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BRÜCK CONJECTURE AND CERTAIN SOLUTION OF SOME DIFFERENTIAL EQUATION

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Abstract. We investigate on the famous Brück conjecture, and improved some of the existing results by extending them up to a differential monomial $M[f]$ sharing small function with certain power f^{d_M} of a meromorphic function. The class of all meromorphic solutions of the differential equation $f^{d_M} \equiv M[f]$ has been explored. For the generalization of our main result, some relevant questions finally have been posed for further study in this direction.

1. Introduction, definitions and main results

In this paper, by a meromorphic function f, we mean a meromorphic function in the whole complex plane. Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. For $a \in \mathbb{C} \cup \{\infty\}$, we say that the meromorphic functions f and g share the value a CM if $f-a$ and $g-a$ have the same set of zeros with counting multiplicities, and we say that f and g share the value a IM if $f-a$ and $g-a$ have the same set of zeros with ignoring multiplicities.

For the standard notations and symbols, we would like to refer our reader to follow the monograph [9]. We need the following in the paper.

Definition 1.1. For a meromorphic function f and for $a \in \mathbb{C} \cup \{\infty\}$, and for a positive integer k

(i) $N_{(k}(r,a;f)(\overline{N}_{(k}(r,a;f))$ denotes the counting function (resp. reduced counting function) of those $a-$points of f whose multiplicities are not less than k;

(ii) $N_{k)}(r,a;f)(\overline{N}_{k)}(r,a;f))$ denotes the counting function (resp. reduced counting function) of those $a-$points of f whose multiplicities are not greater than k;

(iii) $N_k(r,a;f)$ denotes the sum $\overline{N}(r,a;f)+\sum_{j=2}^{k}\overline{N}_{(j}(r,a;f)$.

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It is clear that $N_k(r,a;f) \le k\overline{N}(r,a;f)$.

Definition 1.2 ([4, 5])**.** A function $a \equiv a(z)(\not\equiv 0,\infty)$ is called a small function of a meromorphic function f if $T(r,a) = S(r,f)$.

In 1926, *Nevanlinna* first showed that a non-constant meromorphic function on the complex plane $\mathbb{C}$ is uniqueness determined by the pre-images, ignoring multiplicities, of five distinct values (including infinity). The beauty of this result lies in the fact that there is no counterpart of it in the real function theory. A few years latter, he showed that when multiplicities are taken into consideration, four points are enough and in that case either the two functions coincides or one is a *bilinear transformation* of the other one. Clearly these results initiated the study of the relation between two non-constant meromorphic functions f and g. The study becomes more interesting if the function g is related with the function f.

It was *Brück* [7], who first proved the following result by investigating the uniqueness problems of an entire function sharing a value counting multiplicities with its first derivative.

Theorem A ([7])**.** *Let f be a non-constant entire function. If f and f' share the value 1 CM, and $N(r,0;f') = S(r,f)$, then $f-1 = c(f'-1)$, where c is a non-zero constant.*

Later, *Yang* [13] proved the following result for finite ordered entire function by considering general k-th derivative instead of the first derivative.

Theorem B ([13])**.** *Let f be a non-constant entire function of finite order, and let $a(\neq 0,\infty)$ be a constant. If f and $f^{(k)}$ share the value a CM, then $f-a = a\left(f^{(k)}-a\right)$, where c is a non-zero constant, and $k(\geq 1)$ is an integer.*

Regarding the non-integral hyper order, *Brück* [7] proposed the following famous conjecture which is known as *Brück Conjecture*, which inspired a number of people to work on the topic.

Brück Conjecture. *Let f be a non-constant entire function of finite non-integral hyper order. If f and f' share one finite value a CM, then $f'-a = c(f-a)$ for some constant $c(\neq 0)$.*

There are several results on the *Brück Conjecture* in the literature and many researchers are devoted to solve the conjecture and they put their valuable efforts to find different aspects of it by considering the general $k-$th derivative of an entire or meromorphic function f, or some polynomials expressions in f and its $k-$th derivative or sometimes a differential monomial or polynomials generated by f but till now the original conjecture is open.

We recall here the definition of differential monomials and polynomials.

Definition 1.3 ([4])**.** Let $n_{0j}, n_{1j}, \ldots, n_{kj}$ be non-negative integers.

- The expression $M_j[f] = f^{n_{0j}}\left(f'\right)^{n_{1j}} \ldots \left(f^{(k)}\right)^{n_{kj}}$ is called a *differential monomial* generated by f of degree $d_{M_j} = \sum_{i=0}^{k} n_{ij}$ and weight $\Gamma_{M_j} = \sum_{i=0}^{k}(i+1)n_{ij}$.
- The sum $\mathscr{P}[f] = \sum_{j=1}^{t} b_j M_j[f]$ is called a *differential polynomial* generated by f of degree $\overline{d}(\mathscr{P}) = \max\{d_{M_j} : 1 \le j \le t\}$ and weight $\Gamma_P = \max\{\Gamma_{M_j} : 1 \le j \le t\}$, where $b_j \equiv b_j(z)$ are small functions of f for $j = 1, 2, \ldots, t$.
- The number $\underline{d}(\mathscr{P}) = \min\{d_{M_j} : 1 \le j \le t\}$ and k the highest order of the derivative of f in $\mathscr{P}[f]$ are called respectively the lower of degree and order of $\mathscr{P}[f]$.
- $\mathscr{P}[f]$ is said to be *homogeneous* if $\overline{d}(\mathscr{P}) = \underline{d}(\mathscr{P})$.
- $\mathscr{P}[f]$ is called a *linear differential polynomial* generated by $\overline{d}(\mathscr{P}) = 1$. Otherwise $\mathscr{P}[f]$ is called *non-linear differential polynomial.* We denote by $\mathscr{Q} = \max\{\Gamma_{M_j} - d_{M_j} : 1 \le j \le t\} = \max\{n_{1j} + 2n_{2j} + \ldots + kn_{kj} : 1 \le j \le t\}$.

