TAMKANG JOURNAL OF MATHEMATICS
Volume 52, Number 1, 37-67, March 2021
DOI:10.5556/j.tkjm.52.2021.3045

d-Minimal Surfaces in Three-Dimensional Singular Semi-Euclidean Space $\mathbb{R}^{0,2,1}$

Yuichiro Sato

Abstract. In this paper, we study surfaces in singular semi-Euclidean space $\mathbb{R}^{0,2,1}$ endowed with a degenerate metric. We define d-minimal surfaces, and give a representation formula of Weierstrass type. Moreover, we prove that d-minimal surfaces in $\mathbb{R}^{0,2,1}$ and spacelike flat zero mean curvature (ZMC) surfaces in four-dimensional Minkowski space $\mathbb{R}^4_1$ are in one-to-one correspondence.

1 Introduction

In this paper, we investigate surfaces in three-dimensional singular semi-Euclidean space with the signature $(0, 2, 1)$. The history of surface theory is very long, and there is a lot of research. Minimal surfaces attain stationary values for the volume functional of surfaces. We have many results of the research for minimal surfaces. In particular, they are characterized by having the mean curvature vector field which vanishes identically. Recently, Umehara and Yamada et al. ([24], [10] and [9]) studied the zero mean curvature surfaces in three-dimensional Minkowski space actively. For such surfaces, they showed that singularities appear generically, and relate to the topology of surfaces.

On the other hand, the author [19] classified ruled minimal surfaces in semi-Euclidean space. As a consequence, it was obtained that certain surfaces are included in three-dimensional subspaces whose metrics are degenerate forms. Inspired by this fact, in this work we study the singular differential geometry, i.e. allow to have degenerate metrics. In particular, we consider the surface theory. We introduce a degenerate metric $dx^2 + dy^2$ to three-dimensional vector space $\mathbb{R}^3$ with the coordinates (x, y, z). We call the pair $(\mathbb{R}^3, dx^2 + dy^2)$ singular semi-Euclidean space. It is denoted by $\mathbb{R}^{0,2,1}$. Let M be a surface in $\mathbb{R}^{0,2,1}$. We assume that the induced metric of M is non-degenerate. Actually, this degenerate geometry is equivalent to simply isotropic geometry, which is one of the Cayley-Klein geometries [15]. For isotropic geometry, the well-known

2010 *Mathematics Subject Classification*. 53A10, 53A40.
Key words and phrases. minimal surface, isotropic geometry, semi-Riemannian geometry.

reference is [17]. In terms of the affine geometry with metrics and connections. We reformulate geometrical objects of surface theory such as induced connection and second fundamental form.

Here, we remark how to use the terms. First, in the canonical three-dimensional Euclidean space $\mathbb{R}^3$, surfaces whose mean curvature vanishes identically give stationary values for the volume functional. In a certain situation, its value is minimal, but not extreme in general. Historically, we call such surfaces *minimal*.

Next, in three-dimensional Minkowski space $\mathbb{R}^3_1$, surfaces whose mean curvature vanishes identically change its name with respect to the causal character of the induced metrics. When the induced metric is spacelike, i.e. Riemannian, we call such surfaces *maximal*. This means that, when we consider the volume functional analytically, such surfaces always give maximal values unlike the Euclidean case. On the other hand, when timelike, i.e. Lorentzian, we simply call such surfaces *minimal*. We should remark that timelike minimal surfaces give stationary values for the volume functional, but give neither minimal nor maximal values. We can refer these facts in Remark 32 and Theorem 37 of Chapter 6 of [2]. When connected surfaces have the part of spacelike maximal surfaces and that of timelike minimal surfaces, we call such surfaces *mixed type* [9].

In four-dimensional Minkowski space $\mathbb{R}^4_1$, surfaces whose mean curvature vector field vanishes identically are more complicated. Therefore, in order to treat uniformly, we call all such surfaces *zero mean curvature* when the ambient space is $\mathbb{R}^4_1$. This is why we have to pay attention to the terminology.

The remaining of this work is derived as follows. In the section two, we recall fundamental facts in semi-Riemannian geometry and properties of non-degenerate submanifolds. In particular, we explain the singular semi-Euclidean space. In the section three, we define non-degenerate surfaces in $\mathbb{R}^{0,2,1}$ and study their properties in detail. In addition, we calculate some examples.

The section four is the main section. We consider d-minimal surfaces which we define are analogue objects to classical minimal surfaces. They are called *isotropic minimal surfaces* in terms of simply isotropic geometry [17]. In addition, we show a representation formula of Weierstrass type for d-minimal surfaces (Theorem 4.2), and claim that d-minimal surfaces allow to have isolated singularities. Moreover, we see that spacelike flat ZMC surfaces in $\mathbb{R}^4_1$ are contained in a three-dimensional subspace endowed with a degenerate induced metric (Theorem 5.5).

The section five gives an application. More precisely, we prove that d-minimal surfaces and spacelike flat zero mean curvature (ZMC) surfaces in four-dimensional Minkowski space are in one-to-one correspondence (Corollary 5.6). In particular, we see that there exist infinitely many spacelike flat ZMC surfaces in $\mathbb{R}^4_1$ which are not congruent to each other. In [1] and [12], some representation formulas are known. However, we should remark that singularities do not appear. Actually, since the regularity condition are assumed on surfaces, the possibility of singularities appearing is omitted in the obtained representation formula.

From Table 1, we see that d-minimal surfaces in $\mathbb{R}^{0,2,1}$ have intermediate properties between minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{R}^3_1$. Regarding singularities, they do not appear on minimal surfaces. However, on maximal surfaces, cuspidal edges, swallowtails and cuspidal crosscaps appear in generic case. Refer to [10] in detail. On the other hand, for d-minimal surfaces, isolated singularities are allowed. However, in this paper, these singularities will be not classified.

2 Preliminaries

In this section, we explain the fundamental properties for semi-Riemannian manifolds and their non-degenerate submanifolds.

2.1 Semi-Riemannian manifolds

Let (M, g) be an n-dimensional semi-Riemannian manifold. For each $x \in M$ and a tangent vector $X \in T_xM$, we call X

$$\begin{aligned} &\textit{spacelike} && \text{if} \quad g(X,X) > 0 \text{ or } X = 0,\\ &\textit{timelike} && \text{if} \quad g(X,X) < 0,\\ &\textit{lightlike} \text{ (or } \textit{null}\text{)} && \text{if} \quad g(X,X) = 0. \end{aligned}$$

These are called *causal properties* of tangent vectors [14]. As in the case of Riemannian manifolds, there exists a unique torsion-free and metric connection ∇ for a semi-Riemannian manifold. We call ∇ the *Levi-Civita connection* of (M, g). Hereinafter, we consider that connections for semi-Riemannian manifolds are Levi-Civita connections.

