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A CERTAIN NUMBER OF ZEROS OF $f^n f^{(k)} - a$ AND ITS NORMALITY

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Abstract. In this paper, we study functions of the form $F = f^n f^{(k)} - a$ for integers k, n and non-zero constant a. First, if all the zeros of f have multiplicity at least k, then F has at least a certain number of zeros. As an expected consequence given the Bloch Principle, families of such functions which have less than this number of zeros are normal.

1. Introduction and main results

We shall use the usual notations and classical results of Nevanlinna's theory in [1] such as $T(r,f), N(r,f), \overline{N}(r,f), m(r,f)$. we denote by $\overline{N}_{k)}(r,\frac{1}{f})$ the reduced counting function of those zeros of f which have multiplicities less than or equal to k, where k is a positive integer.

Let $D \subset \mathbb{C}$ be a domain, and let $\mathscr{F}$ be a family of meromorphic functions defined in D. Then $\mathscr{F}$ is said to be normal in D, if for every sequence $f_n \in \mathscr{F}$ there exists a subsequence f_{n_k} converges spherically locally uniformly to a meromorphic function or ∞.

The Bloch principle [2] states that every condition which reduces a meromorphic function in $\mathbb{C}$ to be a constant forces a families of meromorphic functions in D to be normal. The Bloch principle is not true in general, but many authors proved normality criterion for families of meromorphic functions about Picard type theorem.

W. K. Hayman [3] proved the following theorem.

Theorem A. *Let $n \geq 3$ be an integer. Let f be a transcendental meromorphic function, then $f^n f'$ assumes all finite values, except possibly zero, infinitely often.*

Hayman [4] conjectured that Theorem A might be true for $n = 2$ and $n = 1$. E. Mues [5] settled the conjecture for $n = 2$ and the case $n = 1$ was proved by W.Bergweiler and A. Eremenko [6] and by H. H. Chen and M. L. Fang [7].

In 1999, Pang and Zalcman [8] considered the general order derivative of a holomorphic function, and they proved the following result.

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Theorem B. *Let $k(\geq 1), n(\geq 1)$ be two integers. Let f be a transcendental holomorphic function, all of zeros of f have multiplicity at least k, then $f^n f^{(k)}$ assume each nonzero finite value infinitely often.*

In 2003, J. K. Langley [9] proved a stronger conclusion for $n = 1$, $k = 2$.

Theorem C. *Let $a \in \mathbb{C} \setminus \{0\}$. Let f be meromorphic function of positive order $L \leq \infty$ in $\mathbb{C}$, and with few poles, then the zeros sequence of $f f'' - a$ has exponent of convergence L.*

In the paper, we study functions of the form $F = f^n f^{(k)} - a$ for integers k, n and non-zero constant a.

Theorem 1.1. *Let $k(\geq 2), m(\geq 1), n(\geq m+1)$ be three integers, let $a \in \mathbb{C} \setminus \{0\}$. If f be a non-constant meromorphic function, all of whose zeros have multiplicity at least k, then $F = f^n f^{(k)} - a$ has at least $m+1$ distinct zeros.*

The normality corresponding to Theorem A was conjectured by Hayman [4] and studied by L. Yang and G. Zhang [10] (for $n \geq 5$), Y. X. Gu [11] (for $n = 4, 3$), X. C. Pang [12] (for $n \geq 2$) and Chen and Fang [7] (for $n = 1$).

In 1999, Pang and Zalcman considered the general order derivative of a holomorphic function, and they proved the following result.

Theorem D([8]). *Let $a \in \mathbb{C} \setminus \{0\}$ and $k(\geq 1), n(\geq 1)$ be two integers. Let $\mathscr{F}$ be a family of holomorphic functions in a unit disc Δ such that each $f \in \mathscr{F}$ has only zeros of multiplicity at least k. If $f^n f^{(k)} \neq a$ for each $f \in \mathscr{F}$ in Δ, then $\mathscr{F}$ is normal in Δ.*

For the related results, see Zhang [13], Wu and Xu [14], Tan et al.[15], Meng and Hu [16], Yuan et al.[17].

