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BEST APPROXIMATION OF CONJUGATE OF A FUNCTION IN GENERALIZED ZYGMUND CLASS

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Abstract. In this paper, we, for the very first time, study the error estimates of function $\tilde{g}$, conjugate to a function g (2π-periodic) in generalized Zygmund class $Y_z^w (z \geq 1)$ by Matix-Euler (TE^q) product means of its conjugate Fourier series. In fact, we establish two theorems on degree of approximation of a function $\tilde{g}$ of g (2π-periodic) in generalized Zygmund class $Y_z^w (z \geq 1)$ by Matix-Euler (TE^q) product means of its conjugate Fourier series. Our first theorem generalizes three previously known results. Thus, the results of [7], [8] and [26] become the particular cases of our Theorem 1. Some corollaries are also deduced from our Theorem 1.

1. Introduction

The studies of error estimation of conjugate of a function in the different Lipschitz classes ($Lip(\alpha)$, $Lip(\alpha, r)$, $Lip(\xi(t), r)$ and $W(L_r, \xi(t))$ classes) by single summability operators such as Euler, Cesàro, Nörlund, generalized Nörlund, Hölder, Karamata, Riesz, matrix etc. have been a centre of creative study for the several researchers like Kushwaha [14] , Nigam and Sharma [12], Qureshi [15, 16, 17, 18], Lal and Nigam [24], Lal [22] , Mittal et al. [19] , Rhoades [2, 3] Singh and Srivastava [27] and Kranz et al. [21] in last few decades.

The studies of error estimation of a conjugate function $\tilde{g}$ of g (2π-periodic) in different Lipschitz classes using products means have also been of great interest among the researchers like Lal and Singh [25, 26], Dhakal [4], Nigam and Sharma [9, 10, 11, 13] etc. in recent past.

Our motivation to this work is to consider more advance class of function that can provide best approximation of a function $\tilde{g}$ by works a trigonometric polynomial of degree not more than r.

The review of above research works clearly suggests that the studies of error estimation of function $\tilde{g}$, conjugate to a function g (2π-periodic) in generalized Zygmund class $Y_z^w (z \geq 1)$ using Matrix-Euler (TE^q) means of conjugate Fourier series have not been initiated so far.

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Therefore, we study the error estimates of function $\tilde{g}$, conjugate to a function g (2π-periodic) in generalized Zygmund class $Y_z^w (z \geq 1)$ by Matix-Euler (TE^q) product means of its conjugate Fourier series. In fact, we establish two theorems on degree of approximation of conjugate function $\tilde{g}$ of g (2π-periodic) in generalized Zygmund class $Y_z^w (z \geq 1)$ by Matix-Euler (TE^q) product means of its conjugate Fourier series. Our main theorem generalizes three previously known results. Thus, the results of [7], [8] and [26] become the particular cases of our Theorem 1.

Note 1. The conjugate Fourier series is not necessarily a Fourier series for example: The series $\sum_{n=2}^{\infty} \frac{\sin(nx)}{\log n}$ conjugate to the Fourier series $\sum_{n=2}^{\infty} \frac{\cos(nx)}{\log n}$ is not a Fourier series ([1], p. 186).

In view of the above example, a separate study of conjugate series of Fourier series in the present direction of work is so required.

We are not representing here the Fourier series and the conjugate Fourier series as these series are well known and the detailed work on these series can be found in [1].

We denote the r^{th} partial sum of conjugate Fourier series by $(\tilde{g}; x)$, which is given by

$$s_r(\tilde{g}; x) - \tilde{g}(x) = \frac{1}{2\pi} \int_0^{\pi} \psi_x(l) \frac{\cos(r + \frac{1}{2})l}{\sin(\frac{l}{2})} dl,$$

where $\tilde{g}$ is the conjugate function of g and is given by

$$\tilde{g} = -\frac{1}{2\pi} \int_0^{\pi} \psi_x(l) \cot\left(\frac{l}{2}\right) dt.$$