Considering differential polynomial, *Qiu* [12] extended *Theorem A* up to a linear differential polynomial.

In this direction, *Al-Khaladi* [2] in his investigation first observed that in the *Theorem A*, one simply can not replace the value 1 by a small function.

Example 1.1. Let $f(z) = 1 + e^{e^z}$, and $a(z) = \dfrac{e^z}{e^z - 1}$. Then a is a small function of f, and $f - a$ and $f' - a$ share the value 0 CM, and $N\left(r, 0; f'\right) = 0$. Also we see that $f - a = \frac{1}{e^z}(f' - a)$.

Considering the sharing of small functions, *Al-Khaladi* [3] proved the following result.

Theorem C ([3])**.** *Let f be a non-constant entire function satisfying $N\left(r, 0; f'\right) = S(r, f)$ and $a(\not\equiv 0, \infty)$ be a small function of f. If $f - a$ and $f' - a$ share the value 0 CM, then $f - a = \left(1 + \dfrac{c}{a}\right)(f' - a)$, where $1 + \dfrac{c}{a} = e^{\beta}$, c is a constant and β is an entire function.*

For higher order derivative, *Al-khaladi* [3] proved the following result.

Theorem D ([3])**.** *Let f be a non-constant entire function satisfying $\overline{N}\left(r, 0; f^{(k)}\right) = S(r, f)$, $k(> 1)$ be an integer, and let $a(\not\equiv 0, \infty)$ be a meromorphic small function of f. If $f - a$ and $f^{(k)} - a$ share the value 0 CM, then $f - a = \left(1 + \dfrac{P_{k-1}}{a}\right)\left(f^{(k)} - a\right)$, where $P_{k-1}(z)$ is a polynomial of degree at most $k - 1$ and $1 + \dfrac{P_{k-1}}{a} \not\equiv 0$.*

In 2011, *Al-Khaladi* [3] extended *Theorem A* to the class of meromorphic functions and obtained the following result.

Theorem E ([3])**.** *Let f be a non-constant meromorphic function satisfying the condition $N(r, 0; f') = S(r, f)$. If f and f' share the value 1 CM, then $f - 1 = c(f' - 1)$ for some non-zero constant.*

Also in [3], were considered the following examples, showing that the value sharing can not be relaxed from CM to IM, and the condition $N(r,0;f')=S(r,f)$ is essential.

Example 1.2. Let $f(z)=1+\tan z$. Then clearly $f'-1=(f-1)^2$. We also see that $N(r,0;f')=0$. Clearly f and f' share the value 1 IM but the conclusion of *Theorem E* does not hold.

Example 1.3. Let $f(z)=\dfrac{z}{1+e^{-z}}$. Then f and f' share the value 1 CM and $N(r,0;f')\neq S(r,f)$. It is easy to verify that $f'-1=\dfrac{1}{1+e^{z}}(f-1)$.

Thus in order to replace value 1 by a small function some extra conditions are required. We refer our reader for the more details to see ([1, 5, 6])

Chen - Wu [8] extended the result of *Al-Khaladi* up to a linear differential polynomial, and obtained the following result.

Theorem F ([8])**.** *Let f be a noon-constant entire function satisfying $N\left(r,0;f'\right)=S(r,f)$, $a(\not\equiv 0,\infty)$ be a small function of f and $\mathscr{L}\equiv\mathscr{L}(f)=\sum_{j=1}^{k}a_jf^{(j)}$, where $k\in\mathbb{N}$, and $a_1,a_2,\ldots,a_k(\not\equiv 0)$ are small entire functions of f. If $f-a$ and $\mathscr{L}-a$ share 0 CM, then $f-a=\left(1+\dfrac{c}{a}\right)(\mathscr{L}-a)$, where $1+\dfrac{c}{a}=e^{\beta}$, c is a constant and β is an entire function.*

We now define

$$\mathscr{L}\left(f^{(k)}\right)=a_0f^{(k)}+a_1f^{(k+1)}+\ldots+a_pf^{(k+p)},$$

where $a_0,a_1,\ldots,a_p(\neq 0)$ are constants, and $k(\geq 1)$ and $p(\geq 0)$ are integers such that $p=0$ if $k=1$ and $0\leq p\leq k-2$ if $k\geq 2$.

Recently *Lahiri - Pal* [11] considered the problem of sharing small function of a meromorphic function and its linear differential polynomial, and obtained the following result.

Theorem G ([11])**.** *Let f be a transcendental meromorphic function be such that $f-a$ and $\mathscr{L}(f^{(k)})-a$ share the value 0 CM, where $a(\not\equiv 0,\infty)$ is a small function of f. If $N\left(r,0;f^{(k)}\right)=S(r,f)$, then*

$$f-a=\left(1+\frac{P_{k-1}}{a}\right)\left(\mathscr{L}(f^{(k)})-a\right),$$

where P_{k-1} is a polynomial of degree $k-1$ and $1+\dfrac{P_{k-1}}{a}\not\equiv 0$.

The following example shows that the condition $N\left(r,0;f^{(k)}\right)=S(r,f)$ is essential in *Theorem G.*

Example 1.4. Let $f(z) = \dfrac{P(z)e^z}{1+e^z}$, where $P(z)$ is a non-constant polynomial. Then $f' = \dfrac{e^z\left(P+P'+P'e^z\right)}{(1+e^z)^2}$, and hence $N\left(r,0;f'\right) \neq S(r,f)$. Also , $f-P'$ and $f'-P'$ share 0 CM but $f'-P' = \dfrac{1}{e^z+1}\left(f-P'\right)$.

