TAMKANG JOURNAL OF MATHEMATICS
Volume 33, Number 1, Spring 2002

EXISTENCE OF SOLUTIONS OF FUNCTIONAL STOCHASTIC DIFFERENTIAL INCLUSIONS

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Abstract. In this paper, we prove the existence of solutions for functional stochastic differential inclusion via a fixed point analysis approach.

1. Introduction

Random differential and integral inclusions play an important role in characterizing many social, physical, biological and engineering problems. Theory of problems concerning differential and integral inclusions in deterministic cases may be found in several papers and monographs (see for example [4], [6]). A generalization of differential inclusions to 'stochastic differential inclusions' called multivalued stochastic differential equations is obtained by replacing the term $g(t)$ in the differential inclusions

$$x'(t) \in -Ax(t) + f(t, x(t)) + g(t), \quad x(0) = x_0$$

by a matrix G times the generalized derivative of the Brownian motion in which A is a multivalued map, f is a Lipchitz continuous map and g is a locally integrable function. In this case it is convenient to write, analogously to stochastic differential equations as follows:

$$dx(t) \in -Ax(t)dt + f(t, x(t))dt + G(t)dB(t), \quad x(0) = x_0 \tag{1}$$

For $G(t) = G(t, x(t))$, Kree [8] and Pettersson [9, 10] showed the existence of a soultion of (1) in two different ways. The former used a fixed point argument and the latter approximation techniques.

Recently by the Banach fixed point theorem, Ahmed [1,2] obtained the existence of nonlinear stochastic differential inclusions in infinite dimensional spaces. The most important problems examined up to now is one concerning the existence of solutions of differential inclusions, with the basic tools used in solving this problem were mostly the method of approximations or the Banach fixed point principle. In this paper, the

Received February 16, 2001; revised May 8, 2001.
2000 *Mathematics Subject Classification.* 35R10, 60H10.
Key words and phrases. Set valued maps, existence, stochastic differential inclusion, Kakutani's fixed point theorem.

existence of solutions of the following functional stochastic differenital inclusions have been studied via integral inclusions

$$\begin{aligned} &dx(t) \in f(t, x_t)dt + G(t, x_t)dB(t), \quad \text{a.e} \quad t \in I = [0, T] \\ &x(t) = \phi(t), \qquad t \in [-r, 0], \end{aligned} \tag{2}$$

where $x_t : [-r, 0] \to R^n$ is continuous such that $x_t(\theta) = x(t + \theta)$, belongs to the set of all continuous functions defined on $[-r, 0]$ and taking values in R^n this space is denoted by $C_r = C([-r, 0], R^n)$, f is an R^n-valued continuous linear function, G is the set-valued map, $\{B(t)\}_{t \geq 0}$ is a certain R^m-valued Brownian motion or Wiener process and $\phi(t)$ is a suitable initial random variable independent of $B(t)$. First, we convert the functional stochastic differential inclusion (2) into an integral inclusion, then we use Kakutani's fixed point theroem [5] to prove the existence of solutions of the integral inclusion which is the solution of differential inclusion. The considered system is an abstract formulation of many stochastic partial differential equations. In the final section, a physical example is worked out to illustrate the results for the system without delays.

2. Preliminaries

Throughout this paper I and J be the intervals $[0, T]$ and $[-r, T]$ respectively and ' denotes differentiation with respect to t. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a right continuous and complete filtration $\{\mathcal{F}_t, t \in I\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$. Let $L^2(\Omega, \mathcal{F}, \mathbf{P}; R^n)$ be the space of all square integrable random variables with values in R^n, that are measurable with respect to $\{\mathcal{F}_t\}$. Let $\mathcal{B}_r = \mathcal{M}_2(J, R^n)$ and $\mathcal{B} = \mathcal{M}_2(I, R^n)$ respectively denote the classes of R^n-valued stochastic processes $\{\xi(t) : t \in J\}$ and $\{\xi(t) : t \in I\}$ which are $\mathcal{F}_t$-adapted and have finite second moments, that is,

$$||\xi|| = \sup_t (E|\xi(t)|^2)^{1/2} < \infty,$$

here E stands for integration with respect to the probability measure $\mathbf{P}$. It is easy to verify that $\mathcal{B}_r$ and $\mathcal{B}$, furnished with the norm topology as defined above, are Banach spaces. In order to ensure the existence of solutions of the differential inclusion (2), we shall make the following hypotheses:

