ON A PAPER OF S. ZAHID ALI ZENEI

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Abstract. In this note two theorems obtained by S. Zahid Ali Zenei in [1] are refined.

1. Introduction


Definition 1. A null sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class $S$, or briefly $\{a_n\} \in S$ if there exists a monotonically decreasing sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \leq A_n$, for all $n$.

Telyakovskii [6], firstly proved that the Sidon's class is equivalent to the class $S$ and second that $S$ is a $L^1$-integrability class for cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Thus the class $S$ is usually called as Sidon-Telyakovskii class.

Theorem A ([6]) Let the coefficients of the series (1.1) belong to the class $S$. Then the series (1.1) is a Fourier series of some $f \in L^1(0,\pi)$ and the following estimate holds:

$$\int_0^\pi |f(x)| dx \leq C \sum_{n=0}^{\infty} A_n,$$

where $C > 0$.

Similar theorem for sine series

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx$$

is also proved for the class $S$ by Telyakovskii in [6].

Received February 8, 2001; revised May 28, 2001.
2000 Mathematics Subject Classification. 26D15, 42A20.
Key words and phrases. Fourier series, Sidon-Telyakovskii class, embedding relations, inequalities.
Theorem B. ([6]) Let the coefficients of the series (1.2) belong to the class $S$. Then the following relation holds for $p = 1, 2, 3, \ldots$

$$
\int_{\pi/p+1}^{\pi} |g(x)|dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} A_n)
$$

In particular $g(x)$ is a Fourier series iff $\sum_{n=1}^{\infty} \frac{ln1}{n} < \infty$.

For other generalizations of the Theorem A and Theorem B, see [8], [9], [10].

N. Singh and K. M. Sharma [4] defined a class $S'$ as follows:

**Definition 2.** A null sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class $S'$ if there exists a sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} A_n < \infty$, $|\Delta a_n| \leq A_n$ for all $n$, where $\{A_n\}_{n=0}^{\infty}$ is quasi-monotone sequence (i.e. for some constant $\alpha \geq 0$, $\frac{A_n}{(n+1)\alpha} \downarrow 0$).


**Definition 3.** A null sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class $S(\delta)$ if there exists a sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} A_n < \infty$, $|\Delta a_n| \leq A_n$ and $\Delta A_n \geq -\delta_n$, for all $n$, where $\{\delta_n\}$ is a sequence such that $\delta_n \geq 0$, for all $n$ and $\sum_{n=0}^{\infty} (n+1)\delta_n < \infty$.

S. Zahid Ali Zenei [11] verified the following embedding relations

$$S \subset S' \subset S(\delta).$$


**Theorem C.** ([1, 7]) The classes $S$, $S'$ and $S(\delta)$ are identical.

Namely, Telyakovskii [7] supposing that $\{a_n\} \in S(\delta)$ proved that the sequence

$$B_n = A_n + \sum_{k=n}^{\infty} \delta_k$$

satisfies the conditions of the class $S$:

$$B_n \downarrow 0, n \to \infty; \sum_{n=0}^{\infty} B_n < \infty \text{ and } |\Delta a_n| \leq B_n, \text{ for all } n. \quad (1.4)$$

Using this idea, we shall give correct formulations and short proofs of the following two theorems, obtained by S. Zahid Ali Zenei [11].

**Theorem D.** ([11]) Let the coefficients of the series $f(x)$ satisfy the condition $S(\delta)$. Then the series is a Fourier series and the following relation holds:

$$\int_{0}^{\pi} |f(x)|dx \leq C \sum_{n=0}^{\infty} A_n, \text{ where } C > 0.$$
Theorem E. ([11]). Let the coefficients of the series \( g(x) \) satisfy the condition \( S(\delta) \). Then the series converges to a function and the following relation holds for \( p = 1, 2, 3, \ldots \)

\[
\int_{\pi/p+1}^{\pi} |g(x)| \, dx \leq \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} A_n).
\]

2. Results

Theorem 2.1 Let the coefficients of the series (1.1) belong to the class \( S(\delta) \). Then the series (1.1) is a Fourier series of some \( f \in L^1(0, \pi) \) and the following estimate holds:

\[
\int_{0}^{\pi} |f(x)| \, dx \leq C \sum_{n=0}^{\infty} [A_n + (n+1)\delta_n], \quad \text{where } C > 0. \tag{2.1}
\]

Proof. According to the Theorem A and Theorem C it suffices to prove the estimate (2.1). The sequence \( B_n \) defined by (1.3) satisfies the conditions (1.4) of the class \( S \). Applying the Theorem A and Theorem C, again, we obtain

\[
\int_{0}^{\pi} |f(x)| \, dx \leq C \sum_{n=0}^{\infty} B_n = C(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \delta_k)
\]

\[
= C(\sum_{n=0}^{\infty} A_n + \sum_{k=0}^{n} \sum_{n=0}^{k} \delta_k) = C(\sum_{n=0}^{\infty} A_n + \sum_{k=0}^{n} (k+1)\delta_k)
\]

\[
= C \sum_{n=0}^{\infty} [A_n + (n+1)\delta_n].
\]

This proves the Theorem.

Theorem 2.2. Let the coefficients of the series \( g(x) \) satisfy the condition \( S(\delta) \). Then the series converges to a function and the following estimate holds for \( p = 1, 2, 3, \ldots \)

\[
\int_{\pi/p+1}^{\pi} |g(x)| \, dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} nA_n) + O(\sum_{n=1}^{\infty} n\delta_n) \tag{2.2}
\]

Proof. Firstly we note that the convergence of the series \( \sum_{k=0}^{\infty} (k+1)\delta_k < \infty \) implies that \( \sum_{k=1}^{\infty} k\delta_k < \infty \). Similarly as in the proof of the Theorem 2.1 it suffices to prove the estimate (2.2).

Applying the Theorem B and Theorem C, we obtain:

\[
\int_{\pi/p+1}^{\pi} |g(x)| \, dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} B_n)
\]
\[
= \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left( \sum_{n=1}^{\infty} A_n + \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \delta_k \right)
\]
\[
= \sum_{n=1}^{p} \frac{|a_n|}{n} + O\left( \sum_{n=1}^{\infty} A_n \right) + O\left( \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \delta_k \right).
\]

References