Let $T = (a_{r,m})$ be an infinite triangular matrix satisfying the conditions of regularity ([20]) i.e.,

$$\begin{aligned} \sum_{m=0}^{r} a_{r,m} = 1 \quad &\text{as} \quad r \to \infty \\ a_{r,m} = 0 \quad &\text{for} \quad m > r \\ \sum_{m=0}^{r} |a_{r,m}| \leq M, \quad &\text{a finite constant.} \end{aligned} \tag{1}$$

The sequence-to-sequence transformation

$$t_r^T := \sum_{m=0}^{r} a_{r,m} s_m = \sum_{m=0}^{r} a_{r,r-m} s_{r-m} \tag{2}$$

defines the sequence t_r^T of triangular matrix means of the sequence $\{s_r\}$ generated by the sequence of coefficients $(a_{r,m})$.

If $t_r^T \to s$ as $r \to \infty$ then the conjugate Fourier series is said to be summable to s by T-method [1].

Let

$$E_r^q = \frac{1}{(1+q)^r} \sum_{m=0}^{r} \binom{r}{m} q^{r-m} s_m. \tag{3}$$

If $E_r^q \to s$ as $r \to \infty$ then the conjugate Fourier series is said to be summable to s by E^q means [6].

The TE^q means (T-means of E^q means) is given by

$$\begin{aligned} \tilde{t}_r^{T.E^q} &:= \sum_{m=0}^{r} a_{r,m} E_m^q \\ &= \sum_{m=0}^{r} a_{r,m} \frac{1}{(1+q)^m} \sum_{\nu=0}^{m} \binom{m}{\nu} q^{m-\nu} s_\nu. \end{aligned} \tag{4}$$

If $\tilde{t}_r^{T.E^q} \to s$ as $r \to \infty$, then the conjugate Fourier series is said to be summable to s by $T.E^q$ means.

Note 2. The regularity of T and E^q methods implies regularity of $T.E^q$ method.

Remark 1. TE^q means reduces to

(i) $(H, \frac{1}{r+1})E^q$ or $H.E^q$ means if $a_{r,m} = \frac{1}{(r-m+1)\log(r+1)}$.
(ii) $(C,1)E^q$ or C^1E^q means if $a_{r,m} = \frac{1}{r+1}$.
(iii) $(N, p_r)E^q$ or N_pE^q means if $a_{r,m} = \frac{p_{r-m}}{P_r}$ where $P_r = \sum_{m=0}^{r} p_m \neq 0$.
(iv) $(N, p, q)E^q$ or $N_{p,q}E^q$ means if $a_{r,m} = \frac{p_{r-m}q_m}{R_r}$ where $R_r = \sum_{m=0}^{r} p_m q_{r-m}$.
(v) $(\bar{N}, p_r)E^q$ or $\bar{N}_pE^q$ means if $a_{r,m} = \frac{p_m}{P_r}$

Note 3. If $q_m = 1$ for all m then $H.E^1, C^1E^1, N_pE^1, N_{p,q}E^1$ and $\bar{N}_pE^1$ are also the particular cases of the $T.E^q$ means.

Let $L^z[0,2\pi] = \{g : [0,2\pi] \to \mathbb{R} : \int_0^{2\pi} |g(x)|^z dx < \infty,\ z \geq 1\}$ be the space of functions (2π-periodic and integrable).

Now, we define $\|\cdot\|_r$ by

$$\|g\|_z := \begin{cases} \left\{\frac{1}{2\pi}\int_0^{2\pi} |g(x)|^r dx\right\}^{1/r} & \text{for } 1 \leq z < \infty \\ ess \sup\limits_{x\in(0,2\pi)} |g(x)| & \text{for } z = \infty. \end{cases}$$

As defined in [1], $w : [0,2\pi] \to \mathbb{R}$ be an arbitrary function with $w(l) > 0$ for $0 < l \leq 2\pi$ and $\lim\limits_{l\to 0^+} w(l) = w(0) = 0$.