So we see that the Brück result and the research thereafter has a long history. Several special forms on the Brück conjecture such as value sharing, small functions sharing, linear differential polynomial etc. were meticulously investigated by many authors. But to the best of my knowledge, no attempts have so far been mad by any research on the problem *what would happen if one considers certain power of a meromorphic function and its differential monomial sharing small function ?*

From the above discussions, we see that the authors have been obtained some relation between a meromorphic functions and its k-th derivative or linear differential polynomial but no one find the general meromorphic solution of the relation. In the modern uniqueness theory of meromorphic function, finding the class of meromorphic functions which are the solution of some differential equation is interesting and seldom studied.

Regarding the specific form of the function f which is a solution of a differential equation $f^{(k)} = f$, we have the following observations.

Note 1.1. Let us suppose that $f \equiv f^{(k)}$. Clearly the function f can not have any pole. Also we see that no non-constant polynomial satisfies the relation, so it is very natural that the function f must be transcendental entire. So the general solution of $f \equiv f^{(k)}$ will be as follows:

(1) In the case when $N(r,0;f) \neq S(r,f)$.

$$f(z) = d_0\exp(z) + d_1\exp(\zeta z) + d_2\exp(\zeta^2 z) + \ldots + d_{k-1}\exp(\zeta^{k-1}z),$$

(2) In the case when $N(r,0;f) = S(r,f)$, we have

$$f(z) = d\exp(\zeta z),$$

where $\zeta = \cos\left(\dfrac{2\pi}{k}\right) + i\sin\left(\dfrac{2\pi}{k}\right)$ and d $(\neq 0)$, $d_{i-1} \in \mathbb{C}$, not all zero, for $i \in \{1,2,\ldots,k\}$.

Observing Note 1.1, one may ask the natural question as follows:

Question 1.1. Is it possible to extend $f^{(k)}$ up to a general differential monomial $M[f]$ to get a certain form of the function which satisfies the relation $f \equiv M[f]$?

From the next discussions, we will see that the answer of *Question 1.1* is not true in general. Suppose $M[f] = f^{(k)}f^{(s)}$ or $\left(f^{(k)}\right)^{n_k}\left(f^{(s)}\right)^{n_s}$, where k, s and n_k, n_s all are positive integers with $k > s$. It can be easily check that the form of the function in *Note 1.1* does not satisfy the relation $f \equiv M[f]$.

As our main aim is to extend $f^{(k)}$ up to a general differential monomial $M[f]$ and at the same time to find a non-constant meromorphic solution of the relation $f \equiv M[f]$, so it is important to note that we need some power in the first setting of the relation. If we does so, then the natural question arises: '*does it really help us to get the relation of the form* $f^q \equiv M[f]$ *for the function in* Note 1.1 *?*' The answer of this question is NO in general. One can see the fact in the following.

(i). Suppose that $f(z) = d_1 \exp(z)+d_2 \exp(-z)$ and $M[f] = \left(f^{(k)}\right)^{n_k} \left(f^{(s)}\right)^{n_s}$, where $q = n_k+n_s$, k and s are two even positive integers. In this case, we see that $f^p \equiv M[f]$.

(ii) But, if we suppose $f(z) = d_1 \exp(z) + d_2 \exp(-z)$ and $M[f] = \left(f^{(k)}\right)^{n_k} \left(f^{(s)}\right)^{n_s}$, where one of k and r is even and other is odd positive integer. Then clearly we have $f^q \not\equiv M[f]$ for all positive integer q.

Thus, we have the following observation regarding the precise solution.

Note 1.2. For a more general setting $f^{d_M} \equiv M[f]$, where

$$M[f] = (f)^{n_0} (f')^{n_1} (f'')^{n_2} \dots \left(f^{(k)}\right)^{n_k}, \tag{1.1}$$

we see that $f(z) = d \exp(\lambda z)$, $\lambda^{Q_M} = 1$, must be a solution of it, where $Q_M = \Gamma_M - d_M$, $\Gamma_M = \sum_{i=0}^{k}(i+1)n_i$ and $d_M = \sum_{i=0}^{k} n_i$.

In this present paper, we consider the problem of sharing a small function $a(z)$ by certain power f^{d_M} of a meromorphic function and its differential monomial $M[f]$ in conformity with *Brück Conjecture*. Following theorems are the main results of this paper.

Theorem 1.1. *Let f be meromorphic function with $\overline{N}\left(r,0;f'\right) + \overline{N}(r,\infty;f) = S(r,f)$. Suppose that $M[f]$, as defined by (1.1), is a non-constant and $k(\geq 2)$ is a positive integer. Let $a \equiv a(z)(\neq 0,\infty)$ be a small function of f such that $\overline{N}(r,\infty;a) \leq \lambda T(r,a) + S(r,a)$, where $0 < \lambda < 1 - \frac{1}{\Gamma_M}$. If $f^{d_M} - a$ and $M[f] - a$ share 0 CM, then $f^{d_M} \equiv M[f]$. Furthermore $f(z)$ assumes the form $f(z) = ce^{\mu z}$, where $c(\neq 0)$ a constant and $\mu^{Q_M} = 1$.*

Theorem 1.2. *Let f be meromorphic function with $\overline{N}\left(r,0;f^{(2)}\right)+\overline{N}_{(2}(r,\infty;f) = S(r,f)$. Suppose that $M[f]$, as defined by (1.1), is a non-constant, where $n_1 = 0$ and $k(\geq 2)$ is a positive integer. Let $a \equiv a(z)(\neq 0,\infty)$ be a small function of f such that $\overline{N}(r,\infty;a) \leq \lambda T(r,a) + S(r,a)$, where $0 < \lambda < 1 - \frac{1}{\Gamma_M}$. If $f^{d_M} - a$ and $M[f] - a$ share 0 CM, then $f^{d_M} - a \equiv c(M[f] - a)$, where c is a non-zero constant.*

We have the following corollary of *Theorem* 1.1.

