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A NEW CLASS OF DOUBLE INTEGRALS INVOLVING A GENERALIZED HYPERGEOMETRIC FUNCTION ${}_3F_2$

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Abstract. The aim of this research paper is to evaluate fifty double integrals involving a generalized hypergeometric function (25 each) in the form of

$$\int_0^1\int_0^1 x^{c-1}y^{c+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}\,(1-xy)^{c+\ell-\alpha-\beta+1} \times {}_3F_2\left[\begin{matrix} a,\ b,\ 2c+\ell+1 \\ \frac{1}{2}(a+b+i+1),\ 2c+j\end{matrix};xy\right]dxdy$$

and

$$\int_0^1\int_0^1 x^{c+\ell}y^{c+\ell+\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}\,(1-xy)^{c-\alpha-\beta} \times {}_3F_2\left[\begin{matrix} a,\ b,\ 2c+\ell+1 \\ \frac{1}{2}(a+b+i+1),\ 2c+j\end{matrix};1-xy\right]dxdy$$

in the most general form for any $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$. The results are derived with the help of generalization of Edwards's well known double integral due to Kim, *et al.* and generalized classical Watson's summation theorem obtained earlier by Lavoie, *et al.* More than one hundred ineteresting special cases have also been obtained.

1. Introduction

The generalized hypergeometric function with p numeratorial and q denominatorial parameters is defined [1, 9] by

$$ {}_pF_q\left[\begin{matrix} a_1,\ \ldots,\ a_p \\ b_1,\ \ldots,\ b_q\end{matrix};z\right]=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \tag{1.1}$$

where $(a)_n$ denotes the Pochhammer symbol (or shifted factorial or raised factorial, since $(1)_n=n!$) defined for any complex number a by

$$(a)_n=\begin{cases} a(a+1)\ldots(a+n-1), & n\in\mathbb{N} \\ 1, & n=0\end{cases}. \tag{1.2}$$

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The series (1.1) converges for $|z| < \infty$ if $p \leq q$ and for $|z| < 1$ if $p = q+1$, while it is divergent for all z, $z \neq 0$ if $p > q+1$. If $p = q+1$ on $|z| = 1$, the series converges absolutely if $\operatorname{Re}\left(\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i\right) > 0$, converges conditionally if $-1 < \operatorname{Re}\left(\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i\right) \leq 0$, $z \neq 1$ and diverges if $\operatorname{Re}\left(\sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_i\right) \leq -1$.

It is not out of place to mention here that whenever hypergeometric function ${}_2F_1$ or the generalized hypergeometric functions ${}_pF_q$ reduce to gamma function, the results are very important from an application point of view. Thus, the classical summation theorems such as those of Gauss, Gauss's second, Bailey and Kummer for the series ${}_2F_1$; Watson, Dixon and Whipple for the series ${}_3F_2$ and others play an important role.

During 1992-1996, in a series of three interesting research papers, Lavoie, *et al.* [6, 7, 8] have generalized the above mentioned classical summation theorems. For further generalizations and extensions of these summation theorems, see papers by Rakha and Rathie [10] and Kim, *et al.* [5]. However, in our present investigation, we are interested in the following classical Watson's summation theorem [1], viz.

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}; 1\right] = \frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})}, \tag{1.3}$$

provided $\operatorname{Re}(2c-a-b) > -1$ and its following generalization due to Lavoie, *et al.* [6] viz.

$$\begin{aligned} &{}_3F_2\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+i+1), & 2c+j \end{matrix}; 1\right] \\ &= A_{i,j}\, 2^{a+b+i-2}\, \Gamma\left(c - \frac{1}{2}\left(a+b+|i+j|-j-1\right)\right) \frac{\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(c+\left[\frac{j}{2}\right]+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(a)\Gamma(b)} \\ &\times \left\{ \frac{B_{i,j}\,\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1-(-1)^i\right)\right)\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}-\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j}{2}\right]+\frac{1}{2}\right)} \right. \\ &\quad \left. + \frac{C_{i,j}\,\Gamma\left(\frac{a}{2}+\frac{1}{4}\left(1+(-1)^i\right)\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(c-\frac{a}{2}+\left[\frac{j+1}{2}\right]+\frac{(-1)^j}{4}(1-(-1)^i)\right)\Gamma\left(c-\frac{b}{2}+\left[\frac{j+1}{2}\right]\right)} \right\} \\ &= \Omega_{i,j}, \end{aligned} \tag{1.4}$$

for $i, j = 0, \pm 1, \pm 2$. For $i = j = 0$, the result (1.4) reduces to the classical Watson's summation theorem (1.3). Here, $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients $A_{i,j}$, $B_{i,j}$ and $C_{i,j}$ are given in Tables 1, 2 and 3 at the end of this paper.