We define

$$Y_z^{(w)} = \left\{g \in L^z[0,2\pi] : \sup_{l\neq 0} \frac{\|g(\cdot,+l) - g(\cdot,-l)\|_z}{w(l)} < \infty, z \geq 1\right\}$$

and

$$\|.\|_z^{(w)} = \|g\|_z^{(w)} = \|g\|_z + \sup_{l\neq 0} \frac{\|g(\cdot,+l) - g(\cdot,-l)\|_z}{w(l)}; z \geq 1.$$

Hence, the space Y_z^w is considered as a Banach space represented by the norm $\|\|_z^{(w)}$.

The completeness of the space $Z_r^{(w)}$ can be discussed by considering the completeness of L^r, $r \geq 1$.

We define

$$\|g\|_z^{(v)} := \|g\|_z + \sup_{l\neq 0} \frac{\|g(\cdot + l) - g(\cdot - l)\|_z}{v(l)}, \quad z \geq 1.$$

Remark 2. $w(l)$ and $v(l)$ denote moduli of continuity of order 2 [1].

If we consider $\frac{w(l)}{v(l)}$ as positive and non-decreasing,

$$\|g\|_z^{(v)} \leq \max\left(1, \frac{w(2\pi)}{v(2\pi)}\right) \|g\|_z^{(w)} < \infty$$

Thus,

$$Y_z^{(w)} \subset Y_z^{(v)} \subset L^z; z \geq 1.$$

Remark 3.

(i) If $w(l) = l^\alpha$ in $Y^{(w)}$, $Y^{(w)}$ implies Y_α class.

(ii) If $w(l) = l^\alpha$ in $Y_z^{(w)}$, $Y^{(w)}$ implies $Y_{\alpha,z}$ class.

(iii) If $z \to \infty$ in $Y_z^{(w)}$, $Y_z^{(w)}$ implies $Y^{(w)}$ class and $Y_{\alpha,z}$ class implies Y_α class.

The error estimation of function $\tilde{g}$ is given by

$$E_r(\tilde{g}) = \min \|\tilde{g} - t_r\|_z,$$

where $\tilde{t}_r$ is a trigonometric polynomial of degree r, [1].

We write,

$$\psi_x(l) = \psi(x,l) = g(x+l) - g(x-l)$$

$$\tilde{M}_r(l) = \frac{1}{2\pi} \sum_{m=0}^{r} a_{r,m} \frac{1}{m+1} \sum_{v=0}^{m} \frac{\cos(v+\frac{1}{2})l}{\sin(\frac{l}{2})}.$$

2. Main Theorems

Theorem 1. *If $\tilde{g} \in Y_z^{(w)} class; z \geq 1$ and $\frac{w(l)}{v(l)}$ be positive, non-decreasing then the error estimation of $\tilde{g}$ by TE^q means of its conjugate Fourier series is*

$$||\tilde{t}_r^{\,T.E^q} - \tilde{g}||_z^{(v)} = O\left(\frac{\log \pi (r+1)^2}{\log \pi (r+1)} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l\,v(l)} dl\right),$$

where $T = (a_{r,m})$ be an infinite triangular matrix satisfying (1) *and w, v are defined as in Remark 2.*

Theorem 2. *If $\tilde{g} \in Y_z^{(w)} class; z \geq 1$ and in addition of Theorem 1, if $\frac{w(l)}{l v(l)}$ be non-decreasing then the error estimation of $\tilde{g}$ by TE^q means of its conjugate Fourier series is*

$$\| \tilde{t}_r^{\,T.E^q} - \tilde{g} \|_z^{(v)} = O\left(\log \pi (r+1)^2 \frac{w\left(\frac{1}{r+1}\right)}{v\left(\frac{1}{r+1}\right)} \right),$$

where $T = (a_{r,m})$ be an infinite triangular matrix satisfying (1) *and w, v are defined as in Remark 2.*

3. Lemmas

Lemma 1. *Under condition* (1), *$\tilde{M}_r(l) = O(\frac{1}{l})$ for $0 < l < \frac{1}{r+1}$.*