We define the *curvature tensor field* R of a semi-Riemannian manifold (M, g) as

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \quad (X, Y, Z \in \Gamma(TM)).$$

Next, for each $x \in M$, let P be a two-dimensional non-degenerate subspace of the tangent vector space T_xM, and let $\{X, Y\}$ be a basis of P. Then, we define the *sectional curvature* $K(P)$ of P as

$$K(P) := \frac{g(R(X,Y)Y, X)}{g(X,X)g(Y,Y) - g(X,Y)^2},$$

where a subspace $P \subset T_xM$ is called *non-degenerate* if the restriction on P of g is a non-degenerate form and it is called *degenerate* if otherwise. In particular, when the dimension of M is two, sectional curvatures are called *Gaussian curvatures*. We denote the set consisting of smooth functions on M by $C^\infty(M)$. For each $u \in C^\infty(M)$, we define the *gradient vector field* grad$u$ of u as

$$g(\mathrm{grad}u, X) = du(X) \quad (\forall X \in \Gamma(TM)),$$

where du denotes the exterior derivative of u. Next, for each $X \in \Gamma(TM)$, we define the *divergence* div$X$ of X as

$$\mathrm{div}X := \mathrm{tr}((X_1, X_2) \mapsto g(\nabla_{X_1} X, X_2)) \quad (X_1, X_2 \in \Gamma(TM)).$$

For each $u \in C^\infty(M)$, we define the *Laplacian* $\Delta_g u$ of u with respect to g as

$$\Delta_g u := \mathrm{div}(\mathrm{grad}u).$$

When $\Delta_g u \equiv 0$, we say that u is a *harmonic* function.

Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame of (M, g). The gradient vector field and the divergence respectively have the following local expressions

$$\begin{aligned} \mathrm{grad}u &= \sum_{i=1}^{n} \epsilon_i du(e_i) e_i, \\ \mathrm{div}X &= \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} X, e_i), \end{aligned}$$

where $\epsilon_i = g(e_i, e_i) = \pm 1$.

2.2 Non-degenerate submanifolds

Let M be an m-dimensional manifold, and let $(N, \bar{g})$ be an n-dimensional semi-Riemannian manifold. We assume that a C^∞-mapping $f : M \to N$ is an immersion. Then, we call M an *immersed submanifold* in N. In particular, when f is injective, and M is homeomorphic to the image $f(M)$ as the subspace of N, M is said to be a *embedded submanifold* in N.

We denote the induced metric $f^*\bar{g}$ on M by g. For semi-Riemannian manifolds, we remark that g is not always non-degenerate even if f is an immersion. When the induced metric g is non-degenerate, we call (M, g) a *non-degenerate submanifold*, or a *semi-Riemannian submanifold* of $(N, \bar{g})$.

Hereinafter, when we describe submanifolds, unless otherwise stated, we consider immersed, non-degenerate submanifolds. Then, for each $x \in M$, a *normal vector space* $T_x^\perp M$ is defined as

$$T_x^\perp M := \{v \in T_{f(x)}N \mid \bar{g}(df_x(w), v) = 0, \ \forall w \in T_x M\}.$$

We obtain a vector bundle $T^\perp M = \bigcup_{x \in M} T_x^\perp M$ of rank $(n-m)$ over M. This is called a *normal bundle* of M. By this, for each $x \in M$, we have the orthogonal direct sum decomposition

$$T_{f(x)}N = T_x M \perp T_x^\perp M,$$

where $\perp$ stands for the orthogonal direct sum. In particular, we see that, as the orthogonal direct sum of vector bundles, it holds

$$f^*TN = TM \perp T^{\perp}M,$$

where f^*TN is the pull-back bundle over M by f. We denote the Levi-Civita connection of $(N, \bar{g})$ and that of (M, g) by $\bar{\nabla}$ and ∇ respectively. We define $\Gamma(T^{\perp}M)$ as the set of smooth sections of the normal bundle $T^{\perp}M$. This section is said to be a *normal vector field* particularly.

Let $X, Y, \cdots, \xi, \eta, \cdots$, be tangent and normal vector fields on M respectively. By using the orthogonal direct sum decomposition given above, we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.1}$$
$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi, \tag{2.2}$$

where h, A_ξ and $\nabla^{\perp}$ are called the *second fundamental form*, the *shape operator* with respect to ξ and the *normal connection* on M respectively. We call the formula (2.1) and (2.2) *Gauss formula* and *Weingarten formula* of M respectively.

2.3 Singular semi-Euclidean spaces

We define the n-dimensional *singular semi-Euclidean space* with the signature (p, q, r) as

$$\mathbb{R}^{p,q,r} := \left(\mathbb{R}^n, (\cdot,\cdot) = -\sum_{i=1}^{p} dx_i^2 + \sum_{j=p+1}^{p+q} dx_j^2 + \sum_{k=p+q+1}^{n} 0dx_k^2\right), \tag{2.3}$$

where $n = p + q + r$ and $(x_1, \cdots, x_n)$ expresses the canonical coordinates on $\mathbb{R}^n$ [23]. We remark the following statement:

- When $r = 0$, $\mathbb{R}^{p,q,0}$ is called *semi-Euclidean space* having index p, and we denote it by $\mathbb{R}^n_p$.
- When $p = r = 0$, $\mathbb{R}^{0,n,0} = \mathbb{R}^n_0$ is nothing but Euclidean space $\mathbb{R}^n$.

We remark that $r \geq 1$ if and only if the metric $(\cdot,\cdot)$ is degenerate. In the context of isotropic geometry, the notation $\mathbb{R}^{0,n-1,1}$ would denote the simply isotropic n-space $\mathbb{I}^n$ [17].

In the following, the ambient N will be $\mathbb{R}^n_p$. We denote the semi-Euclidean metric by

$$\langle\cdot,\cdot\rangle_p := -\sum_{i=1}^{p} dx_i^2 + \sum_{j=p+1}^{n} dx_j^2.$$

Gauss equation, *Codazzi equation* and *Ricci equation* of M are given by the following

$$\langle R(X,Y)Z, W\rangle_p = \langle h(Y,Z), h(X,W)\rangle_p - \langle h(X,Z), h(Y,W)\rangle_p, \tag{2.4}$$

$$
\begin{aligned}
(\nabla_X h)(Y,Z) &= (\nabla_Y h)(X,Z), &(2.5)\\
\langle R^{\perp}(X,Y)\xi,\eta\rangle_p &= \langle [A_\xi, A_\eta]X, Y\rangle_p, &(2.6)
\end{aligned}
$$

where R and $R^{\perp}$ are curvature tensor fields with respect to connections ∇ and $\nabla^{\perp}$ respectively, and $\nabla_X h$ is the covariant derivative of the second fundamental form h for the tangent vector field X, i.e. it is defined by

$$(\nabla_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Moreover, the normal bundle $T^{\perp}M$ of M is called *flat* if $R^{\perp} \equiv 0$. Finally, we define the mean curvature vector field of M by

$$\vec{H} = \frac{1}{m}\mathrm{trace}_g h. \tag{2.7}$$

3 Surface theory in singular semi-Euclidean space

In this section, we consider three-dimensional singular semi-Euclidean space with the signature $(0,2,1)$, whose canonical coordinates are (x,y,z), and study its surfaces.