Corollary 1.1. *Let f be meromorphic function with $\overline{N}\left(r,0;f'\right)+\overline{N}(r,\infty;f)=S(r,f)$. Suppose that $f^{(k)}$ is non-constant and $k(\geq 2)$ is a positive integer. Let $a\equiv a(z)(\neq 0,\infty)$ be a small function of f such that $\overline{N}(r,\infty;a)\leq \lambda T(r,a)+S(r,a)$, where $0<\lambda<1-\frac{1}{k}$. If $f-a$ and $f^{(k)}-a$ share 0 CM, then $f\equiv f^{(k)}$. Furthermore $f(z)$ assumes the form $f(z)=ce^{\mu z}$, where $c(\neq 0)$ a constant and $\mu^k=1$.*

2. Some useful lemmas

In this section, we present some lemmas which will be needed in sequel.

Lemma 2.1 ([3])**.** *Let $k(\geq 2)$ be a positive integer, and f be a non-constant meromorphic function. If $\overline{N}\left(r,0;f^{(k)}\right)+\overline{N}_{(2}(r,\infty;f)=S(r,f)$, then either $N_{1)}(r,\infty;f)=S(r,f)$ or $f(z)=\frac{-(k+1)^{k+1}}{k!c\{z+d(k+1)\}}$ $+P_{k-1}(z))$, where $c(\neq 0)$, d are constants and $p_{k-1}(z)$ is a polynomial of degree at most $k-1$.*

Lemma 2.2. *Let f be a non-constant meromorphic function, and $k(\geq 2)$ be a positive integer. Suppose that $a(\not\equiv 0,\infty)$ is a small function of f, and $M[f]$, as defined in* Theorem 1.2, *is non-constant. If $\overline{N}\left(r,0;\left(f^{d_M}\right)^{(2)}\right)+\overline{N}_{(2}(r,\infty;f)=S(r,f)$ and $f^{d_M}-a$, $M[f]-a$ share 0 CM, then $\overline{N}(r,\infty;f)=S(r,f)$.*

Proof. If $f^{d_M}=\frac{-27}{2c(z+3d)}+q_1(z)$, then $a(z)$ becomes constant. Therefore, clearly $f^{d_M}-a$ and $M[f]-a$ can not share 0 CM. So from *Lemma 2.1*, we get $\overline{N}(r,\infty;f)=S(r,f)$. □

Lemma 2.3. *Let f be a meromorphic function and $M[f]$ be a differential monomial generated by f, then*

$$T\left(r,M[f]\right)\leq \sum_{j=0}^{k} n_j(j+1)T(r,f)+O(1) \tag{2.1}$$

and hence $S(r,M[f])$ can be replaced by $S(r,f)$.

Proof. We know for a meromorphic function f, f_1 and f_2, $T\left(r,0;f^{(j)}\right)\leq (j+1)\ T(r,f)+S(r.f)$ and $T(r,f_1f_2)\leq T(r,f_1)+T(r,f_2)$. Thus we see that

$$\begin{aligned} T\left(r,M[f]\right) &= T\left(r,\prod_{j=0}^{k}\left(f^{(j)}\right)^{n_j}\right) \\ &\leq \sum_{j=0}^{k} T\left(r,\left(f^{(j)}\right)^{n_j}\right)+O(1) \\ &\leq \sum_{j=0}^{k} n_j\ T\left(r,\left(f^{(j)}\right)\right)+O(1) \\ &\leq \sum_{j=0}^{k} n_j(j+1)\ T\left(r,\left(f\right)\right)+O(1) \end{aligned}$$

and hence $S(r,M[f])$ can be replaced by $S(r,f)$. □

Lemma 2.4 ([9])**.** *Let g be a non-constant meromorphic function, and a_1, a_2, a_3 be distinct meromorphic function small functions of g. Then*

$$T(r,g) \leq \sum_{i=1}^{3} \overline{N}(r,0;g-a_j) + S(r,g).$$

Lemma 2.5 ([10])**.** *Given a transcendental meromorphic function g, and a constant $\Gamma > 1$. Then there exists a set $\mathscr{S}(\Gamma)$ whose upper logarithmic density is at most*

$$\delta(\Gamma) = \min\left\{ \left(2e^{\Gamma-1}\right)^{-1}, (1+e(\Gamma-1))\exp(e(1-\Gamma)) \right\}$$

such that for every positive integer k,

$$\limsup_{r\to\infty, r\in\mathscr{S}(\Gamma)} \frac{T(r,g)}{T\left(r,g^{(k)}\right)} \leq 3e\Gamma.$$

3. Proof of the main result

We prove here *Theorem 1.2* only, as proof of *Theorem 1.1* is similar to the proof of *Theorem 1.2*.

Proof of Theorem 1.2. If f is not a transcendental, since by *Lemma 2.2*, we have $\overline{N}(r,\infty;f) = S(r,f)$, then f must a polynomial. Let $\deg(f) = n$. Thus we see that $M[f]$ would be a polynomial. If $\deg(f) \geq \left[\frac{\Gamma_M}{d_M}\right] + 1$, then $\deg(M[f]) = d_M \deg(f) - \Gamma_M$. If $\deg(f) \leq \left[\frac{\Gamma_M}{d_M}\right]$, then $\deg(M[f]) = 0$, which is impossible as $M[f]$ is non-constant. Since in this case a is a constant, we see that $f^{d_M} - a$ and $M[f] - a$ can not share the value 0 CM, which contradicts our assumption. Thus the function f must be a transcendental meromorphic function.