Proof. For $0 < l \leq \frac{1}{r+1}$, using $\sin(\frac{l}{2}) \geq \frac{l}{\pi}$ and $|\cos rl| \leq 1$

$$\begin{aligned}
\tilde{M}_r(l) &= \frac{1}{2\pi} \sum_{m=0}^{r} a_{r,m} \frac{1}{(1+q)^m} \sum_{v=0}^{m} \binom{m}{v} q^{m-v} \frac{\cos(v+\frac{1}{2})l}{\sin(\frac{l}{2})} \\
|\tilde{M}_r(l)| &\leq \frac{1}{2\pi} \times \frac{\pi}{l} \sum_{m=0}^{r} a_{r,m} \frac{1}{(1+q)^m} \sum_{v=0}^{m} \binom{m}{v} q^{m-v} |\cos(v+\frac{1}{2})l| \\
&\leq \frac{1}{2l} \sum_{m=0}^{r} a_{r,m} \frac{1}{(1+q)^m} \sum_{v=0}^{m} \binom{m}{v} q^{m-v} \\
&\leq \frac{1}{2l} \sum_{m=0}^{r} a_{r,m} \\
\therefore \tilde{H}_r(l) &= O\left(\frac{1}{l}\right).
\end{aligned}$$

□

Lemma 2. *Under condition* (1), *$\tilde{M}_r(l) = O(\frac{1}{t})$ for $\frac{1}{r+1} \leq l \leq \pi$.*

Proof. For $\frac{1}{r+1} \leq l \leq \pi$, using $\sin(\frac{l}{2}) \geq \frac{l}{\pi}$.

$$\begin{aligned}
|\tilde{H}_r(l)| &\leq \frac{1}{2\pi} \times \frac{\pi}{l} \left| \sum_{m=0}^{r} a_{r,m} \frac{1}{(1+q)^m} \sum_{v=0}^{m} \binom{m}{v} q^{m-v} \cos(v+\frac{1}{2})l \right| \\
&\leq \frac{1}{2l} \left| \sum_{m=0}^{r} a_{r,m} \left[\frac{1}{(1+q)^m} \mathrm{Re} \left\{ \sum_{v=0}^{m} \binom{m}{v} q^{m-v} e^{i\left(v+\frac{1}{2}\right)l} \right\} \right] \right| \\
&\leq \frac{1}{2l} \left| \sum_{m=0}^{r} a_{r,m} \left[\frac{1}{(1+q)^m} \mathrm{Re} \left\{ \sum_{v=0}^{m} \binom{m}{v} q^{m-v} e^{ivl} \right\} \right] \right| \left| e^{\frac{il}{2}} \right| \\
&\leq \frac{1}{2l} \left| \sum_{m=0}^{r} a_{r,m} \left[\frac{1}{(1+q)^m} \mathrm{Re} \left\{ \sum_{v=0}^{m} \binom{m}{v} q^{m-v} e^{ivl} \right\} \right] \right| \\
&= \frac{1}{2l} \left| \sum_{m=0}^{\tau-1} a_{r,m} \left[\frac{1}{(1+q)^m} \mathrm{Re} \left\{ \sum_{v=0}^{m} \binom{m}{v} q^{m-v} e^{ivl} \right\} \right] \right|
\end{aligned}$$

$$+\frac{1}{2l}\left|\sum_{m=\tau}^{r} a_{r,m}\left[\frac{1}{(1+q)^m}\mathrm{Re}\left\{\sum_{\nu=0}^{m}\binom{m}{\nu}q^{m-\nu}e^{i\nu l}\right\}\right]\right|. \tag{5}$$

Now considering the first term of (5),

$$\begin{aligned}
&\frac{1}{2l}\left|\sum_{m=0}^{\tau-1} a_{r,m}\left[\frac{1}{(1+q)^m}\mathrm{Re}\left\{\sum_{\nu=0}^{m}\binom{m}{\nu}q^{m-\nu}e^{i\nu l}\right\}\right]\right| \\
&\leq \frac{1}{2l}\left|\sum_{m=0}^{\tau-1} a_{r,m}\left[\frac{1}{(1+q)^m}\left\{\sum_{\nu=0}^{m}\binom{m}{\nu}q^{m-\nu}\right\}\right]\right|\left|e^{i\nu l}\right| \\
&\leq \frac{1}{2l}\left|\sum_{m=0}^{\tau-1} a_{r,m}\right| \\
&= O\left(\tau\sum_{m=0}^{\tau-1} a_{r,m}\right).
\end{aligned} \tag{6}$$