3.1 Preparations

Let M be a two-dimensional manifold, let $f : M \to \mathbb{R}^{0,2,1}$ be a C^{∞}-immersion and let g be the induced metric by f. We assume that the metric g is a positive definite symmetric bilinear form. And, we call f a *non-degenerate immersion* or a *non-degenerate surface*. Then, for each $x \in M$, a *normal vector space* $T^{\perp}_x M$ is defined by

$$T^{\perp}_x M := \{\xi \in \mathbb{R}^3 \mid (df_x(v), \xi) = 0,\ \forall v \in T_x M\} = \mathrm{span}_{\mathbb{R}}\{(0,0,1)\},$$

and we have a vector bundle of rank one over M

$$T^{\perp}M = \bigcup_{x\in M} T^{\perp}_x M.$$

Therefore, we obtain an orthogonal direct sum decomposition

$$T_{f(x)}\mathbb{R}^3 = T_x M \perp T^{\perp}_x M$$

for each $x \in M$. In particular, we see, as a vector bundle decomposition,

$$f^* T\mathbb{R}^3 = TM \perp T^{\perp}M,$$

where TM is the tangent bundle over M and $f^*T\mathbb{R}^3$ is the pull-back bundle by f over M.

Proposition 3.1. *We get an isomorphism as vector bundle*

$$T^{\perp}M \cong M \times \mathbb{R}.$$

Proof. We can take $\xi = (0,0,1) \in \Gamma(T^{\perp}M)$ as a non-vanishing global section. □

Remark 1. For three-dimensional singular semi-Euclidean space with the signature (p,q,r), where $p+q+r=3,\ r \geq 1,\ p \leq q$, we can define non-degenerate surfaces when $r=1$, i.e.

$$(p,q,r) = (0,2,1),\ (1,1,1).$$

When $r \geq 2$, the metric induced on surfaces is degenerate. We remark that $\mathbb{R}^{1,1,1}$ is equivalent to the pseudo-isotropic 3-space $\mathbb{I}^3_1$ (Refer to [20], [21] and [4]). And, as a notation, we define

$$|v| := \sqrt{(v,v)} = \sqrt{v_1^2 + v_2^2}$$

for a vector $v = (v_1, v_2, v_3) \in \mathbb{R}^{0,2,1}$.

On the other hand, how to control null vectors of $\mathbb{R}^{0,2,1}$ is untouched. Since every null vector is proportional to $\xi = (0,0,1)$, it is natural to introduce the *co-metric* $\langle\langle \cdot, \cdot \rangle\rangle$ on the set of null vectors as below

$$\langle\langle (0,0,\alpha), (0,0,\beta) \rangle\rangle := \alpha\beta \in \mathbb{R}.$$

In addition, $\mathbb{R}^{0,2,1}$ can lead to either doubly isotropic $\mathbb{I}^3_{(2)}$, Galilean $\mathbb{G}^3$, or pseudo-Galilean $\mathbb{G}^3_1$ geometries depending on how we deal with null vectors. For example, refer to [7]. One the other hand, in the case $r \geq 2$, this problem is no longer trivial.

Next, we recall affine differential geometry [13]. Let $(\mathbb{R}^{n+1}, d)$ be $(n+1)$-dimensional Euclidean space with the canonical connection d and M be an n-dimensional manifold. A C^{∞}-immersion $f : M \to \mathbb{R}^{n+1}$ is an *affine immersion* if for any $x \in M$ there exists a neighborhood U at x and a vector field ξ on U over $\mathbb{R}^{n+1}$ such that

$$T_{f(y)}\mathbb{R}^{n+1} = T_yM \oplus \mathbb{R}\xi_y \quad (\forall y \in U),$$

where $\oplus$ stands for the direct sum. In particular, when there exists ξ globally on M, it is called a *transversally vector field* on M. Then, a torsion-free connection ∇ is induced on M, and it satisfies

$$d_XY = \nabla_XY + h(X,Y)\xi$$

for any $X, Y \in \Gamma(TM)$. This implies that h is a $(0,2)$-type symmetric tensor field over M, and we call h an *affine fundamental form* (with respect to ξ). In affine differential geometry, we often assume that h is non-degenerate. Moreover, let $f : M \to \mathbb{R}^{n+1}$ be an affine immersion and let ξ be its transversally vector field. We call ξ *equiaffine* when

$$\forall X \in \Gamma(TM), \quad d_X \xi \in \Gamma(TM).$$

Then, f is called an *equiaffine immersion*.

In terms of affine differential geometry, we see the following proposition.

Proposition 3.2. *Let M be a two-dimensional manifold. A non-degenerate immersion $f : M \to \mathbb{R}^{0,2,1}$ is an equiaffine immersion whose transversally vector field over M is $\xi \equiv (0, 0, 1)$.*

Proof. By using the orthogonal direct sum $f^*T\mathbb{R}^3 = TM \perp T^\perp M$, and $d_X \xi = 0$ for all $X \in \Gamma(TM)$, the proof is completed. □

Hereinafter, let ξ be the constant vector field $\xi = (0, 0, 1)$ and let d be the canonical connection as a linear connection, i.e. for all $X, Y \in \Gamma(T\mathbb{R}^{0,2,1})$, identifying Y with the vector-valued function $Y = (Y_1, Y_2, Y_3)$,

$$d_X Y := dX(Y) = (X(Y_1), X(Y_2), X(Y_3)).$$

Then, the connection d is torsion-free and preserves the degenerate metric $(\cdot\, , \cdot\,)$. Thus, the connection d plays the role of the Levi-Civita connection.

We define the automorphism group $\mathrm{Aut}(\mathbb{R}^{0,2,1}, d)$ with respect to $\mathbb{R}^{0,2,1}$ and d as

$$\begin{aligned} \mathrm{Aut}(\mathbb{R}^{0,2,1}, d) &:= \{A \in \mathrm{Diff}(\mathbb{R}^3) \mid A^*d = d,\ A^*(\cdot, \cdot) = (\cdot, \cdot)\} \\ &= O(0, 2, 1) \ltimes \mathbb{R}^3, \end{aligned}$$

where $\mathrm{Diff}(\mathbb{R}^3)$ is the diffeomorphism group of $\mathbb{R}^3$ and

$$O(0, 2, 1) := \left\{ \left(\begin{array}{ccc} \multicolumn{2}{c}{T} & \begin{matrix} 0 \\ 0 \end{matrix} \\ a & b & c \end{array} \right) \,\middle|\, a, b, c \in \mathbb{R},\ c \neq 0,\ T \in O(2) \right\}.$$

We call $\mathrm{Aut}(\mathbb{R}^{0,2,1}, d)$ an *affine isometry group*. In particular, $\mathrm{Aut}(\mathbb{R}^{0,2,1}, d)$ is a seven-dimensional Lie group. From the view of Cayley-Klein geometry, this automorphism group is nothing but the simply isotropic rigid motion group [20]. And, Da Silva studied invariant surfaces generated by subgroups of $O(0, 2, 1)$ [22].