The aim of this paper is to evaluate fifty double integrals (twenty-five each) in the form of the following two general integrals:

$$\int_0^1\int_0^1 x^{c-1}y^{c+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c+\ell-\alpha-\beta+1} \times {}_3F_2\left[\begin{matrix} a,\ b,\ 2c+\ell+1 \\ \frac{1}{2}(a+b+i+1),\ 2c+j\end{matrix};xy\right]dxdy$$

and

$$\int_0^1\int_0^1 x^{c+\ell}y^{c+\ell+\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta} \times {}_3F_2\left[\begin{matrix} a,\ b,\ 2c+\ell+1 \\ \frac{1}{2}(a+b+i+1),\ 2c+j\end{matrix};1-xy\right]dxdy$$

in the most general form for any $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$.

The results are established with the help of the following generalization of a well-known double integral due to Edwards [3], obtained recently by Kim, *et al.* [4], viz.

$$\int_0^1\int_0^1 x^{\gamma-1}y^{\gamma+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{\delta-\alpha-\beta}dxdy=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}, \tag{1.5}$$

provided $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$ $\mathrm{Re}(\gamma)>0$ and $\mathrm{Re}(\delta)>0$. More than one hundred interesting special cases have also been given. The double integrals established in this paper are simple, interesting easily established and may be useful.

Remark 1.1. In (1.5) , if we take $\gamma=\delta=1$, we recover the following double integrals due to Edwards [3]

$$\int_0^1\int_0^1 y^{\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{1-\alpha-\beta}dxdy=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

provided $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$.

2. Main results

The following two general formulas will be evaluated in this paper.

FIRST GENERAL INTEGRAL

$$\int_0^1\int_0^1 x^{c-1}y^{c+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c+\ell-\alpha-\beta+1} \times {}_3F_2\left[\begin{matrix} a,\ b,\ 2c+\ell+1 \\ \frac{1}{2}(a+b+i+1),\ 2c+j\end{matrix};xy\right]dxdy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\Omega_{i,j}, \tag{2.1}$$

for $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$, provided $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(c)>0$ and $\mathrm{Re}(c+\ell+1)>0$.

SECOND GENERAL INTEGRAL

$$\int_0^1\int_0^1 x^{c+\ell}y^{c+\ell+\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta}$$
$$\times\ {}_3F_2\left[\begin{matrix}a,\ \ b,\ \ 2c+\ell+1\\ \frac{1}{2}(a+b+i+1),\ 2c+j\end{matrix};1-xy\right]dxdy$$
$$=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\Omega_{i,j}, \tag{2.2}$$

for $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$, provided $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(c)>0$ $\mathrm{Re}(c+\ell+1)>0$.

Here in both cases, $\Omega_{i,j}$ are the same as given in (1.4).

Proof. In order to evaluate our first integral (2.1), we proceed as follows. Denoting the left-hand side of integral (2.1) by I, expressing ${}_3F_2$ as a series, changing the order of integration and summation (which is easily seen to be justified due to the uniform convergence of the series),

$$I=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(2c+\ell+1)_n}{n!\left(\frac{1}{2}(a+b+i+1)\right)_n(2c+j)_n}$$
$$\times\int_0^1\int_0^1 x^{c+n-1}y^{c+\alpha+n-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c+\ell-\alpha-\beta+1}dxdy.$$

Now, evaluating the double integral with help of result (1.5), we have

$$I=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n(c)_n}{n!\left(\frac{1}{2}(a+b+i+1)\right)_n(2c+j)_n}.$$

Summing up the series, we get

$$I=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\,{}_3F_2\left[\begin{matrix}a,\quad b,\quad c\\ \frac{1}{2}(a+b+i+1),\ 2c+j\end{matrix};1\right].$$

Finally, evaluating ${}_3F_2$ using (1.4), we reach the right-hand side of (2.1). This completes the proof of (2.1).

In exactly the same manner, the result (2.2) can be also established.

We conclude this section by remarking that one hundred interesting special cases will be given in the next section in the form of four results.

