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ON γ-SEMIOPEN SETS

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Abstract. Characterizations of γ-semiopen sets, γ-semiclosed sets, γ-semiclosure and γ-semiinterior are given. Properties of γ-semiopen sets and γ-semiclosed sets are also discussed.

1. Introduction and Preliminaries

In 2005, Güldürdek and Özbakir [3], introduced and studied γ-semiopen sets. In this paper, we further extend the study of γ-semiopen sets. Let X be any nonempty set. We denote by Γ, the collection of all mappings $\gamma\colon \wp(X) \to \wp(X)$ such that $A \subset B$ implies $\gamma(A) \subset \gamma(B)$. As defined in [1], we mention here the following subcollections of Γ.

$$\Gamma_0 = \{\gamma \in \Gamma \mid \gamma(\emptyset) = \emptyset\}$$
$$\Gamma_1 = \{\gamma \in \Gamma \mid \gamma(X) = X\},$$
$$\Gamma_2 = \{\gamma \in \Gamma \mid \gamma^2(A) = \gamma(A) \text{ for every subset } A \text{ of } X\},$$
$$\Gamma_- = \{\gamma \in \Gamma \mid \gamma(A) \subset A \text{ for every subset } A \text{ of } X\} \text{ and}$$
$$\Gamma_+ = \{\gamma \in \Gamma \mid A \subset \gamma(A) \text{ for every subset } A \text{ of } X\}.$$

A subset A of X is said to be $\gamma-open$ [1] if $A \subset \gamma(A)$. B is said to be $\gamma-closed$ [1] if its complement is γ-open. The smallest γ-closed set containing A is called the $\gamma-closure$ of A [1] and is denoted by $c_\gamma(A)$. The largest γ-open set contained in A is called the $\gamma-interior$ of A [1] and is denoted by $i_\gamma(A)$. If $\gamma_1, \gamma_2 \in \Gamma$, then we will denote $\gamma_1 \circ \gamma_2$ by $\gamma_1\gamma_2$. For $\gamma \in \Gamma$, define $\gamma^* : \wp(X) \to \wp(X)$ by $\gamma^*(A) = X - \gamma(X - A)$ [1] for every subset A of X. By a space X, we always mean a topological space (X, τ) with no separation properties assumed. The closure and interior of any subset A of X are denoted by $c(A)$ and $i(A)$ respectively. Moreover, $\Gamma_3 = \{\gamma \in \Gamma \mid G \cap \gamma(A) \subset \gamma(G \cap A)$ for every subset $A \subset X$ and $G \in \tau\}$. If I is a collection of some of the symbols $0, 2, 3, -, +$ and 1, then $\Gamma_I = \{\gamma \in \Gamma \mid \gamma \in \Gamma_i$ for every $i \in I\}$. The following lemmas will be useful in the sequel.

Lemma 1.1. *If $\gamma \in \Gamma$, then $i_\gamma \in \Gamma_{02-}$* [1, Proposition 1.3], *$i_\gamma \in \Gamma_1$ if and only if $\gamma \in \Gamma_1$ and $c_\gamma \in \Gamma_{12+}$* [1, Proposition 1.9].

Lemma 1.2. *If $\gamma \in \Gamma_3$, then $i_\gamma \in \Gamma_3$* [1, Proposition 2.4], *$c_\gamma \in \Gamma_3$* [1, Proposition 2.6] *and $c_\gamma i_\gamma \in \Gamma_3$* [1, Proposition 2.1].

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Lemma 1.3. *If* $\gamma \in \Gamma$, *then* $\gamma^* \in \Gamma$, $(\gamma^*)^* = \gamma$, $(i_\gamma)^* = c_\gamma$ *and* $(c_\gamma)^* = i_\gamma$ [1, Proposition 1.7].