By using the decomposition $f^*T\mathbb{R}^3 = TM \perp T^\perp M$, for each $X, Y \in \Gamma(TM)$, $\alpha\xi \in \Gamma(T^\perp M)$ $(\alpha \in C^\infty(M))$, we have

$$d_X Y = \nabla_X Y + h(X, Y)\xi,$$

$$d_X(\alpha\xi) = X(\alpha)\xi.$$

Then, we see that the connection ∇ is the Levi-Civita connection with respect to the induced metric g on M. And, we call the given affine fundamental form h a *second fundamental form* of the non-degenerate immersion f.

For all $X, Y, Z \in \Gamma(TM)$, since the connection d is flat, we obtain

$$0 = {}^{d}R(X,Y)Z = {}^{\nabla}R(X,Y)Z + \{(\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z)\}\xi,$$

where ${}^{d}R$ and ${}^{\nabla}R$ are the curvature tensor fields for d and ∇ respectively, and we define

$$(\nabla_X h)(Y,Z) := X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

Therefore, we get

$${}^{\nabla}R \equiv 0, \tag{3.1}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{3.2}$$

The formula (3.1) implies that the non-degenerate surface is always flat, and we call the formula (3.2) *Gauss-Codazzi equation* of the non-degenerate surface. These formulas (3.1) and (3.2) were obtained by Sachs in [17].

Let $f : M \to \mathbb{R}^{0,2,1}$ be a non-degenerate immersion. Then, the image of f is locally expressed by the form of a graph surface $\{(u, v, F(u,v)) \in \mathbb{R}^{0,2,1} \mid (u,v) \in U\}$, where F is a smooth function on an open subset $U \subset \mathbb{R}^2$.

4 d-minimal surfaces in $\mathbb{R}^{0,2,1}$

4.1 Properties of d-minimal surfaces

We define some classes of non-degenerate surfaces. Namely,

(i) *d-totally geodesic surface* :$\Leftrightarrow$ the second fundamental form $h \equiv 0$,

(ii) *d-totally umbilical surface* :$\Leftrightarrow$ $\exists\lambda \in C^\infty(M)$ s.t. $h = \lambda g$,

(iii) *d-minimal surface* :$\Leftrightarrow$ $\mathcal{H} := \frac{1}{2}\mathrm{trace}_g h = \frac{1}{2} g^{ij} h_{ij} = 0$,

where g^{ij} is the components of the inverse matrix of $(g_{ij})_{1 \le i,j \le 2}$ and h_{ij} are the coefficients of the second fundamental form h. We call $\mathcal{H}$ the *mean curvature* of the non-degenerate surface. For (ii), we remark that (ii) is equivalent to (i) when $\lambda = 0$.

Proposition 4.1. *Let M be a two-dimensional manifold, and let $f : M \to \mathbb{R}^{0,2,1}$ be connected, not d-totally geodesic and d-totally umbilical surface, that is, there exists a function $\lambda \in C^\infty(M)$ such that $h = \lambda g$ and $\lambda \neq 0$. Then, λ is a constant function, and the image of f is an open subset of a paraboloid of revolution*

$$\left\{ \left(u, v, \frac{\lambda}{2}(u^2+v^2) + Au + Bv + C \right) \in \mathbb{R}^3 \;\middle|\; (u,v) \in \mathbb{R}^2 \right\},$$

where $A, B, C \in \mathbb{R}$ are constant. In particular, it is, up to affine isometry, an open subset of

$$\left\{ (u, v, u^2+v^2) \in \mathbb{R}^3 \;\middle|\; (u,v) \in \mathbb{R}^2 \right\}.$$

Proof. Since non-degenerate surfaces satisfy Gauss-codazzi equation (3.2), the function λ is a constant. Let g be the induced metric by f and let h be its second fundamental form. From the assumption, there exists a non-zero constant number $\lambda \in \mathbb{R}$ such that $h = \lambda g$. Since f is the non-degenerate immersion, for each point of M, there exists a coordinate neighborhood $\{U; (u,v)\}$ such that

$$f(u,v) = (u, v, \varphi(u,v)) \in \mathbb{R}^{0,2,1},$$

where φ is a C^∞-function on U. Then, we get

$$h_{11} = \varphi_{uu}, \; h_{12} = \varphi_{uv}, \; h_{22} = \varphi_{vv}.$$

Therefore, since we have

$$\varphi_{uu} = \lambda g_{11} = \lambda, \; \varphi_{uv} = \lambda g_{12} = 0, \; \varphi_{vv} = \lambda g_{22} = \lambda,$$

there exist constant numbers $A, B, C \in \mathbb{R}$ such that

$$\varphi(u,v) = \frac{\lambda}{2}(u^2+v^2) + Au + Bv + C.$$

Finally, gluing these pieces of surface in the whole of M, we obtain the consequence. □

In the context of isotropic geometry, d-totally umbilical surfaces are known as spheres of parabolic type. See [21] in detail.

Here, we define a *relative Gaussian curvature* $\mathcal{K}$ which is introduced in [17] as

$$\mathcal{K} := \frac{\det h}{\det g} \in C^\infty(M).$$

This quantity expresses the shape of the non-degenerate surface when we look from the ambient space $\mathbb{R}^3$. However, the canonical Gaussian curvature, i.e. the sectional curvature of two-dimensional Riemannian manifolds with respect to the induced metric, identically vanishes.

Proposition 4.2 ([17], Definition 8.11). *Let M be a two-dimensional manifold, and let $f : M \to \mathbb{R}^{0,2,1}$ be a non-degenerate immersion. Let $\mathcal{K}$ be its relative Gaussian curvature. In $\mathbb{R}^{0,2,1}$, we define for each $x \in M$,*

$$\begin{aligned} x &: \textit{elliptic point} \quad \textit{if} \quad \mathcal{K}(x) > 0, \\ x &: \textit{hyperbolic point} \quad \textit{if} \quad \mathcal{K}(x) < 0, \\ x &: \textit{parabolic point} \quad \textit{if} \quad \mathcal{K}(x) = 0. \end{aligned}$$

If we consider f as an immersion to Euclidean space $\mathbb{R}^3$, then the Euclidean Gaussian curvature does not correspond to the relative Gaussian curvature in general, however the two curvatures have the same signal.