As an expected consequence given the Bloch Principle, families of such functions which have less than this number of zeros are normal.

Theorem 1.2. *Let $a \in \mathbb{C} \setminus \{0\}$ and $k(\geq 2), m(\geq 1), n(\geq m+1)$ be three integers. Let $\mathscr{F}$ be a family of meromorphic functions in a domain D such that each $f \in \mathscr{F}$ has only zeros of multiplicity at least k. If $f^n f^{(k)} - a$ has at most m distinct zeros for each $f \in \mathscr{F}$ in D, then $\mathscr{F}$ is normal in D.*

Example 1.3. Let $\Delta = \{z : |z| < 1\}$ and $\mathscr{F} = \{f_j(z)\}$, where

$$f_j(z) = j z^k,\ z \in D,\ j = 1, 2, \ldots.$$

then, $f^m f^{(k)} - a = j^{m+1} k! z^{km} - a$ has $km \geq m$ distinct zeros at least, and $f^{m+1} f^{(k)} - a = j^{m+2} k! z^{k(m+1)} - a$ has $k(m+1) \geq m+1$ distinct zeros at least, but $\mathscr{F}$ is not normal in Δ. This implies that both $n \geq m+1$ and $f^n f^{(k)} - a$ has at most m distinct zeros in Theorem 1.2 are necessary.

2. Some lemmas

A quasi-differential polynomial P of a meromorphic function f is defined by $P(z) = \sum_{i=1}^{n} \varphi_i(z)$, where $\varphi_i(z) = a_i(z)\left(f(z)\right)^{S_{i0}}\left(f'(z)\right)^{S_{i1}} \cdots \left(f^{(t)}(z)\right)^{S_{it}}$, $\gamma_{\varphi_i} = \sum_{j=0}^{t} S_{ij}$ denotes the degree of $\varphi_i(z)$, where $S_{ij}(1 \le i \le n, 0 \le j \le t)$ are non-negative integers, and $\alpha_i(z) \not\equiv 0$ is a meromorphic function such that $m(r, \alpha_i) = S(r, f)$. The number $\gamma_P = \max\limits_{1\le i\le n} \gamma_{\varphi_i}$ is called the degree of quasi-differential polynomial P.

Lemma 2.4 ([18], [19] p.19)**.** *Let f be a non-constant meromorphic function and P, Q be quasi-differential polynomial in f with $Q \not\equiv 0$, let n be a positive integer and $f^n P = Q$. If $\gamma_Q \le n$, then $m(r, P) = S(r, f)$, where γ_Q is the degree of Q.*

Lemma 2.5 ([1, 20, 21])**.** *Let f be a non-constant meromorphic function in $\mathbb{C}$ and $k(\ge 1)$ be a integer, then (i) $m(r, \frac{f^{(k)}}{f}) = S(r, f)$, (ii) $S(r, f^{(k)}) = S(r, f)$.*

Lemma 2.6 ([22])**.** *Let $\mathscr{F}$ be a family of functions meromorphic in the unit disc Δ, all of whose zeros have multiplicity at least k. Then if $\mathscr{F}$ is not normal in any neighbourhood of $z_0 \in \Delta$, there exist, for each α, $0 \le \alpha < k$,(i)points z_n, $z_n \to z_0$, $z_0 \in \Delta$;(ii)functions $f_n \in \mathscr{F}$; and (iii)positive numbers $\rho_n \to 0^+$, such that $g_n(\xi) = \rho_n^{-\alpha} f_n(z_n + \rho_n \xi) \to g(\xi)$ spherically uniformly on compact subsets of $\mathbb{C}$, where g is a non-constant meromorphic function, all of whose zeros have multiplicity at least k.*