(i) $\{B(t)\}_{t \geq 0}$ is an m dimensional $\{\mathcal{F}_t\}$ adapted Brownian motion,

(ii) The functions $f : I \times C_r \to R^n$ are continuous and linear,

(iii) the set-valued map $G : I \times C_r \to 2^{L(R^m, R^n)}/\varphi$, the space of nonempty subsets of the space of linear operators from R^m to R^n, is convex such that for any $(t_0, \phi_0) \in I \times \mathcal{B}$,

$$G(t_0, \phi_0) = \bigcap_{\delta > 0} cl \cup \{G(t_0, \phi) : E|\phi - \phi_0|^2 \leq \delta\}$$

this is, G is upper semicontinuous in the sense of Kuratowski with respect to the variable ϕ. Note that, as the intersection of closed sets, each $G(t_0, \phi_0)$ is closed,

(iv) For the class of all random processes $\{\eta(t), t \geq 0\}$ taking values from the space $L(R^m, R^n)$ there exists a Borel set $\mathcal{A} \subset L(R^m, R^n)$, a constant $M > 0$ with event $\{\eta(t) \in \mathcal{A}\} \in \digamma_t$ for all $s \leq t < \infty$ and for each $\varepsilon > 0$, a function $\sigma_\varepsilon \in L^2(I, L(R^m, R^n))$, $\sigma_\varepsilon(t) > 0$, such that for given $x \in L^2(J, R^n)$ and selection $v(t) \in G(t, x_t)$ there exists a selection $\eta(t) \in \mathcal{A}$ with

$$\int_0^T ||\eta(t)||^2 dt \leq M^2, \qquad |v(t)|^2 \leq \sigma_\varepsilon^2(t) + \varepsilon \eta^2(t).$$

Let $g : I \times C_r$ be continuous and $C = C(J, R^n)$. The variation of parameters formula for the initial value problem (IVP)

$$\begin{aligned} dx(t) &= f(t, x_t)dt + g(t, x_t)dB(t), \ a.e\ t \in I \\ x(t) &= \phi(t), \qquad t \in [-r, 0] \end{aligned}$$

is given by [7]

$$x_t = Z(t, \sigma)\phi(\sigma) + \int_\sigma^t Z(t, s)X_0 g(s, x_s)dB,$$

where the operator $Z(t, \sigma) : C \to C$ is given by

$$Z(t, \sigma)\phi = x_t(\sigma, \phi), \ \sigma \leq t \leq T$$

such that $Z(\sigma, \sigma) = I$, $Z(t, \sigma)Z(\sigma, s) = Z(t, s)$ is a solution of the homogeneous equation

$$x'(t) = f(t, x_t)$$

and X_0 is defined by

$$X_0(\theta) = \begin{cases} 0, & -r \leq \theta \leq 0 \\ I_n, & \theta = 0 \end{cases}$$

where I_n is the identity matrix. Further we assume that

(v) Z is a bounded linear operator with bound N, continuous as follows: For t_1, $t_2 \in I$ and $\delta > 0$, there exists an $\varepsilon > 0$ such that

$$|Z(t_1, s) - Z(t_2, s)| < \varepsilon \qquad \text{for } |t_1 - t_2| < \delta,$$

where δ is independent of s.

Now we can write the equivalent form of the differential inclusion (2) as the integral inclusion

$$\begin{aligned} x(t) &\in Z(t, 0)\phi(0) + \int_0^t Z(t, s)X_0 G(s, x_s)dB, \ t \in I \\ x(t) &= \phi(t) \qquad -r \leq t \leq 0 \end{aligned} \tag{3}$$

So in order to prove the existence of solutions of the differential inclusion (2), we have to prove the existence theorem for the integral inclusion (3). We prove this existence theorem by using the following Bohnenblust-Karlin extension of Kakutani's fixed point theorem.