Proof. Since f is a non-degenerate immersion, for each point of M, there exists a coordinate neighborhood $\{U; (u, v)\}$ such that

$$f(u, v) = (u, v, \varphi(u, v)) \in \mathbb{R}^{0,2,1},$$

where φ is a C^∞-function on U. When we consider f as an immersion to $\mathbb{R}^3$, the Euclidean Gaussian curvature K_G, noting that K_G is different from the canonical Gaussian curvature, is expressed by

$$K_G = \frac{\varphi_{uu}\varphi_{vv} - \varphi_{uv}^2}{(1 + \varphi_u^2 + \varphi_v^2)^2}$$

on U. On the other hand, the relative Gaussian curvature $\mathcal{K}$ is expressed by

$$\mathcal{K} = \varphi_{uu}\varphi_{vv} - \varphi_{uv}^2$$

on U. Therefore, K_G does not correspond to $\mathcal{K}$ in general, but the signs are the same. □

The notion of elliptic, hyperbolic and parabolic points in Euclidean ($\mathbb{R}^3$) and singular semi-Euclidean ($\mathbb{R}^{0,2,1}$) geometry are equivalent.

Remark 2. We consider the sign of the relative Gaussian curvature for some surfaces. First, for d-totally geodesic surfaces, since we have $h = 0$ by definition, it holds

$$\mathcal{K} = \frac{\det h}{\det g} \equiv 0.$$

Next, for d-totally umbilical surfaces, we have, by definition, there exists a constant number $\lambda \in \mathbb{R}$ such that $h = \lambda g$. We assume $\lambda \neq 0$. Then, we obtain

$$\mathcal{K} = \frac{\det h}{\det g} = \frac{\lambda^2 \det g}{\det g} = \lambda^2 > 0,$$

that is, all points are elliptic. Finally, for d-minimal surfaces, we make use of isothermal coordinates, that is, we choose the coordinates in which the coefficients of the induced metric hold

$$g_{11} = g_{22} > 0, \quad g_{12} = 0.$$

Then, since the mean curvature identically vanishes, we have

$$2\mathcal{H} = \text{trace}_g h = \frac{g_{22}h_{11} + g_{11}h_{22}}{g_{11}g_{22}} = \frac{h_{11} + h_{22}}{g_{11}} \equiv 0.$$

Moreover, by using $h_{22} = -h_{11}$, we obtain

$$\mathcal{K} = \frac{\det h}{\det g} = \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22}} = -\frac{h_{11}^2 + h_{12}^2}{g_{11}^2} \leq 0,$$

that is, almost all points are hyperbolic. Actually, we immediately see that $h = 0$ if and only if $\mathcal{K} = 0$. From Theorem 5.5, umbilic points are isolated.

Here, we give some descriptions for curves in $\mathbb{R}^{0,2,1}$. For a connected open interval $I \subset \mathbb{R}$, let c be a C^∞-map $c : I \to \mathbb{R}^{0,2,1}$. We call c a *curve* in $\mathbb{R}^{0,2,1}$. Moreover, we call c a *regular* curve if it holds

$$\forall t \in I, \ c'(t) \neq 0.$$

Next, let π be the projection to xy-plane, i.e.

$$\pi : \mathbb{R}^{0,2,1} \ni (x, y, z) \mapsto (x, y) \in \mathbb{R}^2.$$

From the view of isotropic geometry, π is said to be the *top view* of (x, y, z) [20], [21]. And, we call a parameter s of a curve $c = c(s)$ *arc-length* if it holds

$$|c'(s)| \equiv 1.$$

Then, we obtain the following propositions.

Proposition 4.3. *Let $c = c(t)$ $(t \in I)$ be a regular curve in $\mathbb{R}^{0,2,1}$. The following are equivalent:*

(i) The curve $c = c(t)$ admits an arc-length parameter.

(ii) For all $t \in I$, it holds $|c'(t)| > 0$.

(iii) The mapping $\pi \circ c$ is regular as a planar curve in $\mathbb{R}^2$.

Proof. Easy calculations. □

We call a regular curve $c = c(t)$ $(t \in I)$ in $\mathbb{R}^{0,2,1}$ *null* if it holds

$$|c'(t)| \equiv 0.$$

Proposition 4.4. *A regular curve $c : I \to \mathbb{R}^{0,2,1}$ is null if and only if it is a spacial line which is parallel with the z-axis.*

Proof. Easy calculations. □

Proposition 4.5. *For any connected surfaces in $\mathbb{R}^{0,2,1}$,*

(0) d-totally geodesic surfaces in $\mathbb{R}^{0,2,1}$ are non-degenerate planes only ([17], Theorem 9.4).

(1) a graph surface in $\mathbb{R}^{0,2,1}$

$$\{(u, v, f(u, v)) \in \mathbb{R}^{0,2,1} \mid (u, v) \in U \subset \mathbb{R}^2\}$$

is d-minimal if and only if f is a harmonic function on U ([17], Eq. (9.31)).

(2) non-planar, ruled d-minimal surfaces in $\mathbb{R}^{0,2,1}$ are locally, up to affine isometry and scaling, open subset of

(a) $f(u, v) = (v \cos u, v \sin u, u)$ (refer to Figure 1),

(b) $f(u, v) = (u, v, uv)$ (refer to Figure 1),

where $(u, v) \in \mathbb{R}^2$ ([19], Theorem 6).

(3) non-planar, d-minimal rotational surfaces in $\mathbb{R}^{0,2,1}$ are locally, up to affine isometry and scaling, open subset of

$$f(u, v) = (e^u \cos v, e^u \sin v, u)$$

(refer to Figure 1), where rotational surfaces are the surfaces invariant by the group of rotations around the z-axis, which acts on the xy-plane as Euclidean rotations, i.e. they are $SO(2)$-invariant surfaces.

Proof. (0) and (1) are proved by easy calculations. In case of (2), we apply the method of classification described by [19] since $\mathbb{R}^{0,2,1}$ is isometrically embedded in $\mathbb{R}^4_1$ as a *totally geodesic lightlike submanifold* by the natural way [5]. In fact, the following mapping

$$\iota : \mathbb{R}^{0,2,1} \ni (x, y, z) \mapsto (z, x, y, z) \in \mathbb{R}^4_1 \tag{4.1}$$

is an isometric embedding. We should remark that causal characters in $\mathbb{R}^{0,2,1}$ are ones in $\mathbb{R}^4_1$ because of Eq. (4.1). From the classification of Theorem 6 in [19], non-planar ruled minimal surfaces in the degenerate subspace $\iota(\mathbb{R}^{0,2,1}) \subset \mathbb{R}^4_1$ are locally contained in one of the following:

(a) *An elliptic helicoid of the second kind*

$$f(s,t) = (\cos s e_1 + \sin s e_2)t + s e_3,$$

where $e_1 = (0,1,0,0), e_2 = (0,0,1,0)$ and $e_3 = (1,0,0,1)$.