Now considering the second term of (5) and using Abel's lemma,

$$\begin{aligned}
&\frac{1}{2l}\left|\sum_{m=\tau}^{r} a_{r,m}\left[\frac{1}{(1+q)^m}\mathrm{Re}\left\{\sum_{\nu=0}^{m}\binom{m}{\nu}q^{m-\nu}e^{i\nu l}\right\}\right]\right| \\
&\leq \frac{1}{2l}\left|\sum_{m=\tau}^{r} a_{r,m}\frac{1}{(1+q)^m}\left[\max_{0\leq k\leq m}\left\{\sum_{\nu=0}^{m}\binom{m}{\nu}q^{m-\nu}e^{i\nu l}\right\}\right]\right| \\
&\leq \frac{1}{2l}(1+q)^{\tau}\sum_{m=\tau}^{r} a_{r,m}\frac{1}{(1+q)^m} \\
&\leq \left(\tau(1+q)^{\tau}\sum_{m=\tau}^{r} a_{r,m}\frac{1}{(1+q)^m}\right) \\
&\leq \left[\tau(1+q)^{\tau}\left(a_{r,\tau}\frac{1}{(1+q)^{\tau}}+a_{r,\tau+1}\frac{1}{(1+q)^{\tau+1}}+\cdots+a_{r,r}\frac{1}{(1+q)^{r}}\right)\right] \\
&\leq \left[\tau\left(a_{r,\tau}+a_{r,\tau+1}+\cdots+a_{r,r}\right)\right] \\
&= O\left(\tau\sum_{m=\tau}^{r} a_{r,m}\right).
\end{aligned} \tag{7}$$

Combining (5), (6) and (7),

$$\begin{aligned}
|\tilde{M}_r(l)| &= O\Big(\tau\sum_{m=0}^{\tau-1} a_{r,m}\Big)+O\Big(\tau\sum_{m=\tau}^{r} a_{r,m}\Big) \\
&= O\Big(\tau\sum_{m=0}^{r} a_{r,m}\Big) \\
&= O\Big(\frac{1}{l}\Big).
\end{aligned}$$

□

Lemma 3. *Let* $\tilde{g}\in Y_z{}^{(w)}$ *then for* $0<l\leq\pi$*:*
If $w(l)$ *and* $v(l)$ *are defined as in Remark 2 then*

$$\|\psi(\cdot+y,l)-\psi(\cdot-y,l)\|_z = O\left(v(|y|)\left(\frac{w(l)}{v(l)}\right)\right).$$

Proof. This lemma can be proved along the same line of the proof of lemma proved in ([23], p.93). □

4. Proof of Main Theorems

4.1. Proof of Theorem 1

Proof. The integral representation of $s_r(\tilde{g};x)$ is given by

$$s_r(\tilde{g};x)-\tilde{g}(x)=\frac{1}{2\pi}\int_0^{\pi}\psi_x(l)\frac{\cos(r+\frac{1}{2})l}{\sin(\frac{l}{2})}dl.$$

Now, denoting $T.E^q$ transform of $s_r(\tilde{g};x)$ by $\tilde{t}_r^{\,T.E^q}$, we get

$$\begin{aligned}
\tilde{t}_r^{\,T.E^q}(x)-\tilde{g}(x) &= \sum_{m=0}^{r} a_{r,m}(E_m^q(x)-\tilde{g}(x))\\
&= \sum_{m=0}^{r} a_{r,m}\left(\frac{1}{(1+q)^m}\sum_{v=0}^{m}\binom{m}{v}q^{m-v}s_v(\tilde{g};x)-\tilde{g}(x)\right)\\
&= \frac{1}{2\pi}\int_0^{\pi}\psi_x(l)\left(\sum_{m=0}^{r}a_{r,m}\frac{1}{(1+q)^m}\sum_{v=0}^{m}\binom{m}{v}q^{m-v}\frac{\cos(v+\frac{1}{2})}{\sin(\frac{l}{2})}\right)dl\\
\tilde{t}_r^{\,T.E^q}(x)-\tilde{g}(x) &= \int_0^{\pi}\psi_x(l)\tilde{M}_r(l)dl.
\end{aligned}$$