3. Special cases

In this section, we shall mention more than one hundred interesting special cases of our main results.

In (2.1) and (2.2), let $b = -2n$ and replace a by $a+2n$ or let $b = -2n-1$ and replace a by $a+2n+1$, where n is zero or a positive integer. In both cases, we notice that one of the two terms on the right-hand sides of (1.4) will vanish and hence we get the following one hundred interesting special cases in the form of four general integrals.

$$\begin{aligned}
&\int_0^1\int_0^1 x^{c-1}y^{c+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c+\ell-\alpha-\beta+1}\\
&\qquad\times {}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ 2c+\ell+1\\ \frac{1}{2}(a+i+1),\ 2c+j\end{matrix};xy\right]dx\,dy\\
&= D_{i,j}\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\\
&\qquad\times\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}a-c+\frac{3}{4}-\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{1}{4}(1-(-1)^i)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(1+(-1)^i\right)\right)_n}=\Omega^{(1)}_{i,j},
\end{aligned}\tag{3.1}$$

for $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$, where the coefficients, $D_{i,j}$ are given in Table 4 at the end of this paper.

$$\begin{aligned}
&\int_0^1\int_0^1 x^{c-1}y^{c+\alpha-1}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c+\ell-\alpha-\beta+1}\\
&\qquad\times {}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ 2c+\ell+1\\ \frac{1}{2}(a+i+1),\ 2c+j\end{matrix};xy\right]dx\\
&= E_{i,j}\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\frac{\Gamma(c)\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)}\\
&\qquad\times\frac{\left(\frac{3}{2}\right)_n\left(\frac{1}{2}a-c+\frac{5}{4}+\frac{(-1)^i}{4}-\left[\frac{j}{2}+\frac{1}{4}(1+(-1)^i)\right]\right)_n}{\left(c+\frac{1}{2}+\left[\frac{j+1}{2}\right]\right)_n\left(\frac{1}{2}a+\frac{1}{4}\left(3-(-1)^i\right)\right)_n}=\Omega^{(2)}_{i,j},
\end{aligned}\tag{3.2}$$

for $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$. The coefficients, $E_{i,j}$ are given in Table 5 at the end of this paper.

$$\begin{aligned}
&\int_0^1\int_0^1 x^{c+\ell}y^{c+\ell+\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta}\\
&\qquad\times {}_3F_2\left[\begin{matrix}-2n,\ a+2n,\ 2c+\ell+1;\\ \frac{1}{2}(a+i+1),\ 2c+j;\end{matrix}\,1-xy\right]dx\,dy=\Omega^{(1)}_{i,j},
\end{aligned}\tag{3.3}$$

for $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$. Here, $\Omega^{(1)}_{i,j}$ are the same as defined in (3.1).

$$\begin{aligned}
&\int_0^1\int_0^1 x^{c+\ell}y^{c+\ell+\alpha}(1-x)^{\alpha-1}(1-y)^{\beta-1}(1-xy)^{c-\alpha-\beta}\\
&\qquad\times {}_3F_2\left[\begin{matrix}-2n-1,\ a+2n+1,\ 2c+\ell+1\\ \frac{1}{2}(a+i+1),\ 2c+j\end{matrix};1-xy\right]dx\,dy=\Omega^{(2)}_{i,j},
\end{aligned}\tag{3.4}$$

for $\ell\in\mathbb{Z}$ and $i,j=0,\pm1,\pm2$. Here, $\Omega^{(2)}_{i,j}$ are the same as defined in (3.2).

In particular, in (3.1) and (3.2), if we take $i = j = 0$, since $D_{0,0} = 1$ and $E_{0,0} = 0$, we get

$$\int_0^1 \int_0^1 x^{c-1} y^{c+\alpha-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+\ell-\alpha-\beta+1} \times {}_3F_2\left[\begin{matrix} -2n,\, a+2n,\, 2c+\ell+1 \\ \frac{1}{2}(a+1),\;\; 2c \end{matrix}; xy\right] dx\,dy = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \cdot \frac{\Gamma(c)\,\Gamma(c+\ell+1)}{\Gamma(2c+\ell+1)} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}a-c+\frac{1}{2}\right)_n}{\left(c+\frac{1}{2}\right)_n \left(\frac{1}{2}a+\frac{1}{2}\right)_n} \tag{3.5}$$

and

$$\int_0^1 \int_0^1 x^{c-1} y^{c+\alpha-1} (1-x)^{\alpha-1} (1-y)^{\beta-1} (1-xy)^{c+\ell-\alpha-\beta+1} \times {}_3F_2\left[\begin{matrix} -2n-1,\, a+2n+1,\, 2c+\ell+1 \\ \frac{1}{2}(a+1),\;\; 2c \end{matrix}; xy\right] dx\,dy = 0. \tag{3.6}$$

Similarly, other results can be obtained.