(b) *A minimal hyperbolic paraboloid*

$$f(s,t) = st e_1 + t e_2 + s e_3,$$

where $e_1 = (1,0,0,1), e_2 = (0,0,1,0)$ and $e_3 = (0,1,0,0)$.

These lead to the consequence of the case (2).

In case of (3), we explain the meaning of $SO(2)$-invariant firstly. It is well-known that

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, \theta \in \mathbb{R} \right\}.$$

We realize $SO(2)$ as a subgroup of $\mathrm{Aut}(\mathbb{R}^{0,2,1}, d)$ as below.

$$H := \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Aut}(\mathbb{R}^{0,2,1}, d) \,\middle|\, \theta \in \mathbb{R} \right\}.$$

Then, the group H is isomorphic to $SO(2)$ as a Lie group. We simply denote H as $SO(2)$. A surface is said to be $SO(2)$-invariant if it is invariant under the action of this group. Such surfaces are locally parametrized by

$$f(u,v) = (x(u)\cos v, x(u)\sin v, y(u)) \in \mathbb{R}^{0,2,1},$$

where x, y are real variable functions satisfying $x > 0,\ (x')^2 + (y')^2 = 1$. Then, we have

$$f_u = (x'\cos v, x'\sin v, y'), \quad f_v = (-x\sin v, x\cos v, 0).$$

Thus, we compute

$$g_{11} = (x')^2, \quad g_{12} = 0, \quad g_{22} = x^2.$$

The non-degeneracy implies $x' \neq 0$. Moreover, since we compute

$$\begin{aligned} f_{uu} &= (x''\cos v, x''\sin v, y'') = \frac{x''}{x'} f_u + \left(-\frac{x''}{x'} y' + y'' \right) \xi, \\ f_{uv} &= (-x'\sin v, x'\cos v, 0) = \frac{x'}{x} f_v, \\ f_{vv} &= (-x\cos v, -x\sin v, 0) = -\frac{x}{x'} f_u + \frac{x}{x'} y' \xi, \end{aligned}$$

the coefficients of second fundamental form h hold

$$h_{11} = -\frac{x''}{x'}y' + y'', \quad h_{12} = 0, \quad h_{22} = \frac{x}{x'}y'.$$

Therefore, we compute that the mean curvature of $SO(2)$-invariant d-minimal surfaces is

$$2\mathcal{H} = g^{ij}h_{ij} = \frac{1}{(x')^3}(-x''y' + x'y'') + \frac{y'}{xx'} \equiv 0. \tag{4.2}$$

Since $x' \neq 0$, by the coordinate transformation, we can represent y as a function with respect to x. Then, the equation (4.2) is equal to the following equation

$$\frac{d^2y}{dx^2} = -\frac{1}{x}\frac{dy}{dx}.$$

By solving the ordinary differential equation, we have

$$y(x) = C_1 \log x + C_2 \quad (C_1, C_2 \in \mathbb{R} : \text{constants}).$$

Again, when we replace the parameter x with $x(w) = e^w$, we get $y(w) = C_1 w + C_2$. In particular, if $C_1 = 0$, then it is a plane. So, if it is not a plane, by an affine isometry, we obtain

$$f(u, v) = (e^u \cos v, e^u \sin v, u).$$

The proof is completed. □

In Euclidean space, the only non-planar ruled minimal surfaces are the helicoids, which are invariant surfaces. Proposition 4.5 (2) shows that we still have invariant surfaces as the only non-planar ruled minimal surfaces in $\mathbb{R}^{0,2,1}$. Regarding the surfaces (b) in Proposition 4.5 (2), it is also known as a warped translation surface with a generating curve. Regarding Proposition 4.5 (3), the obtained revolution surfaces are special instances of invariant surfaces. Da Silva classified all invariant minimal simply isotropic surfaces, that is, invariant d-minimal surfaces. See [22] in detail.

Remark 3. We recall that non-degenerate surfaces are locally expressed by graph surfaces. However, (a) of Proposition 4.5 is an example which can not be entirely expressed as a graph surface.

We consider the canonical connection d as a linear connection for $\mathbb{R}^{0,2,1}$. This connection d is a torsion-free connection which is parallel with respect to the degenerate metric $(\cdot, \cdot)$, i.e. d plays the role of Levi-Civita connection. However, since the metric is degenerate, connections having such properties are not unique. Vogel characterized linear connections compatible with a degenerate metric in [25]. For example, let $\lambda \in \mathbb{R}$ be a real parameter, and we define a tensor field $L_\lambda \in \Gamma(S^2T^*\mathbb{R}^3)$ as

$$L_\lambda(X, Y) := \lambda \sum_{i,j} X_i Y_j,$$

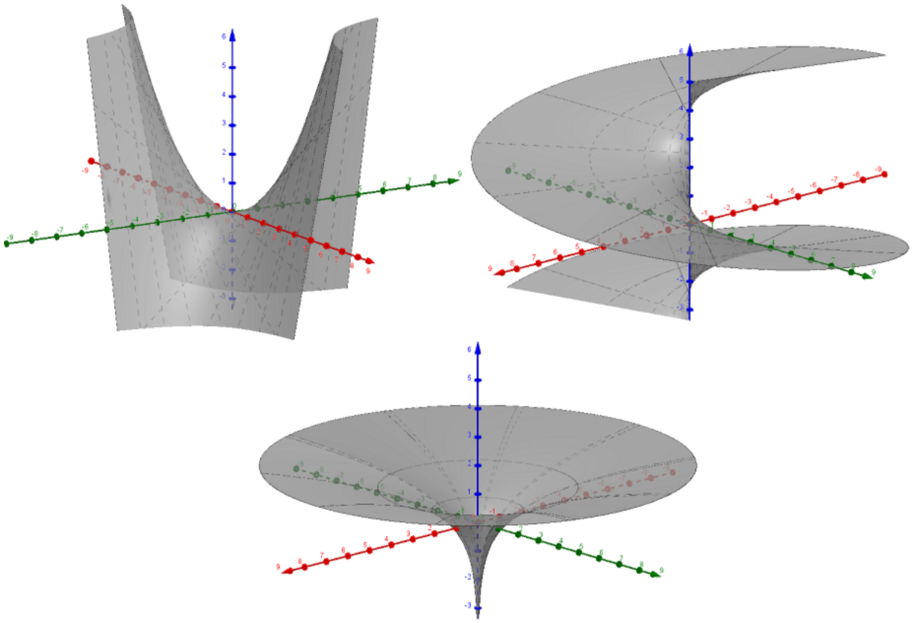

Figure 1: Upper-left: the minimal hyperbolic paraboloid. Upper-right: the elliptic helicoid of the second kind. Lower-middle: the d-minimal rotational surface.