3. Existence Results

Theorem 3.1.(Bohnenblust-Karlin see [5]) *Let Y be a nonempty, closed convex subset of a Banach space X. If $\Gamma : Y \to 2^Y$ is such that*

a) *$\Gamma(v)$ is nonempty and convex for each $v \in Y$,*

b) *the graph of Γ, $G(\Gamma) \subset Y \times Y$ is closed,*

c) *$\cup\{\Gamma(v); v \in Y\}$ is contained in a sequentially compact set $C \subset X$, then the set-valued map Γ has a fixed point, that is, there exists a $v_0 \in Y$ such that $v_0 \in \Gamma(v_0)$.*

Since our interest is to study the existence of the solutions of the differential inclusion (2), we will need to give a precise definition of the term solution.

Definition 3.1. A solution of the differential inclusion (2) is a function x, defined on J with $x_0(\theta) = \phi(\theta)$, $-r \le \theta \le 0$ and $x(t)/I \in C(I, R^n)$, and such that there exists $v \in L^2(I, L(R^m, R^n))$ satisfying the inclusion $v(t) \in G(t, x_t)$ almost everywhere on I and for which

$$x(t) = Z(t,0)\phi(0) + \int_0^t Z(t,s)X_0 v(s)dB,\ 0 \le t \le T$$

Now we may recast the initial value problem as a problem for a fixed point of a set-valued mapping as follows; we introduce two set-valued mappings whose domain $D \subset \mathcal{B}_r$ is defined by

$$D = \{x \in \mathcal{B}_r : x(t) = \phi(t) \text{ for } t \in [-r, 0] \text{ and } x(t) \in C(I, R^n) \text{ for } t \in I\}.$$

Clearly, D is a closed convex set in $\mathcal{B}_r$. We define the set-valued maps $\Phi : D \to L^2(I, L(R^m, R^n))$ and $\Psi : D \to 2^D$ respectively by

$$\Phi(x) = \{v \in L^2(I, L(R^m, R^n)), v(t) \in G(t, x_t) \text{ a.e. } t \text{ on } I\} \tag{4}$$

and

$$\Psi(x) = \{z \in D : z(t) = Z(t,0)\phi(0) + \int_0^t Z(t,s)X_0 v(s)dB,\ v \in \Phi(x),\ z = \phi \text{ on } [-r,0]\} \tag{5}$$

Remark 3.1. Suppose that $x_0 \in D$ is a fixed point of the mapping Ψ defined by the relation (5), that is, suppose $x_0 \in \Psi(x_0)$. Then $x_0 \in D$ is a solution of the integral inclusion on (3).

Theorem 3.2.[3] *Under the hypotheses (iii), for each $x \in D$, $\Phi(x)$ is not empty and the set $\Psi(D)$ defined by the relation (5) is an equi-absolutely integrable set and is weakly compact in $\mathcal{B}$.*

Now we prove the relative compactness of the set $\Psi(D)$ and the convexity of $\Psi(x)$.

Theorem 3.3. *Under the hypotheses (iv) and (v), for each $x \in D$, $\Psi(x)$ is not empty and the set $\Psi(D)$ defined by the relation (5) is a relatively sequentially compact subset of $\mathcal{B}$.*

Proof. First we prove that $\Psi(x)$ is not empty, for all $x \in D$. If we are given $x \in D$, then from Theorem 3.2, $\Phi(x)$ is not empty. We choose $v \in \Phi(x)$ and define

$$y(t) = Z(t,0)\phi(0) + \int_0^t Z(t,s)X_0 v(s)dB, \quad 0 \le t \le T.$$

Let $\varepsilon > 0$, be given and suppose that $\delta_1 < \varepsilon/24N^2M^2$.

