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ON AN EXTENSION OF A BILATERAL GENERATING FUNCTION INVOLVING GENERALISED BESSEL POLYNOMIALS

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Abstract. In this note we obtain an extension of the bilateral generating function involving generalised Bessel polynomials obtained by S. K. Pan from the existence of a quasi-bilinear generating relation involving the polynomial under consideration.

1. Introduction

In [1], quasi-bilateral generating function is defined as follows:

$$G(x,u,w)=\sum_{n=0}^{\infty}a_n p_n^{(\alpha)}(x)q_m^{(n)}(u)w^n \tag{1.1}$$

where $p_n^{(\alpha)}(x)$ and $q_m^{(n)}(u)$ are two polynomials of orders n, m and of parameters α, n respectively. When $q_m^{(n)}(u)\equiv p_m^{(n)}(u)$, then the above generating relation is quasi-bilinear.

In [2], Pan obtained the following theorem on bilateral generating functions involving generalised Bessel Polynomials, $Y_n^{(\alpha)}(x)$ introduced by H. L. Krall and O. Frink [3]:

Theorem 1. *If*

$$G(x,w)=\sum_{n=0}^{\infty}a_n Y_n^{(\alpha)}(x)w^n$$

then

$$e^{-\frac{\beta w}{(1-wx)}}(1-wx)^{\alpha-2}G(x(1-wx),wz)=\sum_{n=0}^{\infty}w^n f_n(x,z)$$

where

$$f_n(x,z)=\sum_{p=0}^{n}a_p\frac{\beta^{n-p}}{(n-p)!}Y_n^{(\alpha-2(n-p))}(x)z^p.$$

The object of the present paper is to obtain the following theorem as an extension of the above theorem from the concept of quasi-bilinear generating function by using one parameter continuous transformations group.

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Theorem 2. *If*

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nY_n^{(\alpha)}(x)Y_m^{(n)}(u)w^n$$

then

$$(1-w)^{-m+1}(1-wx)^{\alpha-2}\exp\Big(-\frac{\beta w}{1-wx}\Big)G\Big(x(1-wx),\frac{u}{1-w},\frac{wv}{1-w}\Big)$$
$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!q!}v^p\beta^pY_{n+p}^{(\alpha-2p)}(x)(m+n-1)_qY_m^{(n+q)}(u).$$

2. Proof of the Theorem

In [4, 5], we see that

$$R_1=x^2y^{-2}z\frac{\partial}{\partial x}+xy^{-1}z\frac{\partial}{\partial y}+\beta y^{-2}z-2xy^{-2}z.$$
$$R_2=ut\frac{\partial}{\partial u}+t^2\frac{\partial}{\partial t}+(m-1)t$$

such that

$$R_1(Y_n^{(\alpha)}(x)y^\alpha z^n)=\beta Y_{n+1}^{(\alpha-2)}(x)y^{\alpha-2}z^{n+1} \tag{2.1}$$
$$R_2(Y_m^{(n)}(u)t^n)=(m+n-1)Y_m^{(n+1)}(u)t^n \tag{2.2}$$

and

$$e^{wR_1}f(x,y,z)=\exp\Big[-\frac{\beta wy^{-2}z}{1-wxy^{-2}z}\Big](1-wxy^{-2}z)^{-2}$$
$$\times f\Big(x(1-wxy^{-2}z),y(1-wxy^{-2}z),z\Big) \tag{2.3}$$
$$e^{wR_2}f(u,t)=(1-wt)^{-m+1}f\Big(\frac{u}{1-wt},\frac{t}{1-wt}\Big). \tag{2.4}$$

Let we consider

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nY_n^{(\alpha)}(x)Y_m^{(n)}(u)w^n. \tag{2.5}$$

Replacing w by $wztv$ and then multiplying both sides of (2.5) by y^α, we get

$$y^\alpha G(x,u,wztv)=\sum_{n=0}^{\infty}a_n(wv)^n(Y_n^{(\alpha)}(x)y^\alpha z^n)(Y_m^{(n)}(u)t^n). \tag{2.6}$$

Operating $e^{wR_1}e^{wR_2}$ on both sides of (2.6), we get

$$e^{wR_1}e^{wR_2}(y^\alpha G(x,u,wztv))$$
$$=e^{wR_1}e^{wR_2}\Big(\sum_{n=0}^{\infty}a_n(wv)^n(Y_n^{(\alpha)}(x)y^\alpha z^n)(Y_m^{(n)}(u)t^n)\Big). \tag{2.7}$$

L.H.S. of (2.7) is

$$\begin{aligned}
&e^{wR_1}e^{wR_2}(y^{\alpha}G(x,u,wztv))\\
&\quad= e^{wR_1}\Big[(1-wt)^{-m+1}y^{\alpha}G\Big(x,\frac{u}{1-wt},\frac{wztv}{1-wt}\Big)\Big]\\
&\quad= (1-wt)^{-m+1}\exp\Big(-\frac{\beta w y^{-2}z}{1-wxy^{-2}z}\Big)(1-wxy^{-2}z)^{-2}\\
&\qquad\times y^{\alpha}(1-wxy^{-2}z)^{\alpha}G\Big(x(1-wxy^{-2}z),\frac{u}{1-wt},\frac{wztv}{1-wt}\Big).
\end{aligned}\tag{2.8}$$