where the set $\Gamma(S^2T^*\mathbb{R}^3)$ expresses the whole of $(0,2)$-type symmetric tensor fields over $\mathbb{R}^3$ and X, Y are vector fields over $\mathbb{R}^3$, and we regard X and Y respectively as vector-valued functions

$$X = (X_1, X_2, X_3), Y = (Y_1, Y_2, Y_3).$$

Then, when we put $d^\lambda := d + L_\lambda \xi$, d^λ is a flat connection over $\mathbb{R}^{0,2,1}$ which has the same properties of Levi-Civita connections. In particular, when $\lambda = 0$, d^λ coincides with the canonical connection d. $(\mathbb{R}^{0,2,1}, d)$ is geodesically complete. However, $(\mathbb{R}^{0,2,1}, d^\lambda)$ is not so if $\lambda \neq 0$. Actually, in $(\mathbb{R}^{0,2,1}, d^\lambda)$ we calculate the geodesic $\gamma(t)$ with the initial data $\gamma(0) = (0,0,0), \gamma'(0) = (1,0,0)$ as

$$\gamma(t) = \left(t, 0, \frac{1}{\lambda}\log|\lambda t + 1| - t\right).$$

Namely, the parameter t is not defined in the whole of real numbers. For $\mathbb{R}^{0,2,1}$, it would be interesting to consider the geometric meaning of torsion-free, metric connections. In [21], the issue is taken into account by Da Silva as well. For example, we can raise the problem of whether a complete connection is d only.

4.2 Representation formula of Weierstrass type for d-minimal surfaces

Let $f : M \to \mathbb{R}^{0,2,1}$ be a non-degenerate immersion. When we set $f = (f_1, f_2, f_3)$, we define Laplacian $\Delta_g f$ of f with respect to the induced metric g as Laplacians of each coordinate functions

f_i $(i = 1, 2, 3)$, i.e.

$$\Delta_g f := (\Delta_g f_1, \Delta_g f_2, \Delta_g f_3).$$

Proposition 4.6. *Let $\mathcal{H}$ be the mean curvature of a non-degenerate immersion f. Then, $2\mathcal{H}\xi \in \Gamma(T^{\perp}M)$ is equal to Laplacian $\Delta_g f$ of f with respect to the induced metric g. In particular, the non-degenerate surface is a d-minimal if and only if coordinate functions of f are all harmonic with respect to g.*

Proof. Since f is non-degenerate, there exists a coordinate neighborhood U of M such that the local expression of f is

$$f(u, v) = (u, v, F(u, v)) \in \mathbb{R}^{0,2,1},$$

where F is a function on U. By using this coordinate, we get

$$2\mathcal{H}\xi = (0, 0, F_{uu} + F_{vv}) = \Delta_g f.$$

The proof is completed. □

In case of graph surfaces, Proposition 4.6 is equivalent to the formula (8) in [16].

Next, we prepare some simple lemmas.

Lemma 4.1. *For a real two variable function $f(u, v)$, we define a complex function $F(w)$ with respect to the complex variable $w = u + iv$ as*

$$F(w) := \frac{\partial f}{\partial u}(u, v) - i\frac{\partial f}{\partial v}(u, v).$$

Then, F is a holomorphic function if and only if $f(u, v)$ is a harmonic function.

Proof. The Cauchy-Riemann's equations imply. □

Lemma 4.2. *In $\mathbb{R}^{0,2,1}$, we consider a surface given by*

$$f(u, v) = (x(u, v), y(u, v), z(u, v)) \in \mathbb{R}^{0,2,1}.$$

We define complex functions φ, ψ with respect to the complex variable $w = u + iv$ as

$$\varphi(w) := \frac{\partial x}{\partial u}(u, v) - i\frac{\partial x}{\partial v}(u, v), \quad \psi(w) := \frac{\partial y}{\partial u}(u, v) - i\frac{\partial y}{\partial v}(u, v).$$

Then, the coordinates (u, v) are isothermal if and only if it holds

$$\varphi^2 + \psi^2 \equiv 0.$$

Proof. By direct calculations, we have

$$\varphi^2 + \psi^2 = |f_u|^2 - |f_v|^2 - 2i(f_u, f_v).$$

This completes the proof. □

Theorem 4.1. *Let U be an open subset of uv-plane. In $\mathbb{R}^{0,2,1}$, let f be an immersion on U which is parametrized by $f(u,v) = (x(u,v), y(u,v), z(u,v))$. We assume that (u,v) are isothermal coordinates and f is d-minimal. Then, complex functions $\varphi_1, \varphi_2, \varphi_3$ with respect to the complex variable $w = u + iv$ defined by*

$$\varphi_1(w) = \frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v}, \quad \varphi_2(w) = \frac{\partial y}{\partial u} - i\frac{\partial y}{\partial v}, \quad \varphi_3(w) = \frac{\partial z}{\partial u} - i\frac{\partial z}{\partial v} \tag{4.3}$$

are all holomorphic, and it holds

$$|\varphi_1|^2 + |\varphi_2|^2 > 0, \quad \varphi_1^2 + \varphi_2^2 = 0. \tag{4.4}$$

Moreover, it holds

$$(f_u, f_u) = (f_v, f_v) = \frac{1}{2}(|\varphi_1|^2 + |\varphi_2|^2).$$

Conversely, let U be a simply-connected domain on $\mathbb{C}$, and we assume that holomorphic functions $\varphi_1(w), \varphi_2(w), \varphi_3(w)$ satisfy the formula (4.4). Then, when we set $w = u + iv \in U$, there exists a d-minimal surface satisfying the formula (4.3) such that, for the parametrized expression $f(u,v) = (x(u,v), y(u,v), z(u,v))$, the coordinates (u,v) are isothermal.

Proof. Since f is d-minimal, each coordinate function is harmonic from Proposition 4.6. Thus, by using Lemma 4.1, each φ_i is holomorphic. And, since (u,v) are isothermal coordinates, it holds $\varphi_1^2 + \varphi_2^2 \equiv 0$ from Lemma 4.2. Next, since we compute

$$|\varphi_1|^2 + |\varphi_2|^2 = x_u^2 + y_u^2 + x_v^2 + y_v^2 = |f_u|^2 + |f_v|^2 = 2|f_u|^2 = 2|f_v|^2 > 0,$$

the former of the claim holds. For the latter, we assume that holomorphic functions $\varphi_1, \varphi_2, \varphi_3$ on a simply-connected domain U satisfy the formula (4.4). We fix a point $w_0 \in U$ and define a real function $x = x(u,v)$ as

$$x(u,v) := \mathrm{Re}\int_{w_0}^{w} \varphi_1(w)dw \quad (w = u + iv \in U).$$

This is well-defined since U is simply-connected. When we act on this formula by the differential operator

$$\frac{\partial}{\partial u} - i\frac{\partial}{\partial v} = 2\frac{\partial}{\partial w},$$

we have

$$\frac{\partial x}{\partial u} - i\frac{\partial x}{\partial v} = 2\frac{\partial}{\partial w}\mathrm{Re}\int_{w_0}^{w} \varphi_1(w)dw = \varphi_1(w).$$