Lemma 1.4. *If* γ_1 *and* $\gamma_2 \in \Gamma$, *then* (i) $\gamma_1\gamma_2 \in \Gamma$, (ii) $\gamma_1, \gamma_2 \in \Gamma_n \Rightarrow \gamma_1\gamma_2 \in \Gamma_n$ *for* $n = 0, 1, +, -$ *and* $(\gamma_1\gamma_2)^* = (\gamma_1)^*(\gamma_2)^*$ [1, Proposition 1.11].

2. γ-semiopen Sets

Let X be any nonempty set and $\gamma \in \Gamma$. A subset A of X is said to be $\gamma - semiopen$ [3] if there exists a γ-open set G such that $G \subset A \subset c_\gamma(G)$. In [3], it is established that X is γ-semiopen, arbitrary union of γ-semiopen sets is γ-semiopen, every γ-open set is γ-semiopen and the intersection of two γ-semiopen sets need not be a γ-semiopen set. The complement of a γ-semiopen set is called a $\gamma - semiclosed$ set [3]. The intersection of all γ-semiclosed sets containing A is called the $\gamma - semiclosure$ [3] of A and is denoted by $sc_\gamma(A)$. It is easy to verify the following.

Theorem 2.1. *Let A be a subset of X and* $\gamma \in \Gamma$. *Then the following hold.*
(a) $sc_\gamma(A)$ *is the smallest* γ*-semiclosed set containing A.*
(b) *A is* γ*-semiclosed if and only if* $A = sc_\gamma(A)$.
(c) $x \in sc_\gamma(A)$ *if and only if for every* γ*-semiopen set G containing x,* $G \cap A \neq \emptyset$.
(d) $sc_\gamma \in \Gamma_{012+}$ ($sc_\gamma \in \Gamma_0$, *since X is* γ*-semiopen* [3, Proposition 1.2] *and so* $\emptyset$ *is* γ*-semiclosed.* $sc_\gamma \in \Gamma_1$, *since* $\emptyset$ *is* γ*-open* [1] *and so is* γ*-semiopen* [3, Proposition 1.3] *which implies that X is* γ*-semiclosed*).

The union of all γ-semiopen sets contained in A is called the $\gamma - semiinterior$ [3] of A and is denoted by $si_\gamma(A)$. It is easy to verify the following.

Theorem 2.2. *Let A be a subset of X and* $\gamma \in \Gamma$. *Then the following hold.*
(a) $si_\gamma(A)$ *is the largest* γ*-semiopen set contained in A.*
(b) *A is* γ*-semiopen if and only if* $A = si_\gamma(A)$.
(c) $x \in si_\gamma(A)$ *if and only if there is a* γ*-semiopen set G containing x such that* $G \subset A$.
(d) $si_\gamma \in \Gamma_{012-}$.

The following Theorem 2.3 gives the relation between sc_γ and si_γ.

Theorem 2.3. *If X is any nonempty set and* $\gamma \in \Gamma$, *then the following hold.*
(a) $(si_\gamma)^* = sc_\gamma$.
(b) $(sc_\gamma)^* = si_\gamma$.
(c) $si_\gamma(X - A) = X - sc_\gamma(A)$ *for every subset A of X.*
(d) $sc_\gamma(X - A) = X - si_\gamma(A)$ *for every subset A of X.*

Proof. (a) Let A be a subset of X. Then $(si_\gamma)^*(A) = X - si_\gamma(X - A)$. Since $si_\gamma(X - A)$ is the largest γ-semiopen set contained in $X - A$, $X - si_\gamma(X - A)$ is the smallest γ-semiclosed set containing A and so $X - si_\gamma(X - A) = sc_\gamma(A)$. Hence $(si_\gamma)^* = sc_\gamma$.
(b) By Lemma 1.3 and (a), $(sc_\gamma)^* = ((si_\gamma)^*)^* = si_\gamma$. This proves (b).

(c) If A is a subset of X, $(si_\gamma)^*(A) = X - si_\gamma(X-A)$ and so by (b), $sc_\gamma(A) = X - si_\gamma(X-A)$ which implies that $si_\gamma(X-A) = X - sc_\gamma(A)$ for every subset A of X.
(d) The proof is similar to the proof of (c).