R.H.S. of (2.7) is

$$\begin{aligned}
&e^{wR_1}e^{wR_2}\Big(\sum_{n=0}^{\infty}a_n(wv)^n(Y_n^{(\alpha)}(x)y^{\alpha}z^n)(Y_m^{(n)}(u)t^n)\Big)\\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!q!}v^nR_1^p(Y_n^{(\alpha)}(x)y^{\alpha}z^n)R_2^q(Y_m^{(n)}(u)t^n)\\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!q!}v^n\beta^pY_{n+p}^{(\alpha-2p)}(x)y^{\alpha-2p}z^{n-2p}(m+n-1)_qY_m^{(n+q)}(u)t^{n+q}.
\end{aligned}\tag{2.9}$$

Equation (2.8) and (2.9) and then putting $y=z=t=1$, we get

$$\begin{aligned}
&(1-w)^{-m+1}(1-wx)^{\alpha-2}\exp\Big(-\frac{\beta w}{1-wx}\Big)G\Big(x(1-wx),\frac{u}{1-w},\frac{wv}{1-w}\Big)\\
&\quad= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!q!}v^n\beta^pY_{n+p}^{(\alpha-2p)}(x)(m+n-1)_qY_m^{(n+q)}(u).
\end{aligned}\tag{2.10}$$

which is Theorem 2.

Particular Case:

If we put $m=0$ in (2.10), we get

$$\begin{aligned}
&(1-w)(1-wx)^{\alpha-2}\exp\Big(-\frac{\beta w}{1-wx}\Big)G\Big(x(1-wx),\frac{wv}{1-w}\Big)\\
&\quad= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}a_n\frac{w^{n+p+q}}{p!q!}v^n\beta^pY_{n+p}^{(\alpha-2p)}(x)(n-1)_q\\
&\quad= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}a_n\frac{w^{n+p}}{p!}v^n\beta^pY_{n+p}^{(\alpha-2p)}(x)\sum_{q=0}^{\infty}\frac{w^q}{q!}(n-1)_q\\
&\quad= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}a_n\frac{w^{n+p}}{p!}v^n\beta^pY_{n+p}^{(\alpha-2p)}(x)(1-w)^{-n+1}\\
&\quad= (1-w)\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}a_n\Big(\frac{v}{1-w}\Big)^n\frac{w^{n+p}}{p!}\beta^pY_{n+p}^{(\alpha-2p)}(x).
\end{aligned}$$

Therefore,

$$(1-wx)^{\alpha-2}\exp\Big(-\frac{\beta w}{1-wx}\Big)G\Big(x(1-wx),\frac{wv}{1-w}\Big)$$

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n w^{n+p} \frac{\beta^p}{p!} Y_{n+p}^{(\alpha-2p)}(x)\left(\frac{v}{1-w}\right)^n.$$

Replacing $\dfrac{v}{1-w}$ by v, we get

$$\begin{aligned}
&(1-wx)^{\alpha-2}\exp\left(-\frac{\beta w}{1-wx}\right)G(x(1-wx), wv)\\
&\quad= \sum_{n=0}^{\infty} w^n \sum_{p=0}^{\infty} a_{n-p}\frac{\beta^p}{p!} Y_n^{(\alpha-2p)}(x)\, v^{n-p}\\
&\quad= \sum_{n=0}^{\infty} w^n \sum_{p=0}^{n} a_p \frac{\beta^{n-p}}{(n-p)!} Y_n^{(\alpha-2(n-p))}(x)\, v^p\\
&\quad= \sum_{n=0}^{\infty} w^n f_n(x, v),
\end{aligned}$$

where

$$f_n(x, v) = \sum_{p=0}^{\infty} a_p \frac{\beta^{n-p}}{(n-p)!} Y_n^{(\alpha-2(n-p))}(x)\, v^p,$$

which is Theorem 1.

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References

[1] S. K. Chatterjea and S. P. Chakraborty, *A unified group-theoretic method of obtaining a more general class of generating relations from a given class of quasi-bilateral* (*or quasi-bilinear*) *generating relations involving some special functions*, Pure Math. Manuscript **8**(1989), 153–162.

[2] S. K. Pan, Application of group-theory in the study of some problems of special functions, Ph.D. Thesis, C.U. (1995).

[3] H. L. Krall and O. Frink, *An orthogonal polynomial; The Bessel Polynomials*, Trans. Amer. Math. Soc. **65**(1949), 100–115.

[4] A. K. Chongdar, *On generating functions of Bessel Polynomials by group-theoretic method*, Boll. U.M.I. **7**(1991), 163–170.

[5] M. P. Chen and C. C. Feng, *Group-theoretic origins of certain of certain generating functions of generalised Bessel polynomials*, Tamkang Jour. Math. **6**(1975), 87–93.

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