Remark 3.1. The result (3.6) is interesting as the right-hand side is independent of $c, \alpha, \beta, a, \ell$ and n.

4. Concluding remark

In this paper, we have obtained fifty interesting double integrals involving generalized hypergeometric function in the form of two general integrals (twenty-five each). The results are established with the help of a generalization of Edwards's double integral and generalization of Watson's summation theorem available in the literature. The results established in this paper may be useful in applied mathematics, engineering mathematics and mathematical physics.

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Table 1: Table for $A_{i,j}$

$i \backslash j$	-2	-1	0	1	2
2	$\frac{1}{2(c-1)(a-b-1)(a-b+1)}$	$\frac{1}{2(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{4(a-b-1)(a-b+1)}$	$\frac{1}{8(c+1)(a-b-1)(a-b+1)}$
1	$\frac{1}{(c-1)(a-b)}$	$\frac{1}{(a-b)}$	$\frac{1}{(a-b)}$	$\frac{1}{2(a-b)}$	$\frac{1}{2(c+1)(a-b)}$
0	$\frac{1}{2(c-1)}$	1	1	1	$\frac{1}{2(c+1)}$
-1	$\frac{1}{(c-1)}$	1	2	2	$\frac{2}{(c+1)}$
-2	$\frac{1}{2(c-1)}$	1	1	2	$\frac{2}{(c+1)}$

Table 2: Table for $B_{i,j}$

$B_{-2,-1} = 2(c-1)(a+b-1)-(a-b)^2+1$
$B_{2,2} = 2c(c+1)[(2c+1)(a+b-1)-a(a-1)-b(b-1)]-(a-b-1)(a-b+1)[(c+1)(2c-a-b+1)+ab]$
$B_{-2,-2} = 2(c-1)(c-2)[(2c-1)(a+b-1)-a(a+1)-b(b+1)+2]-(a-b-1)(a-b+1)[(c-1)(2c-a-b-3)+ab]$

$i \backslash j$	-2	-1	0	1	2
2	$c(a+b-1)-(a+1)(b+1)+2$	$a+b-1$	$a(2c-a)+b(2c-b)-2c+1$	$2c(a+b-1)-(a-b)^2+1$	$B_{2,2}$
1	$c-b-1$	1	1	$2c-a+b$	$2c(c+1)-(a-b)(c-b+1)$
0	$(c-a-1)(c-b-1)+(c-1)(c-2)$	1	1	1	$(c-a+1)(c-b+1)+c(c+1)$
-1	$2(c-1)(c-2)-(a-b)(c-b-1)$	$2c-a+b-2$	1	1	$c-b+1$
-2	$B_{-2,-2}$	$B_{-2,-1}$	$a(2c-a)+b(2c-b)-2c+1$	$a+b-1$	$c(a+b-1)-(a-1)(b-1)$

Table 3: Table for $C_{i,j}$

$C_{-2,-1} = 8c^2 - 2(c-1)(a+b+7) - (a-b)^2 - 7$

$i \backslash j$	-2	-1	0	1	2
2	-4	$-(4c-a-b-3)$	-8	$-[8c^2-2c(a+b-1)-(a-b)^2+1]$	$-4(2c+a-b+1)(2c-a+b+1)$
1	$-(c-a-1)$	-1	-1	$-(2c+a-b)$	$-[2c(c+1)+(a-b)(c-a+1)]$
0	4	1	0	-1	-4
-1	$2(c-1)(c-2)+(a-b)(c-a-1)$	$2c+a-b-2$	1	1	$c-a+1$
-2	$4(2c-a+b-3)(2c+a-b-3)$	$C_{-2,-1}$	8	$4c-a-b+1$	4

Table 4: Table for $D_{i,j}$

$$D_{2,2}=\frac{(a+1)[(a-1)(c+1)(2c-a+1)(2c-a-1)-2an(6c+a+5)(2c-a+1)+4n^2(5a^2+4a-5-4c(3c-a+4))+64n^3(a+n)]}{(c+1)(2c-a+1)(2c-a-1)(a+4n+1)(a+4n-1)}$$

$$D_{-2,-2}=1-\frac{2an(6c+a-7)(2c-a-3)-4n^2[5a^2-4a-21-4c(3c-a-8)]-64n^3(a+n)}{(c-1)(a-1)(2c-a-3)(2c-a-5)}$$