We set $h = \dfrac{f^{d_M} - a}{M[f] - a}$. Let if possible f has a pole at z_0 of order q. Then, an elementary calculation shows that z_0 is a zero of h of multiplicity Γ_M. Again it is clear that h is an entire function. Then by the hypothesis and *Lemma 2.2*, we have

$$\overline{N}(r,0;h) \leq \overline{N}(r,\infty;f) = S(r,f). \tag{3.1}$$

Differentiating $f^{d_M} - a = hM[f] - ha$ twice we get that

$$\left(f^{d_M}\right)^{(2)} - a^{(2)} = \left(hM[f]\right)^{(2)} - (ha)^{(2)}. \tag{3.2}$$

Case 1. We suppose that $a^{(2)} \not\equiv 0$. We set

$$\mathscr{W} = \frac{\left(hM[f]\right)^{(2)}}{h\left(f^{d_m}\right)^{(2)}} - \frac{(ha)^{(2)}}{ha^{(2)}}. \tag{3.3}$$

Subcase 1.1. Suppose that $\mathscr{W} \not\equiv 0$.

Let z_1 be a zero of $\left(f^{d_M}\right)^{(2)} - a^{(2)}$ and $a^{(2)} \not\equiv 0, \infty$. Then from (3.2), we see that z_1 be a zero of $\left(hM[f]\right)^{(2)} - (ha)^{(2)}$, and hence $\mathscr{W}(z_1) = 0$.

Therefore, we see that

$$\begin{aligned} m(r,\mathscr{W}) &\leq m\left(r, \frac{\left(hM[f]\right)^{(2)}}{h\left(f^{d_m}\right)^{(2)}}\right) + m\left(r, \frac{(ha)^{(2)}}{ha^{(2)}}\right) \\ &\leq m\left(r, \frac{\left(hM[f]\right)^{(2)}}{hM[f]}\right) + m\left(r, \frac{M[f]}{\left(f^{d_M}\right)^{(2)}}\right) + m\left(r, \frac{(ha)^{(2)}}{ha}\right) + m\left(r, \frac{a}{a^{(2)}}\right) \\ &= S(r,f). \end{aligned}$$

Therefore

$$\begin{aligned} \overline{N}\left(r,0;\left(f^{d_M}\right)^{(k)} - a^{(k)}\right) &\leq N(r,0;\mathscr{W}) + S(r,f) \\ &\leq T(r,\mathscr{W}) + S(r,\mathscr{W}) \\ &= N(r,\mathscr{W}) + m(r,\mathscr{W}) + S(r,\mathscr{W}) \\ &= N(r,\mathscr{W}) + S(r,f). \end{aligned} \tag{3.4}$$

Let z_2 is a zero of f of multiplicity p, such that $a(z_2) \neq 0, \infty$ and $a^{(2)}(z_2) \neq 0$. Then z_2 is a pole of h of multiplicity Γ_M. Hence z_2 is a pole of $\left(hM[f]\right)^{(2)}$ with multiplicity $(pd_M + \Gamma_M - \Gamma_M) + 2 = pd_M + 2$. Also, z_2 is a pole of $\dfrac{\left(hM[f]\right)^{(2)}}{h\left(f^{d_M}\right)}$ of multiplicity $(pd_M + 2) - (pd_M + 2 - 2) = 2 \leq k$. Then z_2 is a pole of $\mathscr{W}$ with multiplicity at most k. Let z_3 be a zero of $\left(f^{d_M}\right)^{(2)}$ such that $a(z_3) \neq 0, \infty$, $a^{(2)}(z_2) \neq 0$. If $q > \Gamma_M$, then z_3 be a zero of hMf with multiplicity $q - \Gamma_M + 2$, so z_3 is a zero of $\left(hM[f]\right)^{(2)}$ with multiplicity $(q - \Gamma_M + 2) - 2 = q - \Gamma_M$. Hence z_3 be a zero of $\mathscr{W}$ with multiplicity at most $q - (q - \Gamma_M) = \Gamma_M$.

We have

$$\begin{aligned} N(r,\mathscr{W}) &\leq \Gamma_M \overline{N}(r,\infty;f) + N_{\Gamma_M}\left(r,0;\left(f^{d_M}\right)^{(2)}\right) + \overline{N}\left(r,0;\left(f^{d_M}\right)^{(2)}\right) + S(r,f) \\ &= S(r,f). \end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we get $\overline{N}\left(r,0;\left(f^{d_M}\right)^{(2)} - a^{(2)}\right) = S(r,f)$ and

$$\begin{aligned} T\left(r,\left(f^{d_M}\right)^{(2)}\right) &\leq \overline{N}\left(r,\infty;\left(f^{d_M}\right)^{(2)}\right) + \overline{N}\left(r,0;\left(f^{d_M}\right)^{(2)}\right) + \overline{N}\left(r,0;\left(f^{d_M}\right)^{(2)} - a^{(k)}\right) + S\left(r,f^{(2)}\right) \\ &= S(r,f). \end{aligned} \tag{3.6}$$

Let $\mathscr{S}(\Gamma)$ be defined as in *Lemma 2.5*. Then by (3.6), there exists a sequence $r_n \to \infty$, $r_n \notin \mathscr{S}(\Gamma)$ such that $\dfrac{T\left(r_n\left(f^{d_M}\right)^{(2)}\right)}{T(r_n,f)} \to 0$ as $n \to \infty$. This contradicts the *Lemma 2.5*. Thus we have $\mathscr{W}\equiv 0$, and so from the equations (3.2) and (3.3), we get

$$\left(\left(f^{d_M}\right)^{(2)}-a^{(2)}\right)a^{(2)}=(ha)^{(2)}\left(\left(f^{d_M}\right)^{(2)}-a^{(2)}\right).$$

Since $\left(f^{d_M}\right)^{(2)}\neq a^{(2)}$, we obtained that $(ha)^{(2)}=a^{(2)}$. On integration, we obtained that $ha=a+d_1z+d_0$, $d_1,d_0\in\mathbb{C}$ and so $h=1+\dfrac{d_1z+d_0}{a}$.