The following Theorem 2.4 and Theorem 2.5(a) give characterizations of γ-semiopen sets in terms of γ-interior and γ-closure operators. Theorem 2.5(b), (c) and (d) give properties of γ-semiinterior and γ-semiclosure operators. In Theorem 2.6, we prove that the closure of every γ-semiopen set is a γ-semiopen set.

Theorem 2.4. *Let A be a subset of X and $\gamma \in \Gamma$. Then the following are equivalent.*
(a) *A is γ-semiopen.*
(b) $A \subset c_\gamma i_\gamma(A)$.
(c) $c_\gamma(A) = c_\gamma i_\gamma(A)$.

Proof. (a)⇒(b). Suppose A is γ-semiopen. Then there exists a γ-open set G such that $G \subset A \subset c_\gamma(G)$. Since G is γ-open, $G = i_\gamma(G)$ and so $A \subset c_\gamma i_\gamma(G)$. Since $c_\gamma i_\gamma \in \Gamma$, by Lemma 1.4 and $G \subset A$, it follows that $A \subset c_\gamma i_\gamma(A)$ which proves (b).
(b)⇒(c). Since $c_\gamma \in \Gamma$ and $i_\gamma(A) \subset A$, we have $c_\gamma i_\gamma(A) \subset c_\gamma(A)$. By hypothesis and Lemma 1.1, $c_\gamma(A) \subset c_\gamma c_\gamma i_\gamma(A) = c_\gamma i_\gamma(A)$. Therefore, $c_\gamma(A) = c_\gamma i_\gamma(A)$.
(c)⇒(a). Since $i_\gamma(A)$ is a γ-open set such that $i_\gamma(A) \subset A \subset c_\gamma i_\gamma(A)$, A is γ-semiopen.

Theorem 2.5. *Let A be a subset of X and $\gamma \in \Gamma$. Then the following hold.*
(a) *A is γ-semiopen if and only if A is $c_\gamma i_\gamma$-open if and only if $A = i_{c_\gamma i_\gamma}(A)$.*
(b) *$si_\gamma = i_{c_\gamma i_\gamma}$ and $sc_\gamma = c_{c_\gamma i_\gamma}$.*
(c) $si_\gamma(A) = A \cap c_\gamma i_\gamma(A)$.
(d) $sc_\gamma(A) = A \cup i_\gamma c_\gamma(A)$.

Proof. The proof of (a) follows from Theorem 2.4(a) and (b).
(b) If $x \in si_\gamma(A)$, then there exists a γ-semiopen set B such that $x \in B \subset A$. By (a), B is a $c_\gamma i_\gamma$−open set and so $x \in i_{c_\gamma i_\gamma}(A)$. Hence $si_\gamma(A) \subset i_{c_\gamma i_\gamma}(A)$. Similarly, we can prove that $i_{c_\gamma i_\gamma}(A) \subset si_\gamma(A)$. Therefore, $si_\gamma = i_{c_\gamma i_\gamma}$. Again, $sc_\gamma = (si_\gamma)^*$, by Theorem 2.3(a) and so $sc_\gamma = (i_{c_\gamma i_\gamma})^* = c_{c_\gamma i_\gamma}$, by Lemma 1.3.
(c) Since $i_\gamma(i_\gamma(A)) = i_\gamma(A)$ and $i_\gamma(A) \subset i_\gamma c_\gamma(A)$ for every subset A of X, by Theorem 1.3 of [2], we have $i_{c_\gamma i_\gamma}(A) = A \cap c_\gamma i_\gamma(A)$ and so, by (b), $si_\gamma(A) = A \cap c_\gamma i_\gamma(A)$.
(d) Since $i_{c_\gamma i_\gamma}(A) = A \cap c_\gamma i_\gamma(A)$, by Theorem 3.1 of [2], $c_{c_\gamma i_\gamma}(A) = A \cup (c_\gamma i_\gamma)^*(A) = A \cup (c_\gamma)^*(i_\gamma)^*(A) = A \cup i_\gamma c_\gamma(A)$, by Lemmas 1.3 and 1.4. By (b), $sc_\gamma(A) = A \cup i_\gamma c_\gamma(A)$.