3. Proof of Theorem 1.1

Proof. Set

$$F = f^n f^{(k)} - a. \tag{3.1}$$

By (3.1) we have

$$T(r, F) = O(T(r, f)). \tag{3.2}$$

Rewriting (3.1) as $F - f^n f^{(k)} = -a$, which leads to

$$(-a)\frac{F'}{F} = f^n[-\frac{F'}{F} f^{(k)} + n\frac{f'}{f} f^{(k)} + f^{(k+1)}],$$

then

$$f^n \phi = (-a)\frac{F'}{F}, \tag{3.3}$$

where

$$\phi = -\frac{F'}{F} f^{(k)} + n\frac{f'}{f} f^{(k)} + f^{(k+1)}. \tag{3.4}$$

We shall show that $\phi \not\equiv 0$. Otherwise we have $\phi \equiv 0$, then $F \equiv c$, where c is a constant. So $f^n f^{(k)} \equiv a + c$.

If $a+c=0$, then $f^n f^{(k)} \equiv 0$. Since f has only zeros of multiplicity at least k, we obtain $f \equiv$ constant, which is a contradiction.

If $a+c \neq 0$, noting $f^n f^{(k)} \equiv a+c$, and f has only zeros of multiplicity at least k, we get $f \neq 0$, which means that $\frac{1}{f^n}(\equiv \frac{f^{(k)}}{a+c})$ must be an entire function. Thus, we have

$$\begin{aligned} nT(r,f) &= T(r,\tfrac{f^{(k)}}{a+c}) \leq T(r,f^{(k)}) + O(1) \\ &\leq N(r,f^{(k)}) + m(r,f^{(k)}) + O(1) \\ &\leq N(r,f) + k\overline{N}(r,f) + m(r,f) + S(r,f) \\ &\leq T(r,f) + S(r,f), \end{aligned}$$

this is $(n-1)T(r,f) \leq S(r,f)$. Since $n \geq m+1$, we conclude that $T(r,f) = S(r,f)$, which is a contradiction. So $\phi \not\equiv 0$.

By Lemma 2.2, in view of (3.2) and by Lemma 2.1 applied to (3.3), we have $m(r,\phi) = S(r,f)$.

From (3.2), (3.3) and Nevanlinna's first fundamental theorem, we get

$$\begin{aligned} nm(r,f) &\leq m(r,\tfrac{F'}{F}) + m(r,\tfrac{1}{\phi}) + S(r,f) \\ &= m(r,\tfrac{1}{\phi}) + S(r,f) \\ &= N(r,\phi) + m(r,\phi) - N(r,\tfrac{1}{\phi}) + S(r,f) \\ &= N(r,\phi) - N(r,\tfrac{1}{\phi}) + S(r,f). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &N(r,\phi) \leq \overline{N}_{k)}(r,\tfrac{1}{f}) + \overline{N}(r,\tfrac{1}{F}) + S(r,f), \\ &N(r,\tfrac{1}{\phi}) \geq nN(r,f) - \overline{N}(r,f). \end{aligned}$$

This implies

$$nT(r,f) \leq \overline{N}_{k)}(r,\frac{1}{f}) + \overline{N}(r,\frac{1}{F}) + \overline{N}(r,f) + S(r,f). \tag{3.5}$$

Noting that f has only zeros with multiplicity at least k, thus we obtain

$$\overline{N}_{k)}(r,\frac{1}{f}) \leq \frac{1}{k}N(r,\frac{1}{f}) \leq \frac{1}{k}T(r,f) + S(r,f). \tag{3.6}$$

From (3.5) and (3.6) we have $(n-1-\frac{1}{k})T(r,f) \leq \overline{N}(r,\frac{1}{F}) + S(r,f)$.

Suppose $f^n f^{(k)} - a$ has $l \leq m$ distinct zeros, then we get

$$(n-1-\frac{1}{k})T(r,f) \leq l\,logr + S(r,f).$$

Considering $n \geq m+1$, it follows that

$$T(r,f) \leq \frac{kl}{km-1}logr + S(r,f) \leq \frac{km}{km-1}logr + S(r,f). \tag{3.7}$$

Thus, we deduce that f is a non-constant rational function satisfying $deg\, f \le 2$ for $k = 2, m = 1$, and $deg\, f \le 1$ for $km > 2$. Next, we consider two cases.

