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ON A PAPER OF S. ZAHID ALI ZENEI

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Abstract. In this note two theorems obtained by S. Zahid Ali Zenei in [11] are refined.

1. Introduction

In 1973, S. A. Telyakovskii [6] introduced a Sidon-type condition [3] described by class S in his paper.

Definition 1. A null sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class S, or briefly $\{a_n\} \in S$ if there exists a monotonically decreasing sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} A_n < \infty$ and $|\Delta a_n| \leq A_n$, for all n.

Telyakovskii [6], firstly proved that the Sidon's class is equivalent to the class S and second that S is a L^1-integrability class for cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

Thus the class S is usually called as Sidon-Telyakovskii class.

Theorem A.([6]) *Let the coefficients of the series* (1.1) *belong to the class S. Then the series* $(1,1)$ *is a Fourier series of some $f \in L^1(0,\pi)$ and the following estimate holds:*

$$\int_0^{\pi} |f(x)| dx \leq C \sum_{n=0}^{\infty} A_n, \text{ where } C > 0.$$

Similar theorem for sine series

$$g(x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{1.2}$$

is also proved for the class S by Telyakovskii in [6].

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Theorem B.([6]) *Let the coefficients of the series* (1.2) *belong to the class S. Then the following relation holds for $p = 1, 2, 3, \ldots$*

$$\int_{\pi/p+1}^{\pi} |g(x)|dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} A_n)$$

In particular $g(x)$ is a Fourier series iff $\sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty$.

For other generalizations of the Theorem A and Theorem B, see [8], [9], [10].
N. Singh and K. M. Sharma [4] defined a class S' as follows:

Definition 2. A null sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class S' if there exists a sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} A_n < \infty$, $|\Delta a_n| \leq A_n$, for all n, where $\{A_n\}_{n=0}^{\infty}$ is quasi-monotone sequence (i.e. for some constant $\alpha \geq 0$, $\frac{A_n}{(n+1)^{\alpha}} \downarrow 0$).

Singh and Sharma [5] have proved Theorem A and Theorem B for the class S'. In [11] S. Zahid Ali Zenei considered the class $S(\delta)$, introduced in [2].

Definition 3. A null sequence $\{a_n\}_{n=0}^{\infty}$ belongs to the class $S(\delta)$ if there exists a sequence $\{A_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} A_n < \infty$, $|\Delta a_n| \leq A_n$ and $\Delta A_n \geq -\delta_n$, for all n, where $\{\delta_n\}$ is a sequence such that $\delta_n \geq 0$, for all n and $\sum_{n=0}^{\infty}(n+1)\delta_n < \infty$.

S. Zahid Ali Zenei [11] verified the following embedding relations

$$S \subset S' \subset S(\delta).$$

Very recently, S. A. Telyakovskii [7] and L. Leindler [1] proved the following theorem.

Theorem C.([1, 7]) *The classes S, S' and $S(\delta)$ are identical.*

Namely, Telyakovskii [7] supposing that $\{a_n\} \in S(\delta)$ proved that the sequence

$$B_n = A_n + \sum_{k=n}^{\infty} \delta_k \tag{1.3}$$

satisfies the conditions of the class S:

$$B_n \downarrow 0, n \to \infty; \ \sum_{n=0}^{\infty} B_n < \infty \text{ and } |\Delta a_n| \leq B_n, \text{ for all } n. \tag{1.4}$$

Using this idea, we shall give correct formulations and short proofs of the following two theorems, obtained by S. Zahid Ali Zenei [11].

Theorem D.([11]) *Let the coefficients of the series $f(x)$ satisfy the condition $S(\delta)$. Then the series is a Fourier series and the following relation holds:*

$$\int_0^{\pi} |f(x)|dx \leq C \sum_{n=0}^{\infty} A_n, \text{ where } C > 0.$$

Theorem E.([11]). *Let the coefficients of the series $g(x)$ satisfy the condition $S(\delta)$. Then the series converges to a function and the following relation holds for $p = 1, 2, 3, \ldots$*

$$\int_{\pi/p+1}^{\pi} |g(x)|dx \leq \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} A_n).$$

2. Results

Theorem 2.1 *Let the coefficients of the series* (1.1) *belong to the class $S(\delta)$. Then the series* (1.1) *is a Fourier series of some $f \in L^1(0, \pi)$ and the following estimate holds:*

$$\int_{0}^{\pi} |f(x)|dx \leq C \sum_{n=0}^{\infty} [A_n + (n+1)\delta_n], \text{ where } C > 0. \tag{2.1}$$

Proof. According to the Theorem A and Theorem C it suffices to prove the estimate (2.1). The sequence B_n defined by (1.3) satisfies the conditions (1.4) of the class S. Applying the Theorem A and Theorem C, again, we obtain

$$\int_{0}^{\pi} |f(x)|dx \leq C \sum_{n=0}^{\infty} B_n = C(\sum_{n=0}^{\infty} A_n + \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \delta_k)$$
$$= C(\sum_{n=0}^{\infty} A_n + \sum_{k=0}^{\infty} \sum_{n=0}^{k} \delta_k) = C(\sum_{n=0}^{\infty} A_n + \sum_{k=0}^{\infty} (k+1)\delta_k)$$
$$= C \sum_{n=0}^{\infty} [A_n + (n+1)\delta_n].$$

This proves the Theorem.

Theorem 2.2. *Let the coefficients of the series $g(x)$ satisfy the condition $S(\delta)$. Then the series converges to a function and the following estimate holds for $p = 1, 2, 3, \ldots$*

$$\int_{\pi/p+1}^{\pi} |g(x)|dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} A_n) + O(\sum_{n=1}^{\infty} n\delta_n) \tag{2.2}$$

Proof. Firstly we note that the convergence of the series $\sum_{k=0}^{\infty} (k+1)\delta_k < \infty$ implies that $\sum_{k=1}^{\infty} k\delta_k < \infty$. Similarly as in the proof of the Theorem 2.1 it suffices to prove the estimate (2.2).

Applying the Theorem B and Theorem C, we obtain:

$$\int_{\pi/p+1}^{\pi} |g(x)|dx = \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} B_n)$$

$$= \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} A_n + \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \delta_k)$$
$$= \sum_{n=1}^{p} \frac{|a_n|}{n} + O(\sum_{n=1}^{\infty} A_n) + O(\sum_{n=1}^{\infty} n\delta_n).$$

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