Let

$$\tilde{R}_r(x)=\tilde{t}_r^{T.E^q}(x)-\tilde{g}(x)=\int_0^{\pi}\psi_x(l)\tilde{M}_r dl.$$

Then

$$\tilde{R}_r(x+y)-\tilde{R}_r(x-y)=\int_0^{\pi}\{\psi_x(x+y,l)-\psi_x(x-y,l)\}\tilde{M}_r(l)dl.$$

Using generalized Minkowski's inequality Chui [5], we get

$$\begin{aligned}
\|\tilde{R}_r(\cdot+y)-\tilde{R}_r(\cdot-y)\|_z &\le \int_0^{\pi}\|\psi_x(\cdot+y,l)\|_z\tilde{M}_r(l)dl\\
&= \left(\int_0^{\frac{1}{r+1}}+\int_{\frac{1}{r+1}}^{\pi}\right)\|\psi(\cdot+y,l)-\psi(\cdot-y,l)\|_z\tilde{M}_r(l)dl\\
&= I_1+I_2. \qquad (8)
\end{aligned}$$

Using Lemmas 1 and 3, we have

$$I_1=\int_0^{\frac{1}{r+1}}\|\psi(\cdot+y,l)-\psi(\cdot-y,l)\|_z\tilde{M}_r(l)dl$$

$$= O\left(v(|y|)\frac{w(\frac{1}{r+1})}{v(\frac{1}{r+1})}\int_0^{\frac{1}{r+1}}\frac{1}{l}dl\right)$$

$$= O\left(v(|y|)\frac{w(\frac{1}{r+1})}{v(\frac{1}{r+1})}\log(r+1)\right). \tag{9}$$

Using Lemmas 2 and 3, we have

$$I_2 = \int_{\frac{1}{r+1}}^{\pi}\|\psi(\cdot+y,l)-\psi(\cdot-y,l)\|_z\tilde{M}_r(l)dl$$

$$= O\left(\int_{\frac{1}{r+1}}^{\pi}v(|y|)\frac{w(l)}{lv(l)}dl\right). \tag{10}$$

Using (8), (9) and (10), we have

$$\sup_{y\neq0}\frac{\|\tilde{R}_r(\cdot+y)-\tilde{R}_r(\cdot-y)\|_z}{v(|y|)} = O\left(\frac{w(\frac{1}{r+1})}{v(\frac{1}{r+1})})\log(r+1)\right)+O\left(\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{lv(l)}dl\right). \tag{11}$$

Again using Lemmas 1 and 2 and $\|\psi(\cdot,l)\|_z = O(w(l))$,

$$\|\tilde{R}_r(\cdot)\|_z = \|\tilde{t}_r^{\,T.E^q}-\tilde{g}\|_z \le \left(\int_0^{\frac{1}{r+1}}+\int_{\pi}^{\frac{1}{r+1}}\right)\|\psi(\cdot,l)\|_z\,\tilde{M}_r(l)dl$$

$$= O\left(\int_0^{\frac{1}{r+1}}\frac{w(l)}{l}dl\right)+O\left(\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{l}dl\right)$$

$$= O\left(w\left(\frac{1}{r+1}\right)\log(r+1)\right)+O\left(\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{l}dl\right). \tag{12}$$

Now, we have

$$\|\tilde{R}_r(\cdot)\|_z^{(v)} = \|\tilde{R}_r(\cdot)\|_z+\sup_{y\neq0}\frac{\|\tilde{R}_r(\cdot+y)-\tilde{R}_r(\cdot-y)\|_z}{v(|y|)}.$$

