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ON COEFFICIENTS OF α-SHAPIRO SHIELDS FUNCTIONS

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Abstract. We investigate the Taylor coefficients of α-Shapiro Shields functions and some applications to α-Bloch functions and Bergman functions are given.

1. Introduction

Let $\mathcal{D} = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$. A function f analytic in $\mathcal{D}$ is called to be *α-Shapiro Shields* $(\alpha > 0)$ [4, p.224] if f satisfies

$$\sup\{(1-|z|)^{\alpha}|f(z)| : z \in \mathcal{D}\} < \infty \tag{1.1}$$

The set of all α-Shapiro Shields functions is denoted by $S^{-\alpha}$. Let $S_0^{-\alpha}$ be the set of all functions f analytic in $\mathcal{D}$ (little α-Shapiro Shields functions) such that

$$(1-|z|)^{\alpha}|f(z)| \to 0,\ |z| \to 1^-.$$

A function f analytic in $\mathcal{D}$ is called an *α-Bloch function* $(\alpha > 0)$ [5] if $f' \in S^{-\alpha}$, denoted by $f \in B^{\alpha}$. A function f analytic in $\mathcal{D}$ is called a little *α-Bloch function* $(\alpha > 0)$ if $f' \in S_0^{-\alpha}$, denoted by $f \in B_0^{\alpha}$.

If a function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ belongs to a certain $S^{-\alpha}$ space, what can be said about its Taylor coefficients a_n? Ideally, one would like to find a condition on the a_n which is both sufficient and necessary for f to be in $S^{-\alpha}$. But the general situation is much more complicated, and no complete answer is available. In this paper, under an argument condition (see expression (2.1)) on coefficients, a sufficient and necessary condition on coefficients for functions to be α-Shapiro Shields is proved (see Theorem 2.1). An example shows that the argument condition cannot be substantially relaxed. In Section 3 we obtain two sufficient conditions and one necessary condition on coefficients of α-Shapiro Shields functions, and show by examples that any one of these conditions can not be both sufficient and necessary. However, if we consider the special case, Hadamard gap series, a sufficient and necessary condition on coefficients for functions to be α-Shapiro Shields is easily obtained (see Theorem 3.4). In the last section we mention some applications to α-Bloch functions and Bergman functions.

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2. A Sufficient and Necessary Condition on Coefficients of α-Shapiro Shields Functions

Theorem 2.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If there is a $\theta \in [0, 2\pi)$ such that the argument condition:*

$$\theta \leq \arg a_n \leq \theta + \frac{\pi}{2} \tag{2.1}$$

holds for each $n \in \mathbb{N}$, then for $\alpha > 0$ the following are equivalent:

(1) $f \in S^{-\alpha}$.

(2) *For each nonnegative integer J we have*

$$\sum_{k=J}^{n} k^J |a_k| = O(n^{J+\alpha}). \tag{2.2}$$

(3) *There exists a nonnegative integer J such that* (2.2) *holds.*

Proof. We first prove that (1) implies (2). There is no loss of generality in assuming that $\theta = 0$. Assume $f \in S^{-\alpha}$, we have

$$\sum_{k=0}^{\infty} a_k z^k = O((1 - |z|)^{-\alpha})$$

and

$$\left| \sum_{k=0}^{\infty} \bar{a}_k z^k \right| = |f(\bar{z})| = O((1 - |z|)^{-\alpha}).$$

Therefore,

$$\sum_{k=0}^{\infty} Re(a_k) z^k = O((1 - |z|)^{-\alpha}),$$

where $0 \leq Re(a_k)$ since $0 \leq \arg a_n \leq \frac{\pi}{2}$, $k = 0, 1, \cdots$. Similarly

$$\sum_{k=0}^{\infty} Im(a_k) z^k = O((1 - |z|)^{-\alpha}),$$

where $0 \leq Im(a_k)$, $k = 0, 1, \cdots$. Thus

$$\sum_{k=0}^{\infty} |a_k| z^k = O((1 - |z|)^{-\alpha}).$$

For each positive integer n, we choose $z = 1 - 1/n$. Then

$$\sum_{k=0}^{n} |a_k| (1 - 1/n)^k = O(n^{\alpha}).$$

For $0 \le k \le n$, we have

$$(1-1/n)^k \ge (1-1/n)^n \to 1/e, \qquad n \to \infty.$$

Therefore,

$$\sum_{k=0}^{n} |a_k| = O(n^\alpha), \qquad n \to \infty. \tag{2.3}$$

For each nonnegative integer J, and $n \ge J$, it follows from (2.3) that

$$\sum_{k=J}^{n} k^J |a_k| \le n^J \sum_{k=J}^{n} |a_k| = O(n^{\alpha+J}).$$

Hence (2) holds.

(2) implies (3) is obvious.