We again note that h is an entire and the zeros of h are precisely the poles of f. Also we note that zeros of h is of multiplicity Γ_M. Let $d_1\neq 0$. Then, we have $T(r,h)=T(r.a)+O(\log r)$. Also $\overline{N}(r,1;h)=\overline{N}(r,\infty;a)+O(\log r)$ and $\overline{N}(r,0;h)=\frac{1}{\Gamma_M}N(r,0;h)$. Thus by the Second Fundamental Theorem, we have

$$\begin{aligned}T(r,h)&\leq\overline{N}(r,1;h)+\overline{N}(r,0;h)+\overline{N}(r,\infty;h)+S(r,h)\\&=\overline{N}(r,\infty;a)+\frac{1}{\Gamma_M}N(r,0;h)+O(\log r)+S(r,h)\\&\leq\lambda\,T(r,a)+\frac{1}{\Gamma_M}T(r,h)+O(\log r)+S(r,h)\\&=\left(\lambda+\frac{1}{\Gamma_M}\right)T(r,h)+O(\log r)+S(r,h),\end{aligned}$$

and so we get by our assumption that $T(r,h)=O(\log r)+S(r,h)$. This implies that $h-1$ is a polynomial, say $P(z)$.

If $P(z)\equiv 0$, then $h\equiv 1$, and we get the result. We suppose that $P(z)\not\equiv 0$. Then $h=1+\dfrac{d_1z+d_0}{a}$ implies that $a=\dfrac{d_1z+d_0}{P(z)}$.

We suppose that d_1z+d_0 is a factor of $P(z)$. Then $a=\dfrac{1}{\mathscr{Q}(z)}$, where $P(z)=(d_1z+d_0)\mathscr{Q}(z)$. This implies that $T(r,a)=(\deg(\mathscr{Q})\log r+O(1)=N(r,\infty;a)+O(1)$, a contradiction. So d_1z+d_0 is not a factor of $P(z)$. Then $T(r,a)=\max\{\deg(P),1\}\log r+O(1)$ and $N(r,\infty;a)=(\deg(P))\log r+O(1)$. Therefore by the hypothesis, we obtained that $\deg(P)\leq\lambda\max\{\deg(P),1\}$. This implies that $\deg(P)=0$, and so $a=\dfrac{d_1z+d_0}{d}$, where $d(\neq 0)$ a constant.

Let $d_1=0$. Then $h=\dfrac{a+d_0}{a}$. Since h is entire and each zero of h is of multiplicity Γ_M, we have $\overline{N}(r,0;a)=0$ and $\overline{N}(r,0;a+d_0)\leq\dfrac{1}{\Gamma_M}N(r,0;a+d_0)$. Therefore if $d_0\neq 0$, we get by the Second Fundamental Theorem,

$$\begin{aligned}T(r,a)&=\overline{N}(r,\infty;a)+\overline{N}(r,0;a)+\overline{N}(r,0;a+d_0)+S(r,f)\\&\leq\left(\lambda+\frac{1}{\Gamma_M}\right)T(r,a)+S(r,a),\end{aligned}$$

which contradicts $0<\lambda<1-\dfrac{1}{\Gamma_M}$. So $d_0=0$ and hence $h=1$.

Case 2. Let $a^{(2)} \equiv 0$. Then $a(z) = a_1 z + a_0$, where $a_0, a_1 \in \mathbb{C}$. Then from (3.2), we see that

$$\left(f^{d_M}\right)^{(2)} = \left(hM[f]\right)^{(2)} - (ah)^{(2)}.$$

i.e.,

$$\frac{1}{h} = \frac{\left(hM[f]\right)^{(2)}}{h\left(f^{d_M}\right)^{(2)}} - \frac{(ah)^{(2)}}{h\left(f^{d_M}\right)^{(2)}}. \tag{3.7}$$

We set $\mathscr{F} = \left(f^{d_M}\right)^{(2)}$, $\mathscr{G} = \frac{\left(hM[f]\right)^{(2)}}{h\left(f^{d_M}\right)^{(2)}}$ and $b = \frac{(ah)^{(2)}}{h}$. Therefore, from (3.7), we have $\frac{1}{h} = \mathscr{G} - \frac{b}{\mathscr{F}}$. On differentiating, we have

$$-\frac{1}{h}\frac{h'}{h} = \mathscr{G}' - \frac{b'}{\mathscr{F}} + \frac{b}{\mathscr{F}}.\frac{\mathscr{F}'}{\mathscr{F}}. \tag{3.8}$$

From (3.7) and (3.8), we have

$$\frac{\mathscr{A}}{\mathscr{F}} = \mathscr{G}' + \mathscr{G}\frac{h'}{h}, \tag{3.9}$$

where $\mathscr{A} = b\frac{h'}{h} + b' - b\frac{\mathscr{F}'}{\mathscr{F}}$.

We now discuss the following cases.

Subcase 2.1. Let $\mathscr{G} \equiv 0$. i.e., $\left(hM[f]\right)^{(2)} = 0$. On integration, we get $hM[f] = b_1 z + b_0$, $b_0, b_1 \in \mathbb{C}$. Putting $h = \frac{f^{d_M} - a}{M[f] - a}$, we get

$$\left(f^{d_M} - a\right)M[f] = (M[f] - a)(b_1 z + b_0). \tag{3.10}$$

Since a is a polynomial, so from (3.10), we see that f is an entire function. Therefore, h is an entire function having no zero. We set $h = ce^{\alpha}$, where $c \neq 0$, α is an entire function.