Using (11) and (12), we get

$$\|\tilde{R}_r(\cdot)\|_z^{(v)} = O\left(w\left(\frac{1}{r+1}\right)\log(r+1)\right)+O\left(\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{l}dl\right)+O\left(\frac{w(\frac{1}{r+1})}{v(\frac{1}{r+1})}\log(r+1)\right)+O\left(\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{lv(l)}dl\right).$$

By the Monotonicity of $v(l)$, we have $w(l) = \frac{w(l)}{v(l)}v(l) \le v(\pi)\frac{w(l)}{v(l)}$, $0<l\le\pi$. Hence,

$$\|\tilde{R}_r(\cdot)\|_z^{(v)} = O\left(\frac{w(\frac{1}{r+1})}{v(\frac{1}{r+1})}\log(r+1)\right)+O\left(\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{lv(l)}dl\right). \tag{13}$$

Using the fact that $\frac{w(l)}{v(l)}$ is positive and non-decreasing, we have

$$\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{lv(l)}dl \ge \frac{w(\frac{1}{r+1})}{v(\frac{1}{r+1})}\int_{\frac{1}{r+1}}^{\pi}\frac{1}{l}dl$$

$$\ge \frac{w(\frac{1}{r+1})}{2v(\frac{1}{r+1})}\log\pi(r+1).$$

Then,

$$\frac{w(\frac{1}{r+1})}{v(\frac{1}{r+1})} = O\left(\frac{\int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{lv(l)}dl}{\log \pi(r+1)}\right). \tag{14}$$

From (13) and (14), we get

$$\|\tilde{R}_r(\cdot)\|_z^{(v)} = O\left(\frac{\log(r+1)}{\log\pi(r+1)}\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{lv(l)}dl\right) + O\left(\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{lv(l)}dl\right)$$

$$\therefore \|\tilde{t}_r^{\,T.E^q} - \tilde{g}\|_z^{(v)} = O\left(\frac{\log\pi(r+1)^2}{\log\pi(r+1)}\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{lv(l)}dl\right). \qquad \square$$

4.2. Proof of Theorem 2

Proof. Following the proof of the Theorem 1,

$$\|\tilde{t}_r^{\,T.E^q} - \tilde{g}\|_z^{(v)} = O\left(\frac{\log\pi(r+1)^2}{\log\pi(r+1)}\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{lv(l)}dl\right).$$

Since $\frac{w(l)}{lv(l)}$ is positive, non-increasing, then by second mean value theorem of integral calculus,

$$\begin{aligned}
\|\tilde{t}_r^{\,T.E^q} - \tilde{g}\|_z^{(v)} &= O\left(\frac{\log\pi(r+1)^2}{\log\pi(r+1)}\int_{\frac{1}{r+1}}^{\pi}\frac{w(l)}{lv(l)}dl\right)\\
&= O\left(\frac{\log\pi(r+1)^2}{\log\pi(r+1)}\frac{w\left(\frac{1}{r+1}\right)}{v\left(\frac{1}{r+1}\right)}\int_{\frac{1}{r+1}}^{\pi}\frac{1}{l}dl\right)\\
&= O\left(\frac{\log\pi(r+1)^2}{\log\pi(r+1)}\frac{w\left(\frac{1}{r+1}\right)}{v\left(\frac{1}{r+1}\right)}\log\pi(r+1)\right)\\
\|\tilde{t}_r^{\,T.E^q} - \tilde{g}\|_z^{(v)} &= O\left(\log\pi(r+1)^2\frac{w\left(\frac{1}{r+1}\right)}{v\left(\frac{1}{r+1}\right)}\right).
\end{aligned} \qquad \square$$