Case 1. $deg\, f \le 1$ for $km > 2, k > 2$.

Sub-case 1.1. If $f \ne 0$, then write $f = \frac{1}{Az+C}, A \ne 0$.

Obviously, $f^n f^{(k)} - a = \frac{(-1)^k k! A^k - a(Az+C)^{n+k+1}}{(Az+C)^{n+k+1}}$ has $m+2+k$ distinct zeros at least, a contradiction.

Sub-case 1.2. If f has at least one zero, noting that f has only zeros with multiplicity at least $k > 2$, we can get $deg\, f > 1$, which is a contradiction since $deg\, f \le 1$.

Case 2. $deg\, f \le 2$ for $k = 2, m = 1$.

Sub-case 2.1. If $f \ne 0$, from (3.7) we have $(n-1)T(r, f) \le mlogr + S(r, f)$, this is $T(r, f) \le \log r + S(r, f)$.

Hence f is a non-constant rational function satisfying $deg\, f \le 1$, then $f = \frac{A}{Bz+C}, A, B \ne 0$. we obtain $f^n f^{(2)} - a = \frac{2A^{n+1}B^2}{(Bz+C)^{n+3}} - a$ has $n+3$ distinct zeros at least, a contradiction.

Sub-case 2.2. If f has at least one zero, since f has only zeros with multiplicity at least $k = 2$, then we get $f = \frac{(z-z_0)^2}{Az^2+Bz+C}$.

Sub-case 2.2.1. $A = 0$.

Sub-case 2.2.1.1. $B = 0$. Then $f = D(z-z_0)^2$, $D \ne 0$, it follows that $f^n f^{(2)} - a = 2D^{n+1}(z-z_0)^{2n} - a$ has $2n \ge 4$ distinct zeros at least, a contradiction.

Sub-case 2.2.1.2. $B \ne 0$. Then $f = \frac{(z-z_0)^2}{Bz+C}$. Obviously, $\overline{N}(r, \frac{1}{f}) \le \frac{1}{2}T(r, f) + O(1)$, $\overline{N}(r, f) = \log r$, and $T(r, f) = 2\log r + O(1)$.

On the other hand, from (3.5), we get $(n - \frac{1}{2})T(r, f) \le 2\log r + S(r, f)$. Since $n \ge m+1 = 2$, then

$$T(r, f) \le \frac{4}{3}\log r + S(r, f),$$

we also get a contradiction.

Sub-case 2.2.2. $A \ne 0$.

Sub-case 2.2.2.1. If $Az^2 + Bz + C = A(z-z_1)^2$, then $f = \frac{(z-z_0)^2}{A(z-z_1)^2}$, $z_0 \ne z_1$.

From (3.7) and $n \ge m+1$, we have

$$T(r, f) \le \frac{2}{n} logr + S(r, f).$$

This shows that $deg\, f \le 1$, which contradicts the fact $deg\, f = 2$.

Sub-case 2.2.2.2. If $Az^2+Bz+C=A(z-z_1)(z-z_2), z_1\neq z_2$, then

$$f=\frac{1}{A}(1+\frac{A_1}{z-z_1}+\frac{A_2}{z-z_2}),$$

where $A_1=\frac{(z_0-z_1)^2}{z_1-z_2}, A_2=-\frac{(z_0-z_2)^2}{z_1-z_2}$. It follows that

$$f^nf^{(2)}-a=\frac{Q(z)}{A^{n+1}(z-z_1)^{n+3}(z-z_2)^{n+3}},$$

where $Q(z)=2[A_1(z-z_2)^3+A_2(z-z_1)^3](z-z_0)-aA^{n+1}(z-z_1)^{n+3}(z-z_2)^{n+3}$

Obviously, $f^nf^{(2)}-a$ has one zero at least.