Thus we see that $f^{d_M} = a + (b_1 z + b_0) - ace^{\alpha}$ and $M[f] = c\,(b_1 z + b_0)\,e^{-\alpha}$. An elementary calculation shows that $M[f] = \mathscr{P}(\alpha, \alpha', \ldots, \alpha^{(k)})e^{\alpha}$, where $\mathscr{P}(\alpha, \alpha', \ldots, \alpha^{(k)})$ is a differential polynomial in $\alpha, \alpha', \ldots, \alpha^{(k)}$. This shows that

$$\begin{aligned} 2T\left(r, e^{\alpha}\right) &= T\left(r, e^{2\alpha}\right) \\ &= T\left(r, \frac{c(b_1 z + b_0)}{\mathscr{P}(\alpha, \alpha', \ldots, \alpha^{(k)})}\right) \\ &= S(r, f), \end{aligned}$$

which is not possible.

Subcase 2.2. Let $\mathscr{G} \not\equiv 0$.

Now we have the following two possibilities.

Subcase 2.2.1. If h is constant, then we get our result.

Subcase 2.2.2. If h is non-constant. Suppose that $b=0$ i.e., $(ah)^{(2)}=0$. Then on integration, we have $ah=e_1z+e_0$, where $e_1,e_0\in\mathbb{C}$. i.e., $h=\frac{e_1z+e_0}{a}$. Since h is entire, and a is a polynomial of degree 1, thus it is clear that a is a factor of the polynomial e_1z+e_0, and hence

$$h=\mathscr{Q}_1, \tag{3.11}$$

where $\mathscr{Q}_1\equiv\mathscr{Q}_1(z)$ is a polynomial of degree at most 1. Since each pole of f is a zero of h of multiplicity $\Gamma_M(\geq 2)$, by (3.11), we see that f must be an entire function. So, h is an entire function having no zero and which by (3.11) implies that h must be a constant, a contradiction. Thus we have, $b\equiv 0$.

Subcase 2.2.3. Suppose $\mathscr{A}\equiv 0$.

Then from (3.9), we obtained $\dfrac{\mathscr{G}'}{\mathscr{G}}+\dfrac{h'}{h}=0$, on integration, we get $\mathscr{G}h=\mathscr{D}$ such that

$$\left(hM[f]\right)^{(2)}\equiv\mathscr{B}\left(f^{d_M}\right)^{(2)}, \tag{3.12}$$

where $\mathscr{B}$ is an arbitrary constant of integration. Again since,

$$\frac{\mathscr{A}}{b}=\frac{h'}{h}+\frac{b'}{b}-\frac{\mathscr{F}'}{\mathscr{F}},$$

so on integration, we get $hb=\mathscr{D}\mathscr{F}$, and so

$$(ah)^{(2)}=\mathscr{D}\left(f^{d_M}\right)^{(2)}, \tag{3.13}$$

where $\mathscr{D}$ is an arbitrary constant of integration.

Since a is a polynomial, h is an entire function, then from (3.13), we see that f is an entire function and so $h=e^{\alpha}$, where $\alpha\equiv\alpha(z)$ is an entire function. Again integrating (3.12) twice we obtained

$$hMf=\mathscr{B}f^{d_M}+P_1, \tag{3.14}$$

where $P_1\equiv P_1(z)$ is a polynomial of degree at most 1. Since $hM[f]=f^{d_M}-a+ah$, so from (3.14) we obtained

$$(1-\mathscr{B})f^{d_M}=a(1-e^{\alpha})+P_1. \tag{3.15}$$

If $\mathscr{B}=1$, then from (3.15), we get $e^{\alpha}=1+\frac{P_1}{a}$, a contradiction. Hence, $\mathscr{B}\neq 1$, and from (3.15) can be written as

$$f^{d_M}=\frac{ae^{\alpha}}{\mathscr{B}-1}-\frac{a+P_1}{\mathscr{B}-1}. \tag{3.16}$$

From the definition of differential monomial $M[f]$, we have

$$M[f]=\mathscr{P}(\alpha,\alpha',\ldots,\alpha^{(k)})e^{\alpha}, \tag{3.17}$$

where $\mathscr{P}(\alpha,\alpha',\dots,\alpha^{(k)})(\not\equiv)$ is a differential polynomial in $\alpha,\alpha',\dots,\alpha^{(k)}$ with the coefficients as polynomials.

From (3.15) and (3.17), we have

$$M[f]=\frac{a\mathscr{B}}{\mathscr{B}-1}-\frac{(a\mathscr{B}+P_1)}{\mathscr{B}-1}e^{-\alpha}. \tag{3.18}$$

Again from (3.17) and (3.18), we have

$$\mathscr{P}(\alpha,\alpha',\dots,\alpha^{(k)})e^{2\alpha}=\frac{a\mathscr{B}e^{\alpha}}{\mathscr{B}-1}-\frac{(a\mathscr{B}+P_1)}{\mathscr{B}-1}, \tag{3.19}$$

which implies that

$$\begin{aligned}2T\left(r,e^{\alpha}\right) &= T\left(r,e^{2\alpha}\right)\\ &= T\left(r,\frac{a\mathscr{B}e^{\alpha}}{(\mathscr{B}-1)\mathscr{P}(\alpha,\alpha',\dots,\alpha^{(k)})}-\frac{(a\mathscr{B}+P_1)}{(\mathscr{B}-1)\mathscr{P}(\alpha,\alpha',\dots,\alpha^{(k)})}\right)\\ &\le T\left(r,\frac{e^{\alpha}}{\mathscr{P}(\alpha,\alpha',\dots,\alpha^{(k)})}\right)+T\left(r,\frac{a\mathscr{B}+P_1}{\mathscr{P}(\alpha,\alpha',\dots,\alpha^{(k)})}\right)\\ &= S(r,f),\end{aligned}$$

and this is absurd.