Theorem 2.6. *If $\gamma \in \Gamma$, $A \subset B \subset c_\gamma(A)$ and A is γ-semiopen, then B is γ-semiopen. In particular, the γ-closure of every γ-semiopen set is a γ-semiopen set.*

Proof. Since A is γ-semiopen, by Theorem 2.4(c), $c_\gamma(A) = c_\gamma i_\gamma(A)$ and so $c_\gamma(A) \subset c_\gamma i_\gamma(B)$. Since $B \subset c_\gamma(A)$, $B \subset c_\gamma i_\gamma(B)$ and so by Theorem 2.4, B is γ-semiopen.

The following Theorem 2.7 gives characterizations of γ-semiclosed sets.

Theorem 2.7. *Let A be a subset of X and $\gamma \in \Gamma$. Then the following are equivalent.*

(a) *A is γ-semiclosed.*
(b) $i_\gamma c_\gamma(A) \subset A$.
(c) $i_\gamma c_\gamma(A) = i_\gamma(A)$.
(d) *There exists a γ-closed set F such that $i_\gamma(F) \subset A \subset F$.*

Proof. (a)⇒(b). A is γ-semiclosed ⇒ $X - A$ is γ-semiopen ⇒ $X - A \subset c_\gamma i_\gamma(X - A)$, by Theorem 2.4(b). By Lemma 1.3, it follows that $c_\gamma i_\gamma(X - A) = X - i_\gamma c_\gamma(A)$ and so $i_\gamma c_\gamma(A) \subset A$.
(b)⇒(c). $i_\gamma c_\gamma(A) \subset A \Rightarrow i_\gamma c_\gamma(A) \subset i_\gamma(A)$ and so $i_\gamma c_\gamma(A) = i_\gamma(A)$.
(c)⇒(d). If $F = c_\gamma(A)$, then F is a γ-closed set such that $i_\gamma(F) = i_\gamma c_\gamma(A) = i_\gamma(A) \subset A \subset F$, which proves (d).
(d)⇒(a). If there exists a γ-closed set F such that $i_\gamma(F) \subset A \subset F$, then $X - F \subset X - A \subset X - i_\gamma(F) = c_\gamma(X - F)$. Since $X - F$ is γ-open, $X - A$ is γ-semiopen and so A is γ-semiclosed.

We say that A is $c_\gamma - dense$ if $c_\gamma(A) = X$. The following Theorem 2.8 characterizes $c_\gamma-$dense subsets.

Theorem 2.8. *If X is any nonempty set, A is a subset of X and $\gamma \in \Gamma$, then the following are equivalent.*
(a) $c_\gamma(A) = X$.
(b) $sc_\gamma(A) = X$.
(c) *If B is any γ-semiclosed subset of X such that $A \subset B$, then $B = X$.*
(d) *Every nonempty γ-semiopen set has a nonempty intersection with A.*
(e) $si_\gamma(X - A) = \emptyset$.

Proof. (a)⇒(b). Suppose $x \not\in sc_\gamma(A)$. Then there exists a γ-semiopen set G containing x such that $G \cap A = \emptyset$. Since G is a nonempty γ-semiopen set, there is a nonempty γ-open set H such that $H \subset G$ and so $H \cap A = \emptyset$ which implies that $c_\gamma(A) \neq X$, a contradiction. Hence $sc_\gamma(A) = X$.
(b)⇒(c). If B is any γ-semiclosed set such that $A \subset B$, then $X = sc_\gamma(A) \subset sc_\gamma(B) = B$ and so $B = X$.
(c)⇒(d). If G is any nonempty γ-semiopen set such that $G \cap A = \emptyset$, then $A \subset X - G$ and $X - G$ is γ-semiclosed. By hypothesis, $X - G = X$and so $G = \emptyset$, a contradiction. Therefore, $G \cap A \neq \emptyset$.
(d)⇒(e). Suppose $si_\gamma(X - A) \neq \emptyset$. Then $si_\gamma(X - A)$ is a nonempty γ-semiopen set such that $si_\gamma(X - A) \cap A = \emptyset$, a contradiction. Therefore, $si_\gamma(X - A) = \emptyset$.
(e)⇒(a). $si_\gamma(X - A) = \emptyset \Rightarrow X - si_\gamma(X - A) = X \Rightarrow sc_\gamma(A) = X$. By Theorem 2.5(a), $sc_\gamma(B) \subset c_\gamma(B)$ for every subset B of X. Therefore, $sc_\gamma(A) = X$ implies that $c_\gamma(A) = X$.

