamma_m)\right) + \sum_{m=\alpha}^{n-1} f_m \phi(u_m), \, n \geq \alpha.$$

In view of Lemma 2,

$$u_n \leq \Phi^{-1}\left(\Phi\left(\gamma_\alpha + \sum_{m=\alpha}^{n-1} (g_m \psi(u_m) + \Delta \gamma_m)\right) + \sum_{m=\alpha}^{n-1} f_m\right) \qquad (7)$$

for $n = \alpha, \alpha + 1, \dots, \beta$, provided

$$\Phi\left(\gamma_\alpha + \sum_{m=\alpha}^{n-1} (g_m \psi(u_m) + \Delta \gamma_m)\right) + \sum_{m=\alpha}^{n-1} f_m \in \Phi(0, \infty), \, n = \alpha, \alpha + 1, \dots, \beta.$$

Since $\lim_{u \to \infty} \Phi(u) = +\infty$, the last condition holds for any β. Thus (7) holds for all $n \geq \alpha$. Next, note that the conditions imposed on Φ imply $\Phi^{-1}(u+v) \leq \Phi^{-1}(u) + \Phi^{-1}(v)$ for all $u, v \in \Phi(0, \infty)$. Thus,

$$u_n \leq \gamma_\alpha + \sum_{m=\alpha}^{n-1} (g_m \psi(u_m) + \Delta\gamma_m) + \Phi^{-1}\left(\sum_{m=\alpha}^{n-1} f_m\right)$$

$$= \left(\gamma_n + \Phi^{-1}\left(\sum_{m=\alpha}^{n-1} f_m\right)\right) + \sum_{m=\alpha}^{n-1} g_m \psi(u_m).$$

Since $\left\{\gamma_n + \Phi^{-1}\left(\sum_{m=\alpha}^{n-1} f_m\right)\right\}$ is a positive and nondecreasing sequence, according to Lemma 1,

$$x_n \leq u_n \leq \Psi^{-1}\left(\Psi\left(\gamma_n + \Phi^{-1}\left(\sum_{m=\alpha}^{n-1} f_m\right)\right) + \sum_{m=\alpha}^{n-1} g_m\right)$$

for $n = \alpha, \alpha+1, \ldots, \beta$, provided

$$\Psi\left(\gamma_n + \Phi^{-1}\left(\sum_{m=\alpha}^{n-1} f_m\right)\right) + \sum_{m=\alpha}^{n-1} g_m \in \Psi(0, \infty), \ n = \alpha, \alpha+1, \ldots, \beta.$$

The proof is complete.

As an application, consider the functional relation

$$x_n \leq \left(a_n + \sum_{m=\alpha}^{n-1} F_m w(x_m)\right)\left(b_n + \sum_{m=\alpha}^{n-1} G_m w(x_m)\right), \ n \geq \alpha, \tag{8}$$

where $\{F_n\}_{n=\alpha}^{\infty}$, $\{G_n\}_{n=\alpha}^{\infty}$, $\{a_n\}_{n=\alpha}^{\infty}$, $\{b_n\}_{n=\alpha}^{\infty}$, $\{\Delta a_n\}_{n=\alpha}^{\infty}$ and $\{\Delta b_n\}_{n=\alpha}^{\infty}$ are nonnegative sequences and w is a positive, nondecreasing and continuous function defined on $[0, \infty)$, and $\alpha_\alpha b_\alpha > 0$. Denote the right hand side of (8) by z_n. Suppose $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative solution of (8). Then

$$\Delta z_n = \left(a_n + \sum_{m=\alpha}^{n-1} F_m w(x_m)\right)(\Delta b_n + G_n w(x_n))$$

$$+ (\Delta a_n + F_n w(x_n))\left(b_{n+1} + \sum_{m=\alpha}^{n} G_m w(x_m)\right)$$

$$= a_n \Delta b_n + b_{n+1} \Delta a_n + (a_n G_n + b_{n+1} F_n) w(x_n)$$

$$+ \Delta b_n \sum_{m=\alpha}^{n-1} F_m w(x_m) + \Delta a_n \sum_{m=\alpha}^{n} G_m w(x_m)$$