Suppose that $f^nf^{(2)}-a$ has only one zero, say z_0^*, where $z_0^*\neq z_1,z_2$. Then we get $Q(z)=-aA^{n+1}(z-z_0^*)^{2n+6}$.

Set $Z=z-z_0^*$, we can obtain $(Z+z_0^*-z_1)^{n+3}(Z+z_0^*-z_2)^{n+3}=Z^{2n+6}$. Therefore $z_0^*=z_1=z_2$, which is a contradiction. Thus we deduce that $f^nf^{(2)}-a$ has 2=m+1 distinct zeros at least, a contradiction. This proves Theorem 1.1. □

4. Proof of Theorem 1.2

Proof. Suppose that $\mathscr{F}$ is not normal at $z_0\in D$. Let $\alpha=\frac{k}{n+1}$. Then by Lemma 2.3, there exists a sequence of complex numbers $z_j\to z_0$, a sequence of functions $f_j\in\mathscr{F}$ and a sequence of positive numbers $\rho_n\to 0^+$ such that

$$g_j(\xi)=\rho_j^{-\frac{k}{n+1}}f_j(z_j+\rho_j\xi)$$

converges locally uniformly with respect to the spherical metric to a nonconstant meromorphic functions $g(\xi)$ in $\mathbb{C}$. Also the order of g does not exceed two and by Hurwitz's theorem g has no zero of mulitiplicity less than k.

If $g^n(\xi)g^{(k)}(\xi)-a\equiv 0$, noting $n\geq m+1$, and $g(\xi)$ has only zeros of multiplicity at least k ,then $g(\xi)$ has no poles, and thus $g(\xi)$ is entire function. So, we have

$$\begin{aligned}nT(r,g)=T(r,\frac{g^{(k)}}{a})&\leq T(r,g^{(k)})+O(1)\\&\leq N(r,g^{(k)})+m(r,g^{(k)})+O(1)\\&\leq N(r,g)+k\overline{N}(r,g)+m(r,g)+S(r,g)\\&\leq T(r,g)+S(r,g),\end{aligned}$$

this implies $(n-1)T(r,g)\leq S(r,g)$. Since $n\geq m+1$, this is $T(r,g)=S(r,g)$, which is a contradiction. So $g^n(\xi)g^{(k)}(\xi)-a\not\equiv 0$.

Next, we shall show that $g^n(\xi)g^{(k)}(\xi)-a$ has at most m distinct zeros. Suppose that $g^n(\xi)g^{(k)}(\xi)-a$ has $m+1$ distinct zeros $\xi_i(i=1,2,\cdots,m+1)$.

Since on every compact subset of $\mathbb{C}$ which contains no poles of g, we get

$$g_j^n(\xi)g_j^{(k)}(\xi) - a = \rho_j^{-\frac{nk}{n+1}}\left\{f_j^n(z_j+\rho_j\xi)f_j^{(k)}(z_j+\rho_j\xi)\right\} - a \to g^n(\xi)g^{(k)}(\xi) - a,$$

also locally uniformly with respect to the spherical metric.

By Hurwitz's theorem, there exist points $\xi_{j,i} \to \xi_i$, $i = 1,2,\cdots,m+1$ such that

$$f_j^n(z_j+\rho_j\xi_{j,i})f_j^{(k)}(z_j+\rho_j\xi_{j,i}) = a.$$

Since $f_j^n(\xi)f_j^{(k)}(\xi) - a$ has at most m distinct zeros and $z_j + \rho_j\xi_{j,i} \to z_0$, it follows that $z_j + \rho_j\xi_{j,i_0} = z_j + \rho_j\xi_{j,l_0}$ for some i_0 and l_0. Thus $\xi_{j,i_0} = \xi_{j,l_0}$ and hence $\xi_{i_0} = \xi_{l_0}$. This is a contradiction.

However, by Theorem 1.1, there do not exist non-constant meromorphic functions that have the above properties

Therefore, $\mathscr{F}$ is normal in D and hence Theorem 1.2 is proved. □

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