Theorem 2.9. *If X is any nonempty set, A is a subset of X and $\gamma \in \Gamma$, then the following hold.*
(a) $si_\gamma(sc_\gamma(A)) = sc_\gamma(A) \cap c_\gamma i_\gamma c_\gamma(A)$.
(b) $sc_\gamma(si_\gamma(sc_\gamma(A))) = si_\gamma(sc_\gamma(A))$.
(c) $A \cup si_\gamma(sc_\gamma(A)) = sc_\gamma(A)$.
(d) $sc_\gamma(si_\gamma(A)) = si_\gamma(A) \cup i_\gamma c_\gamma i_\gamma(A)$.
(e) $si_\gamma(sc_\gamma(si_\gamma(A))) = sc_\gamma(si_\gamma(A))$.

(f) $A \cap sc_\gamma(si_\gamma(A)) = si_\gamma(A)$.

Proof. (a) $si_\gamma(sc_\gamma(A)) = sc_\gamma(A) \cap c_\gamma i_\gamma(sc_\gamma(A))$ by Theorem 2.5(c) and so $si_\gamma(sc_\gamma(A)) \subset sc_\gamma(A) \cap c_\gamma i_\gamma c_\gamma(A)$. Again, $si_\gamma(sc_\gamma(A)) = sc_\gamma(A) \cap c_\gamma i_\gamma(sc_\gamma(A)) = sc_\gamma(A) \cap c_\gamma i_\gamma(A \cup i_\gamma c_\gamma(A)) \supset sc_\gamma(A) \cap c_\gamma i_\gamma i_\gamma c_\gamma(A) = sc_\gamma(A) \cap c_\gamma i_\gamma c_\gamma(A)$ and so (a) follows.
(b) Since every γ-closed set is γ-semiclosed, $sc_\gamma(A) \cap c_\gamma i_\gamma c_\gamma(A)$ is γ-semiclosed and so $si_\gamma(sc_\gamma(A))$ is γ-semiclosed. Therefore, (b) follows.
(c) By (a), $A \cup si_\gamma(sc_\gamma(A)) = A \cup (sc_\gamma(A) \cap c_\gamma i_\gamma c_\gamma(A)) = (A \cup sc_\gamma(A)) \cap (A \cup c_\gamma i_\gamma c_\gamma(A)) \supset sc_\gamma(A) \cap (A \cup i_\gamma c_\gamma(A)) = sc_\gamma(A) \cap sc_\gamma(A) = sc_\gamma(A)$ and so $A \cup si_\gamma(sc_\gamma(A)) \supset sc_\gamma(A)$. The reverse direction is clear.
(d), (e) and (f) can be similarly proved.

Example 1.6 of [3] says that the intersection of two γ-semiopen sets in a space need not be γ-semiopen. The following Theorem 2.10 says that if any one of the sets is open and $\gamma \in \Gamma_3$, then the intersection is a γ-semiopen set.