$$+ G_n w(x_n) \sum_{m=\alpha}^{n-1} F_m w(x_m) + F_n w(x_n) \sum_{m=\alpha}^{n} G_m w(x_m)$$

$$\geq 0$$

for $n \geq \alpha$. If we assume further that w is nondecreasing, then

$$\Delta z_n \leq \Delta(a_n b_n) + (a_n G_n + b_{n+1} F_n) w(z_n)$$
$$+ \left(\Delta b_n \sum_{m=\alpha}^{n-1} F_m + \Delta a_n \sum_{m=\alpha}^{n} G_m \right) w(z_n)$$
$$+ \left(G_n \sum_{m=\alpha}^{n-1} F_m + F_n \sum_{m=\alpha}^{n} G_m \right) w^2(z_n).$$

Let

$$f_n = a_n G_n + b_{n+1} F_n + \Delta b_n \sum_{m=\alpha}^{n-1} F_m + \Delta a_n \sum_{m=\alpha}^{n} G_m, \ n \geq \alpha,$$

$$g_n = G_n \sum_{m=\alpha}^{n-1} F_m + F_n \sum_{m=\alpha}^{n} G_m, \ n \geq \alpha,$$

$$W_1(u) = \int_{u_0}^{u} \frac{dt}{w(t)}, \ u, u_0 > 0,$$

and

$$W_2(u) = \int_{u_0}^{u} \frac{dt}{w^2(t)}, \ u, u_0 > 0.$$

According to Theorem 2, if we make the additional assumption that W_1 satisfies $W_1(u + v) \geq W_1(u) + W_1(v)$ for $u, v > 0$, and $\lim_{u \to +\infty} W(u) = +\infty$, then

$$x_n \leq W_2^{-1} \left(W_2 \left(a_n b_n + W_1^{-1} \left(\sum_{m=\alpha}^{n-1} f_n \right) \right) + \sum_{m=\alpha}^{n-1} g_m \right)$$

for $n = \alpha, \alpha + 1, \dots, \beta$, provided that

$$W_2 \left(a_n b_n + W_1^{-1} \left(\sum_{m=\alpha}^{n-1} f_n \right) \right) + \sum_{m=\alpha}^{n-1} g_m \in W_2(0, \infty), \ n = \alpha, \alpha + 1, \dots, \beta.$$

As another example, we consider our equation (3). It is reasonable to assume that ϕ, ψ are continuous and nonnegative functions defiend on $[0, \infty)$, ϕ is increasing and ψ is nondecreasing. For simplicity, we will also assume that $\varepsilon_n = \varepsilon > 0$ for all n. Then letting

$$\Omega(r) = \int_{\beta}^{r} \frac{dt}{\psi(\phi^{-1}(t))}, \ r, \beta > 0,$$

and

$$\Gamma_n = \gamma_n + \sum_{m=\alpha}^{n-1} \gamma_m f_m \prod_{j=m+1}^{n-1} (1 + f_j)$$
$$= n\varepsilon + mf\varepsilon \sum_{m=0}^{n-1} (1 + f)^{n-m-1}$$
$$= n\varepsilon + m\varepsilon \left((1 + f)^n - 1 \right),$$

if $\{x_n\}_{n=\alpha}^{\infty}$ is a nonnegative solution of (2), then

$$x_n \le \phi^{-1}\left(\Omega^{-1}\left[\Omega(\Gamma_n) + g\sum_{m=0}^{n-1}(1+f)^{n-m-1}\right]\right)$$

$$= \phi^{-1}\left(\Omega^{-1}\left[\Omega(\Gamma_n) + g\frac{(1+f)^n - 1}{f}\right]\right)$$

for $n = \alpha, \alpha+1, \ldots, \beta$, provided

$$\Omega(\Gamma_n) + g\frac{(1+f)^n - 1}{f} \in \Omega(0, \infty), \; n = \alpha, \alpha+1, \ldots, \beta.$$

References

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