5. Corollaries

Corollary 1. *Let* $0 \le \beta < \alpha \le 1$ *and* $\tilde{g} \in Y_{(\alpha),z}$, $z \ge 1$. *Then*

$$\|\tilde{t}_r^{\,T.E^q} - \tilde{g}\|_{(\beta),z} = \begin{cases} O\left[\frac{(\log\pi(r+1)^2)(r+1)^{\beta-\alpha}}{\log\pi(r+1)}\right] & \text{if } 0 \le \beta < \alpha < 1.\\ O\left[\frac{(\log\pi(r+1)^2)}{(r+1)\log\pi(r+1)}\right] & \text{if } \beta = 0,\ \alpha = 1.\end{cases}$$

Proof. The proof is obtained by putting $w(l) = l^\alpha, v(l) = l^\beta, 0 \le \beta < \alpha \le 1$ in Theorem 1. $\square$

Corollary 2. *If* $a_{r,m} = \frac{1}{\log(r+1)} \sum_{m=0}^{r} \frac{1}{r-m+1}$ *then* $T.E^q$ *means reduces to* $\left(H, \frac{1}{r+1}\right) \cdot E^q$ *means and the error estimation of* $\tilde{g} \in Y_z^{(w)}$ *by* $\left(H, \frac{1}{r+1}\right).E^q$ *means of conjugate Fourier series is*

$$\| \tilde{t}_r^{(H, \frac{1}{r+1})} - \tilde{g} \|_z^{(v)} == O\left(\frac{\log \pi (r+1)^2}{\log \pi (r+1)} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l v(l)} dl \right).$$

Corollary 3. *If* $a_{r,m} = \frac{p_{r-m}}{P_r}$ *then* $T.E^q$ *means reduces to* $N_p.E^q$ *and the error estimation of* $\tilde{g} \in Y_z^{(w)}$ *by* $N_p.E^q$ *means of conjugate Fourier series is*

$$\| \tilde{t}_r^{N_p.E^q} - \tilde{g} \|_z^{(v)} == O\left(\frac{\log \pi (r+1)^2}{\log \pi (r+1)} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l v(l)} dl \right).$$

Corollary 4. *If* $a_{r,m} = \frac{p_{r-m} q_m}{R_r}$ *then* $T.E^q$ *means reduces to* $N_{p,q}.E^q$ *and the error estimation of* $\tilde{g} \in Y_z^{(w)}$ *by* $N_{p,q}.E^q$ *means of conjugate Fourier series is*

$$\| \tilde{t}_r^{N_{p,q}.E^q} - \tilde{g} \|_z^{(v)} == O\left(\frac{\log \pi (r+1)^2}{\log \pi (r+1)} \int_{\frac{1}{r+1}}^{\pi} \frac{w(l)}{l v(l)} dl \right).$$

Remark 4.

(i) In our Theorem 1, if $z \to \infty$ in $Y_z^{(w)}$ class then $Y_z^{w)}$ class reduces to $Y^{(w)}$ class. Also putting $w(l) = l^{\alpha}$ and $v(l) = l^{\beta}$ in our Theorem 1, $Y^{(w)}$ class reduces to Y_{α} class then by putting $\beta = 0$ in Y_{α} class, Y_{α} class reduces to Lipα class.

(ii) In our Theorem 1, by putting $w(l) = l^{\alpha}$, $v(l) = l^{\beta}$ in $Y_z^{(w)}$ class, $Y_z^{(w)}$ class reduces to $Y_{\alpha,z}$ then by putting $\beta = 0$ in $Y_{\alpha,z}$ class, $Y_{\alpha,z}$ class reduces to Lip(α, z) class.

6. Particular cases

(i) Using remark 4(i) and putting $a_{r,m} = \frac{1}{r+1}$ and $q_m = 1$ for all m in our Theorem 1, our Theorem 1 becomes a particular case of the Theorem 2.1 of Nigam [7].

(ii) Using remark 4(ii) and putting $a_{r,m} = \frac{1}{r+1}$ in our Theorem 1, our Theorem 1 becomes a particular case of the Theorem 1 of Nigam [8].

(iii) Using remark 4(ii) and putting $a_{r,m} = \frac{1}{r+1}$ and $q_m = 1$ for all m in our Theorem 1, our Theorem 1 becomes a particular case of Lal and Singh [26].

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