Theorem 2.10. *Let (X,τ) be any space and $\gamma \in \Gamma_3$. If A is open and B is γ-semiopen, then $A \cap B$ is γ-semiopen.*

Proof. Since B is γ-semiopen, there exists a γ-open set G such that $G \subset B \subset c_\gamma(G)$ and so $A \cap G \subset A \cap B \subset A \cap c_\gamma(G)$. By Proposition 2.2 of [1], $A \cap G$ is γ-open and so $A \cap G = i_\gamma(A \cap G)$. By Lemma 1.2, $c_\gamma \in \Gamma_3$ and so $A \cap c_\gamma(G) \subset c_\gamma(A \cap G)$. Therefore, $A \cap B \subset A \cap c_\gamma(G) \subset c_\gamma(A \cap G) = c_\gamma i_\gamma(A \cap G) \subset c_\gamma i_\gamma(A \cap B)$. By Theorem 2.4, $A \cap B$ is γ-semiopen.

If A is any nonempty semiopen set in a space X, then $i(A) \neq \emptyset$ [4, Lemma 1.1]. Similar property will not hold for γ-semiopen sets. The following Example 2.11 gives a γ-semiopen set G for which $i_\gamma(G) = \emptyset$. Theorem 2.12 shows that every open set is a γ-semiopen set, if $\gamma \in \Gamma_3$.

Example 2.11. [1, Example 1.12] Let $\mathbb{R}$ be the set of all real numbers with the usual topology. If γ is defined by $\gamma(A) = \{0\}$ if $0 \in A$, $\gamma(A) = \emptyset$ if $0 \notin A$, then $\gamma \in \Gamma_{023-}$ and $\emptyset$ and $\{0\}$ are the only γ-open sets. If $G = \mathbb{Q} - \{0\}$, where $\mathbb{Q}$ is the set of all rational numbers, then $i_\gamma(G) = \emptyset$ but $c_\gamma i_\gamma(G) = \mathbb{R} - \{0\}$ implies that G is γ-semiopen. This also shows that a γ-semiopen set need not be open.

Theorem 2.12. *Let X be a space and $\gamma \in \Gamma_3$. Then every open set G is a γ-semiopen set.*

Proof. Now $G = G \cap X \subset G \cap c_\gamma i_\gamma(X)$, since X is γ-semiopen. Therefore, $G \subset G \cap c_\gamma i_\gamma(X) \subset c_\gamma i_\gamma(G \cap X) = c_\gamma i_\gamma(G)$, by Lemma 1.2, which implies that G is γ-semiopen.

Theorem 2.13. *If X is a space and $\gamma \in \Gamma_3$, then $si_\gamma \in \Gamma_3$ and $sc_\gamma \in \Gamma_3$.*

Proof. Let G be an open set and A be any subset of X. Then $G \cap si_\gamma(A)$ is a γ-semiopen set by Theorem 2.10 and $G \cap si_\gamma(A) \subset G \cap A$. Since $si_\gamma(G \cap A)$ is the largest γ-semiopen set contained in $G \cap A$, we have $G \cap si_\gamma(A) \subset si_\gamma(G \cap A)$ and so $si_\gamma \in \Gamma_3$. Again, $si_\gamma \in \Gamma_3 \Rightarrow (si_\gamma)^* \in \Gamma_3 \Rightarrow sc_\gamma \in \Gamma_3$.

Corollary 2.14. *If X is a space, $\gamma \in \Gamma_3$ and G is open, then the following hold.*

(a) *For every subset A of X*, $sc_\gamma(G\cap A) = sc_\gamma(G\cap sc_\gamma(A))$.
(b) *If* $sc_\gamma(A) = X$, *then* $sc_\gamma(G\cap A) = sc_\gamma(G)$.

Proof. (a) Since $sc_\gamma \in \Gamma_3$, by Theorem 2.13, $G\cap sc_\gamma(A) \subset sc_\gamma(G\cap A)$ and so $sc_\gamma(G\cap sc_\gamma(A)) \subset sc_\gamma(G\cap A)$. But $G\cap A \subset G\cap sc_\gamma(A) \Rightarrow sc_\gamma(G\cap A) \subset sc_\gamma(G\cap sc_\gamma(A))$. Therefore, $sc_\gamma(G\cap A) = sc_\gamma(G\cap sc_\gamma(A))$.
(b) The proof follows from (a).

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