Now for any t_1, $t_2 \in I$,

$$\begin{aligned}
E|y(t_1) - y(t_2)|^2 &\le 2|Z(t_1,0) - Z(t_2,0)|^2 E|\phi(0)|^2 \\
&\quad +2\int_0^{t_1} |Z(t_1,s) - Z(t_2,s)|^2 |X_0|^2 E|v(s)|^2 ds \\
&\quad +2\int_{t_1}^{t_2} |Z(t_2,s)||X_0|^2 E|v(s)|^2 ds \\
&\le 2|Z(t_1,0) - Z(t_2,0)|^2 E|\phi(0)|^2 \\
&\quad +2\int_0^{t_1} |Z(t_1,s) - Z(t_2,s)|^2 |X_0|^2 E[\sigma_{\delta_1}^2(s) + \delta_1\eta^2(s)]ds \\
&\quad +2\int_{t_1}^{t_2} |Z(t_2,s)||X_0|^2 E[\sigma_{\delta_1}^2(s) + \delta_1\eta^2(s)]ds \\
||y(t_1) - y(t_2)||^2 &\le 2|Z(t_1,0) - Z(t_2,0)|^2 ||\phi(0)||^2 \\
&\quad +2\int_0^{t_1} |Z(t_1,s) - Z(t_2,s)|^2 ||\sigma_{\delta_1}(s)||^2 ds + 4N^2\delta_1 \int_0^T ||\eta(s)||^2 ds \\
&\quad +2\int_{t_1}^{t_2} |Z(t_2,s)|^2 ||\sigma_{\delta_1}(s)||^2 ds + 2N^2\delta_1 \int_0^T ||\eta(s)||^2 ds \\
&\le 2|Z(t_1,0) - Z(t_2,0)|^2 ||\phi(0)||^2 \\
&\quad +2\int_0^{t_1} |Z(t_1,s) - Z(t_2,s)|^2 ||\sigma_{\delta_1}(s)||^2 ds + 6\delta_1 N^2 M^2 \\
&\quad +2N^2\int_{t_1}^{t_2} ||\sigma_{\delta_1}(s)||^2 ds \\
&\le I_1 + I_2 + \varepsilon/4 + I_3.
\end{aligned}$$

Now from the hypothesis (v), there exists a $\delta_2 > 0$ such that

$$|Z(t_1,0) - Z(t_2,0)|^2 < \varepsilon/8||\phi(0)||^2 \text{ if } |t_1 - t_2| < \delta_2$$

that is, $I_1 < \varepsilon/4$ if $|t_1 - t_2| < \delta_2$.

Also since Z is bounded and $\sigma_{\delta_1} \in L^2(I, L(R^m, R^n))$ then by using the Lebesque dominated convergence theorem, for a sufficiently small $\delta_3 > 0$, $I_2 < \varepsilon/4$ if $|t_1 - t_2| < \delta_3$.

Moreover, since σ_{δ_1} is integrable, we may conclude that there is a $\delta_4 > 0$ such that if $|t_1 - t_2| < \delta_4$, then

$$\int_{t_1}^{t_2} ||\sigma_{\delta_1}(s)||^2 ds < \varepsilon/8N^2 \text{and therefore } I_3 < \varepsilon/4.$$

Therefore $||y(t_1) - y(t_2)||^2 \leq \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon$.

Hence the elements of $\Psi(D)$ restricted to the interval I form an equicontinuous family. Now choose $\delta < \min\{\delta_2, \delta_3, \delta_4\}$. Then the piecewise continuous function z defined by

$$z(t) = \begin{cases} \phi(t) & -r \leq t \leq 0 \\ y(t) & 0 \leq t \leq T \end{cases}$$

lies in D. Hence $\Psi(x)$ is not empty.