Therefore $\mathscr{A}\equiv 0$. Again since $\mathscr{A}=b\left(\frac{h'}{h}+\frac{b'}{b}-\frac{\mathscr{F}'}{\mathscr{F}}\right)$, clearly we have $m(r,\mathscr{A})=S(r,f)$. Also we note that the poles of $\mathscr{A}$ are coming from (i) the poles of $b=\frac{(ah)^{(2)}}{h}$, (ii) the poles of $\frac{h'}{h}$ and (iii). the poles of $\frac{\mathscr{F}'}{\mathscr{F}}=\frac{\left(f^{d_M}\right)^{(3)}}{\left(f^{d_M}\right)^{(2)}}$. Since h is an entire function and the zeros of h are precisely the poles of f, and each zero of h is of multiplicity Γ_M, we get by the hypothesis and *Lemma* 2.2

$$\begin{aligned}N(r,\mathscr{A}) &\le (\Gamma_M+1)\,\overline{N}(r,\infty;f)+\overline{N}\left(r,0;\left(f^{d_M}\right)^{(2)}\right)+S(r,f)\\ &= S(r,f).\end{aligned}$$

Therefore, we have $T(r,\mathscr{A})=m(r,\mathscr{A})+N(r,\mathscr{A})=S(r,f)$.

Next by (3.9), we obtained that

$$\begin{aligned}m\left(r,\frac{1}{\mathscr{F}}\right) &\le m\left(r,\frac{1}{\mathscr{A}}\right)+m\left(r,\mathscr{G}'+\mathscr{G}\frac{h'}{h}\right)\\ &\le T(r,\mathscr{A})+m(r,\mathscr{G})+m\left(r,\frac{\mathscr{G}'}{\mathscr{G}}+\frac{h'}{h}\right)\\ &= m(r,\mathscr{G})+S(r,h)\\ &= m\left(r,\frac{(hM[f])^{(2)}}{hM[f]}.\frac{M[f]}{\left(f^{d_M}\right)^{(2)}}\right)+S(r,f)\end{aligned}$$

$$
\begin{aligned}
&\leq m\left(r, \frac{\left(hM[f]\right)^{(2)}}{hM[f]}\right) + m\left(r, \frac{M[f]}{\left(f^{d_M}\right)^{(2)}}\right) \\
&= S(r,f).
\end{aligned} \tag{3.20}
$$

In view of (3.1), we get that

$$
\begin{aligned}
T(r,f) &= N(r,b) + S(r,f) \\
&= N\left(r, \frac{(ah)^{(2)}}{h}\right) + S(r,f) \\
&\leq 2\overline{N}(r,0;h) + S(r,f) \\
&= S(r,h).
\end{aligned} \tag{3.21}
$$

Let z_4 be a zero of $\mathscr{F} = \left(f^{d_M}\right)^{(2)}$ such that $a(z_4) \neq 0$. Then z_4 will be a zero of $\left(hM[f]\right)^{(2)}$ with multiplicity at least $q - (\Gamma_M - 2) - 2 = q - \Gamma_M$. So, z_4 is a zero of $\mathscr{F}\mathscr{G} = \dfrac{\left(hM[f]\right)^{(2)}}{h}$ with multiplicity at least $q - \Gamma_M$. Hence z_4 is a zero of $b = \mathscr{F}\mathscr{G} - \frac{\mathscr{F}}{h}$ with multiplicity $q - \Gamma_M$.

Therefore by (3.21), we get

$$
\begin{aligned}
N_{(\Gamma_M+1}\left(r,0;\left(f^{d_M}\right)^{(2)}\right) &\leq N(r,0;b) + \Gamma_M \overline{N}_{(\Gamma_M+1}\left(r,0;\left(f^{d_M}\right)^{(2)}\right) \\
&= \Gamma_M\left(r,0;\left(f^{d_M}\right)^{(2)}\right).
\end{aligned}
$$

Hence we have

$$
\begin{aligned}
N\left(r, \frac{1}{\mathscr{F}}\right) &= N\left(r,0;\left(f^{d_M}\right)^{(2)}\right) \\
&= N_{\Gamma_M)}\left(r,0;\left(f^{d_M}\right)^{(2)}\right) + \Gamma_M \overline{N}_{(\Gamma_M+1}\left(r,0;\left(f^{d_M}\right)^{(2)}\right) + S(r,f) \\
&= \Gamma_M \overline{N}_{(\Gamma_M+1}\left(r,0;\left(f^{d_M}\right)^{(2)}\right) + S(r,f) \\
&= S(r,f).
\end{aligned} \tag{3.22}
$$

Then from (3.20) and (3.22), and by applying the *First Fundamental Theorem*, we get $T\left(r,\left(f^{d_M}\right)^{(2)}\right) = S(r,f)$ which is (3.6), and likewise we get a contradiction.

This completes the proof. □

4. Concluding remarks and open questions

From our previous discussions, we have noticed that solution of the relation $f^{d_M} \equiv M[f]$ is of the form $f(z) = ce^{\mu z}$, where $c(\neq 0)$ a constant and $\mu^{Q_M} = 1$. Now for the generalization of our results from differential monomial $M[f]$ up to a differential polynomial $P[f]$, we understand that only a suitable power (like d_M for the case of differential monomial $M[f]$) of the function f is not enough to get a certain solution. We need some extra supposition for the solution.

Thus we posed the following questions for the further study in this direction.

Question 4.1. Is it possible to prove the main results of this paper up to a general differential polynomial $P[f]$?

Question 4.2. What should we set with the function f, so that when that setting shares a small function with its differential polynomial $P[f]$ we get a certain solution of the identical relation?

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