Finally we show that (3) implies (1). We may suppose that (2.2) holds for a fixed nonnegative integer J. Then there exists a constant C such that

$$\sum_{k=J}^{n} k^J |a_k| \le C_n^{\alpha+J}, \ n \in \mathbb{N}. \tag{2.4}$$

Here and elsewhere constants are denoted by C which is positive and may indicate different from one occurrence to the next. Since

$$\frac{1}{(1-|z|^2)^\lambda} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda)}{n!\Gamma(\lambda)} |z|^{2n}, \ \lambda > 0,$$

and $\frac{\Gamma(n+\lambda)}{n!} \approx n^{\lambda-1}$ as $n \to \infty$ by Stirling's formula, it follows from (2.4) that

$$\begin{aligned}
|f^{(J)}(z)| &\le C \sum_{n=J}^{\infty} n^J |a_n| |z|^{n-J} \sum_{n=J}^{\infty} |z|^{n-J} (1-|z|) \\
&= C(1-|z|) \sum_{n=J}^{\infty} \Big(\sum_{k=J}^{n} k^J |a_k| \Big) |z|^{n-J} \\
&\le C(1-|z|) \sum_{n=J}^{\infty} n^{\alpha+J} |z|^{n-J} \\
&\le C(1-|z|)^{-(\alpha+J)}.
\end{aligned} \tag{2.5}$$

By successive integration this shows that f is α-Shapiro Shields. This completes the proof of Theorem 2.1.

It follows from Theorem 2.1 that

Corollary 2.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_n \ge 0$ and $\alpha > 0$. Then the following statements are equivalent:*

(1) $f \in S^{-\alpha}$.
(2) *For each nonnegative integer* J *we have*

$$\sum_{k=J}^{n} k^J a_k = O(n^{J+\alpha}). \tag{2.6}$$

(3) *There exists a nonnegative integer* J *such that* (2.6) *holds.*

Corollary 2.3. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$*. If there exists a* $\theta \in [0, 2\pi)$ *such that the argument condition:*

$$\theta \leq \arg a_n \leq \theta + \frac{\pi}{2}$$

holds for each $n \in \mathbb{N}$*, then for* $\alpha > 0$ *the following are equivalent:*
(1) $f \in B^{\alpha}$.
(2) *For each nonnegative integer* J *we have*

$$\sum_{k=J}^{n} k^{J+1} |a_k| = O(n^{J+\alpha}). \tag{2.7}$$

(3) *There exists a nonnegative integer* J *such that* (2.7) *holds.*

The following example shows that the argument condition (2.1) can not be substantially relaxed in Theorem 2.1. Let us first state an earlier result which is due to Hardy; see [2].

Lemma. *Let* $f(z) = \sum_{n=0}^{\infty} n^{-b} e^{in^a} z^n$, $0 < a < 1$*. If* $1 - b - a/2 > 0$*, then* f *is unbounded and*

$$|f(z)| = O((1 - |z|)^{-(1-b-a/2)}).$$

Example 1. For $\alpha > 0$ there exists an α-Shapiro Shields function f which does not satisfy the condition (2.2) for any given nonnegative integer J.

Proof. For $\alpha > 0$, we choose $f(z) = \sum_{n=0}^{\infty} n^{\alpha-1+\frac{a}{2}} e^{in^a} z^n$, where $0 < a < 1$. By Lemma, $|f(z)| = O((1 - |z|)^{-\alpha})$, so $f \in S^{-\alpha}$. However, for given nonnegative integer J, we have

$$\begin{aligned} \frac{1}{n^{\alpha+J}} \sum_{k=J}^{n} k^J |a_k| &= \frac{1}{n^{\alpha+J}} \sum_{k=J}^{n} k^{J+\alpha-1+\frac{a}{2}} \\ &= \frac{1}{n^{\alpha+J}} \Big(\sum_{k=0}^{n} k^{J+\alpha-1+\frac{a}{2}} - \sum_{k=0}^{J-1} k^{J+\alpha-1+\frac{a}{2}} \Big) \\ &\geq \frac{1}{n^{\alpha+J}} \Big(C n^{J+\alpha+\frac{a}{2}} - \sum_{k=0}^{J-1} k^{J+\alpha-1+\frac{a}{2}} \Big) \to \infty,\ n \to \infty. \end{aligned}$$

This example also indicates that the argument condition (2.1) cannot be substantially relaxed in Theorem 2.1.

3. Two Sufficient Conditions and One Necessary Condition on Coefficients of α-Shapiro Shields Functions

Theorem 3.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathcal{D}$ and $\alpha > 0$. If the coefficient condition* (2.2) *is satisfied for some nonnegative integer J, then $f \in S^{-\alpha}$.*

The proof of Theorem 3.1 is given implicitly in the proof of Theorem 2.1 in which we proved (3) $\Longrightarrow$ (1) without the argument condition; hence the coefficient condition (2.2) is sufficient for a given analytic function f to be α-Shapiro Shields. However it is not necessary; see the example 1 above.

Theorem 3.2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $\mathcal{D}$. If $\alpha > 0$ and*

$$\sum_{k=1}^{n} k^{q-1} |a_k|^q = O(n^{\alpha q}) \tag{3.1}$$

for some q, $1 \leq q \leq \infty$, then $f \in S^{-\alpha}$.