To prove the theorem, it remains to show that $\Psi(D)$ is equibounded. For a given $t_0 \in I$,

$$E|y(t_0)|^2 \leq 2|Z(t_0, 0)|^2 E|\phi(0)|^2 + \int_0^{t_0} |Z(t_0, s)|^2 |X_0|^2 E|v(s)|^2 ds.$$

Taking $\varepsilon = 1$, in the hypothesis (iv), we have

$$||y(t_0)||^2 \leq 2N||\phi(0)||^2 + 2N(M^2 + K^2) < \infty, \text{ since } \int_0^T ||\sigma_1(t)||^2 dt = K^2 < \infty \text{ (see [2])}.$$

Therefore $\Psi(D)$ is equibounded. Then by the Arzela-Ascoli theorem, any sequence $\{z_k\}$ in $\Psi(D)$ restricted to I has a uniformly convergent subsequence. Hence the set $\Psi(D)$ is relatively sequentially compact in $\mathcal{B}$.

Theorem 3.4. *For each $x \in D$, the set $\Psi(x)$ defined by the relation* (5) *is convex.*

Proof. Let y_1, $y_2 \in \Psi(x)$. Then there exists $v_1(t)$, $v_2(t) \in G(t, x_t)$ such that

$$y_i(t) = Z(t, 0)\phi(0) + \int_0^t Z(t, s) X_0 v_i(s) ds, \ i = 1, 2.$$

And so for $0 < \lambda < 1$, we have

$$\lambda y_1(t) + (1 - \lambda) y_2(t) = \int_0^t Z(t, s) X_0 [\lambda v_1(s) + (1 - \lambda) v_2(s)] ds.$$

Since $G(t, x_t)$ is convex, $\lambda v_1(t) + (1 - \lambda) v_2(t) \in G(t, x_t)$ a.e in I. Therefore

$$\lambda y_1(t) + (1 - \lambda) y_2(t) \in \Psi(x). \text{ Hence } \Psi(x) \text{is convex.}$$

Next we will prove that the graph of Ψ, $\mathcal{G}(\Psi)$ is closed. For that we use the following closure theorem.

Theorem 3.5. (see [2]). *Consider the set-valued mapping* $G : I \times C_r \to 2^{L^2(R^n,R^m)}$ *and assume that* G *satisfies the hypothesis* (iii) *with respect to* ϕ. *Let* v, v_k, x *and* x_k *be functions measurable on* I, x, x_k *bounded and let* v, $v_k \in L^2(I, L(R^n, R^n))$. *Then if* $v_k(t) \in G(t, x_{k_t})$ *a.e. in* I *and* $v_k \to v$ *weakly in* $L^2(I, L(R^n, R^n))$ *while* $x_k \to x$ *uniformly on* I, *then* $v(t) \in G(t, x_t)$ *a.e. in* I.

Theorem 3.6. *Under the hypotheses* (iii), (iv) *and* (v) *the map* $\Psi : D \to 2^D$ *has a closed graph, that is* $\{(x,y) \in D \times D : y \in \Psi(x)\}$ *is closed.*

Proof. Let (x_k, y_k) be a sequence of functions, $y_k \in \Psi(x_k)$, which converges to a limit point (x, y) of $\mathcal{G}(\Psi)$. Thus $x_k \to x$ and $y_k \to y$ uniformly on I. We have to prove that $y \in \Psi(x)$.

By definition of Ψ there exists a sequence $\{v_k\}$ with $v_k \in \Phi(x_k)$ such that

$$y_k(t) = Z(t,0)\phi(0) + \int_0^t Z(t,s)X_0 v_k(s)ds.$$

Without loss of generality, we may assume that $v_k \to v$ weakly in $L^2(I, L(R^n, R^m))$ and from Theorem 3.5, $v(s) \in G(s, x_s)$.

To prove $y \in \Psi(x)$ we wish to show that y satisfies the equation

$$y(t) = Z(t,0)\phi(0) + \int_0^t Z(t,s)X_0 v(s)ds$$

which, for convenience, we will write symbolically as

$$y - v = 0 \tag{6}$$

Recognizing that (y_k, v_k) satisfying the above relation, we may write

$$|y - v|^2 \leq 2|y - v_k|^2 + 2|v_k - v|^2.$$

It is enough to show that the relation (6) holds pointwise.
Let us fix $t_0 \in I$. Since $\{y_k\}$ uniformly converges to y, we have that $|y(t_0) - y_k(t_0)|^2 < \varepsilon/4$. Also since Z is bounded in $(L^2(I, R^n))'$ and $\{v_k\}$ weakly converges in $L^2(I, L(R^n, R^m))$, $|v_k(t_0) - v(t_0)|^2 < \varepsilon/4$. Therefore given $\varepsilon > 0$,

$$|y(t) - Z(t,0)\phi(0) + \int_0^t Z(t,s)X_0 b(s)ds|^2 < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence $y \in \Psi(x)$ and therefore $(x, y) \in \mathcal{G}(\Psi)$, that is the graph of Ψ is closed and the proof is complete.