$i \backslash j$	-2	-1	0	1	2
2	$\frac{(a+1)[(c-1)(a-1)+2n(a+2n)]}{(c-1)(a+4n-1)(a+4n+1)}$	$\frac{(a+1)(a-1)}{(a+4n+1)(a+4n-1)}$	$\frac{(a+1)[(a-1)(2c-a-1)-4n(a+2n)]}{(2c-a-1)(a+4n+1)(a+4n-1)}$	$\frac{(a+1)[(a-1)(2c-a-1)-8n(a+2n)]}{(2c-a-1)(a+4n+1)(a+4n-1)}$	$D_{2,2}$
1	$\frac{a(c+2n-1)}{(c-1)(a+4n)}$	$\frac{a}{a+4n}$	$\frac{a}{a+4n}$	$\frac{a(2c-a-4n)}{(2c-a)(a+4n)}$	$\frac{a[(c+1)(2c-a)-2n(2c+a+4n+2)]}{(c+1)(2c-a)(a+4n)}$
0	$1-\frac{2n(a+2n)}{(c-1)(2c-a-3)}$	1	1	1	$1-\frac{2n(a+2n)}{(c+1)(2c-a+1)}$
-1	$1-\frac{2n(2c+a+4n-2)}{(c-1)(2c-a-4)}$	$1-\frac{4n}{(2c-a-2)}$	1	1	$1+\frac{2n}{(c+1)}$
-2	$D_{-2,-2}$	$1-\frac{8n(a+2n)}{(a-1)(2c-a-3)}$	$1-\frac{4n(a+2n)}{(a-1)(2c-a-1)}$	1	$1+\frac{2n(a+2n)}{(c+1)(a-1)}$

Table 5: Table for $E_{i,j}$

$$E_{2,1} = \frac{(a+1)[(4c+a+3)(2c-a-1)-8n(a+2n+2)]}{(a+4n+1)(a+4n+3)(2c+1)(2c-a-1)} \; ; \; E_{-2,-1} = -\frac{[(4c+a-1)(2c-a-3)-8n(a+2n+2)]}{(a-1)(2c-1)(2c-a-3)}$$

$$E_{-1,-2} = -\frac{[(c+a)(2c-a-4)-2n(3a-2c+4n+6)]}{a(c-1)(2c-a-4)}$$

i \ j	-2	-1	0	1	2
2	$\frac{(a+1)(2c-a-3)}{(c-1)(a+4n+1)(a+4n+3)}$	$\frac{(a+1)(4c-a-3)}{(a+4n+1)(a+4n+3)(2c-1)}$	$\frac{2(a+1)}{(a+4n+1)(a+4n+3)}$	$E_{2,1}$	$\frac{(a+1)(2c+a+4n+3)(2c-a-4n-1)}{(c+1)(2c-a-1)(a+4n+1)(a+4n+3)}$
1	$\frac{(c-a-2n-2)}{(c-1)(a+4n+2)}$	$\frac{2c-a-2}{(a+4n+2)(2c-1)}$	$\frac{1}{a+4n+2}$	$\frac{(2c+a+4n+2)}{(2c+1)(a+4n+2)}$	$\frac{(c+a+2)(2c-a)-2n(3a-2c+4n+2)}{(c+1)(2c-a)(a+4n+2)}$
0	$\frac{-1}{(c-1)}$	$\frac{-1}{(2c-1)}$	0	$\frac{1}{(2c+1)}$	$\frac{1}{(c+1)}$
-1	$E_{-1,-2}$	$\frac{-(2c+a+4n)}{a(2c-1)}$	$\frac{-1}{a}$	$\frac{-(2c-a)}{a(2c+1)}$	$\frac{-(c-a-2n)}{a(c+1)}$
-2	$\frac{-(2c+a+4n-1)(2c-a-4n-5)}{(a-1)(c-1)(2c-a-5)}$	$E_{-2,-1}$	$\frac{-2}{(a-1)}$	$\frac{-(4c-a+1)}{(a-1)(2c+1)}$	$\frac{-(2c-a+1)}{(a-1)(c+1)}$

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