Proof. By Theorem 2.1 and Theorem 3.1 we only need to consider the case $1 < q < \infty$. Applying Theorem 3.1 it suffices to show that $\sum_{k=1}^{n} k|a_k| = O(n^{\alpha+1})$. By Hölder inequality, for $1/p + 1/q = 1$, we have

$$\sum_{k=1}^{n} k|a_k| \leq \Big(\sum_{k=1}^{n} k^{p-1} \Big)^{1/p} \Big(\sum_{k=1}^{n} k^{q-1} |a_k|^q)^{1/q} = O(n^{1+\alpha} \Big).$$

Example 2. For $\alpha > 0$ there exists a function $f \in S^{-\alpha}$, but f does not satisfy (3.1) for any $q \geq 1$.

Proof. We consider the function $f(z)$ in Example 1 again. So $f \in S^{-\alpha}$. However, for any $q \geq 1$, we have

$$\sum_{k=1}^{n} k^{q-1} |a_k|^q = \sum_{k=1}^{n} k^{\alpha q - 1 + q\frac{a}{2}} \approx n^{\alpha q + q\frac{a}{2}} \neq O(n^{\alpha q}).$$

Theorem 3.3. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a function in $S^{-\alpha}$, then we have*

$$a_n = O(n^{\alpha}),\ n \to \infty. \tag{3.2}$$

Proof. From Cauchy's formula

$$
\begin{aligned}
|a_n| &= \Big|\frac{1}{2\pi i}\int_{|z|=r}\frac{f(z)}{z^n}dz\Big| \\
&\le \frac{C}{2\pi}\Big(\frac{r^{1-n}}{(1-r)^\alpha}\Big) \\
&= O\Big(\frac{r^{1-n}}{(1-r)^\alpha}\Big), \qquad r\to 1^-.
\end{aligned}
$$

The minimum value of the last term occurs at $r_0 = 1-\frac{\alpha}{n-1}$. Evaluating for this r_0, we obtain

$$\frac{r_0^{1-n}}{(1-r_0)^\alpha} = (1-\frac{\alpha}{n-1})^{1-n}(\frac{n-1}{\alpha})^\alpha = O(n^\alpha)$$

as $n\to\infty$, and the result follows.

Example 3. There is a function $f(z)=\sum_{n=0}^\infty a_n z^n$ with (3.2), but $f\notin S^{-\alpha}$.

Proof. It is easy to check that the function $f(z)=\sum_{n=0}^\infty n^\alpha z^n$ satisfies (3.2), but

$$\sum_{k=0}^n k a_k = \sum_{k=0}^n k^{\alpha+1} \approx n^{\alpha+2}+O(1).$$

It shows by Corollary 2.2 that f is not α-Shapiro Shields.

Although (3.2) is a necessary condition not a sufficient condition for f to be α-Shapiro Shields, in case f is a Hadamard gap series we have the following result.

Theorem 3.4. *Let $f(z)=\sum_{k=0}^\infty a_{n_k}z^{n_k}$ be analytic in $\mathcal{D}$ and $\alpha>0$. If f has Hadamard gaps, that is*

$$n_{k+1}/n_k \ge \lambda > 1, \qquad k=0,1,\cdots,$$

then $f\in S^{-\alpha}$ if and only if f satisfies

$$a_{n_k} = O(n_k^\alpha),\ n\to\infty. \tag{3.3}$$

Proof. It suffices to prove that if f satisfies (3.3), then $f\in S^{-\alpha}$. For each large positive integer m, put $i_m=\max\{j: n_j\le m\}$. Then

$$
\begin{aligned}
\frac{1}{m^\alpha}\sum_{n_j\le m}|a_{n_j}| &\le \frac{C}{m^\alpha}\sum_{j=0}^{i_m} n_j^\alpha \\
&\le \frac{C}{m^\alpha}\sum_{j=0}^{i_m}\lambda^{\alpha(j-i_m)}n_{i_m}^\alpha \\
&= O\Big(\sum_{j=0}^{i_m}\lambda^{\alpha(j-i_m)}\Big) = O(1).
\end{aligned}
$$

Applying Theorem 2.1, $f \in S^{-\alpha}$.

4. Notes

1. If there is a result on the coefficients for α-Shapiro Shields functions in terms of its Taylor coefficients a_n with magnitude big oh, then there is a similar result for little α-Shapiro Shields functions in terms of a_n with magnitude little oh and vice versa. In this way we may establish some results on the coefficients of little α-Shapiro Shields functions.

2. All results in this paper can be applied to α-Bloch functions since $f \in B^{\alpha}$ if and only if $f' \in S^{-\alpha}$ for analytic functions f. For example, Yamashita's result [4] can be obtained by Theorem 3.4.

3. Let $L_a^p(\mathcal{D})(1 \leq p \leq \infty)$ be the Bergman space [3] of all analytic functions f such that $|f|^p$ is integrable in $\mathcal{D}$. Since $L_a^p(\mathcal{D}) \subset S^{-2/p}$ and $S^{-\alpha} \subset L_a^{1/\alpha+\varepsilon}(\mathcal{D})$ for $\varepsilon > 0$ (see [1]), some results on the coefficients for α-Shapiro Shields functions can be held for the functions of Bergman space.

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