So far we have verified that all of the hypotheses of Theorem 3.1 are satisfied. We may thus consider the following existence theorem.

Theorem 3.7. *Under the hypotheses* (iii)-(v) *the set-valued map* $\Psi : D \to 2^D$ *has a fixed point in* D; *consequently, the integral inclusion* (3) *has a solution in* D.

Since the existence of solution to the integral inclusion (3) is equivalent to the existence of solution to the differential inclusion (2), we state our main theorem.

Theorem 3.8. *Under the hypotheses (ii)-(iv), the differential inclusion* (2) *has a solution.*

4. Example

Example 4.1. (Coulomb damping)

An equation for describing a mechanical system with both linear viscous damping and friction forces is as follows:

$$mx'' + \beta x' + kx = r\eta_2(x')B'(t), \quad m, r, \beta, k > 0 \tag{7}$$

where $B'(\cdot)$ is an excitation force, here assumed to be white Gaussian noise; further $\eta_2(x') \in Gx'$ for the maximal monotone set-valued map G on R defined by

$$G(z) = \begin{cases} \text{sign } z, & \text{if } z \neq 0 \\ [-1,1], & \text{if } z = 0. \end{cases}$$

For simplicity we assume the mass m is equal to 1. As customary, we rewrite the second degree equation (7) into a first degree system: for $u = (u_1, u_2)$ in R^2 and $y \in rGu_2$ let

$$f(t,u) = \begin{pmatrix} u_2 \\ -ku_1 - \beta u_2 \end{pmatrix}. \qquad v = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

Then with $\xi = (x, x')$, the second order equation (7) may be reformulated into a multivalued stochastic differential equation of the form (2). Let f and G satisfy the hypotheses (ii) and (iii). Then an application of Theorem 3.8, there exists a solution for the quation (7) in $D \subset \mathcal{B}$.

References

[1] N. U. Ahmed, *Nonlinear stochastic differential inclusions on Banach space*, Stochastic Analysis and Applications, **12**(1994), 1-10.

[2] N. U. Ahmed, *Existence of solutions of nonlinear stochastic differential inclusions on Banach space*, Proceedings of the first world congress of nonlinear analysis, Tampa, Florida, August 19-26, 1992, (Editor V. Lakshikantham) Offprint, Walter de Gruyter and Co., Berlin, New York, 1996, 1699-1712.

[3] T. S. Angell, *Existence of multivalued Urysohn integral equations*, Journal of Optimization Theory and Applications, **46**(1995), 129-150.

[4] J. P. Aubin and A. Cellina, Differential Inclusions, Springer, Berlin, 1984.

[5] H. F. Bohnenblust and S. Karlin, *"On a Theorem of Ville" Contributions to the theory of games:* I, Edited by H. W. Kuhn and A. W. Tucker, Princeton University press, Princeton, New Jersey, 1950, 155-160.

[6] K. Deimling, Nonlinear Functional Analysis, Springer-verlag, New York, 1985.

[7] J. K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[8] P. Kree, *Diffusion equation for multivalued stochastic differential equations*, Journal of Functional Analysis, **49**(1982), 73-90.

[9] R. Pettersson, *Yosida approximations for multivalued stochastic differential equations*, Stochastics and Stochastics Reports, **52**(1995), 107-120.

[10] R. Pettersson, *Existence theorem and Wong-Zakai approximations for multivalued stochastic differential equations.*, Probability and Mathematical Statistics, **17**(1997), 29-45.

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