As above, when we define $y = y(u, v)$ and $z = z(u, v)$, we have

$$\frac{\partial y}{\partial u} - i\frac{\partial y}{\partial v} = \varphi_2(w), \quad \frac{\partial z}{\partial u} - i\frac{\partial z}{\partial v} = \varphi_3(w).$$

From Lemma 4.1 again, we see that $x(u, v), y(u, v), z(u, v)$ are harmonic functions on U. Next, we prove that the mapping $f = f(u, v)$ gives a surface, i.e. two-dimensional manifold. For the purpose of that, we prove that the Jacobi matrix

$$\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

has rank two for any point $w \in U$. We prove by using contradiction, i.e. we assume that there is a point $w' \in U$ such that the rank of its Jacobi matrix is less than two. Since we have

$$0 < |\varphi_1|^2 + |\varphi_2|^2 = (x_u)^2 + (x_v)^2 + (y_u)^2 + (y_v)^2,$$

at the point w', we see that either of column vectors

$$\begin{pmatrix} x_u \\ x_v \end{pmatrix}, \begin{pmatrix} y_u \\ y_v \end{pmatrix}$$

is not the zero vector. So, we suppose that the former is not the zero vector. From the assumption of contradiction, since we may set

$$\exists \lambda \in \mathbb{R} \text{ s.t. } \begin{pmatrix} y_u \\ y_v \end{pmatrix} = \lambda \begin{pmatrix} x_u \\ x_v \end{pmatrix},$$

by using $\varphi_2 = \lambda\varphi_1$, we compute

$$\{\varphi_1(w')\}^2 + \{\varphi_2(w')\}^2 = (1 + \lambda^2)\{\varphi_1(w')\}^2 \neq 0$$

at w'. This contradicts the formula (4.4). Thus, since f is a C^∞-immersion,

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

gives a surface in $\mathbb{R}^{0,2,1}$, and $(u, v) \in U$ are isothermal coordinates from the condition (4.4). In particular, f is a d-minimal surface satisfying the formula (4.3). □

In Theorem 4.1, the function φ_3 seems not to play any role in the characterization of isothermal coordinates because of Eq. (4.4). Does this *apparent independence* bring any kind of symmetry or freedom to construct d-minimal surfaces? It would be interesting to research the geometrical interpretation of such a symmetry, or freedom.

Theorem 4.2 (Weierstrass-type representation formula for d-minimal surfaces)**.** *Let $U \subset \mathbb{C}$ be a simply-connected domain and F, G be a holomorphic and meromorphic function on U respectively such that F does not have zero points on U and FG be a holomorphic function on U. Then, a mapping*

$$f(u,v) = Re \int_w (F, iF, 2FG)dw \quad (w := u + iv \in U)$$

gives a d-minimal surface in $\mathbb{R}^{0,2,1}$, and the coordinates $(u,v) \in U$ are isothermal. Moreover, it holds

$$(f_u, f_u) = (f_v, f_v) = |F|^2.$$

Conversely, a d-minimal surface in $\mathbb{R}^{0,2,1}$ locally has the expression as above.

Proof. For the former of the claims, when we set $\varphi_1 := F, \varphi_2 := iF, \varphi_3 := 2FG$, it immediately holds from Theorem 4.1. For the latter of the claims, given a d-minimal surface, it is locally considered on a simply-connected domain. From Theorem 4.1 again, we have the parametrized expression

$$f(u,v) = \mathrm{Re} \int (\varphi_1, \varphi_2, \varphi_3)dw.$$

Since it satisfies

$$|\varphi_1|^2 + |\varphi_2|^2 > 0, \ \varphi_1^2 + \varphi_2^2 = 0,$$

setting $F := \varphi_1, G := \dfrac{\varphi_3}{2F}$, we obtain the expression which we want. □

Zero points of F correspond to singularities of d-minimal surfaces. For example, we see cross-caps on d-minimal surfaces. We remark that there exist other types of singularities not only cross-caps. Here, we recall the definition of singularities. Let M, N be manifolds, and f be an immersion from M into N. A point $x \in M$ is a *singularity* of f if the differential map df_x is not injective. This means that there is no tangent plane on the singularity. For a d-minimal surface $f : M \to \mathbb{R}^{0,2,1}$ and a point $x \in M$, we see that F has a zero point at x if and only if f has a singularity at x by easy calculation. We describe other types in the next section.

At the end of this section, for Weierstrass type expression formula for d-minimal surfaces

$$f(u,v) = \mathrm{Re} \int_w (F, iF, 2FG)dw \quad (w := u + iv \in U),$$

the function F expresses the induced metric g, i.e. it holds

$$g = |F|^2(du^2 + dv^2).$$

On the other hand, the function G is concerned with the second fundamental form h by the following proposition.

Proposition 4.7. *Under the situation stated above, it holds*

$$h = -\frac{1}{|F|^2}\left\{ReFG\frac{\partial}{\partial u}|F|^2 + ImFG\frac{\partial}{\partial v}|F|^2 + 2\frac{\partial}{\partial u}(ReFG)\right\}(du^2 - dv^2)$$
$$-\frac{1}{|F|^2}\left\{ImFG\frac{\partial}{\partial u}|F|^2 + ReFG\frac{\partial}{\partial v}|F|^2 + 2\frac{\partial}{\partial v}(ReFG)\right\}(2dudv).$$

Proof. By direct calculations, we obtain

$$\begin{pmatrix} f_{uu} \\ f_{uv} \\ f_{vv} \end{pmatrix} = \begin{pmatrix} \Gamma^1_{11} & \Gamma^2_{11} & h_{11} \\ \Gamma^1_{12} & \Gamma^2_{12} & h_{12} \\ \Gamma^1_{22} & \Gamma^2_{22} & h_{22} \end{pmatrix} \begin{pmatrix} f_u \\ f_v \\ \xi \end{pmatrix},$$

where Γ^i_{jk} are Christoffel symbols with respect to ∇. □

Remark 4. The pair (F, G) is called a *Weierstrass data.* And, for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$,

$$f_\theta(s,t) = \cos\theta\left(\mathrm{Re}\int(F, iF, 2FG)dw\right) + \sin\theta\left(\mathrm{Im}\int(F, iF, 2FG)dw\right)$$

is a d-minimal surface in $\mathbb{R}^{0,2,1}$ and this gives an isometric deformation.

In fact, it follows

$$\mathrm{Re}\int_{w_0}^{w}(-iF, F, -2iFG)dw = \mathrm{Im}\int_{w_0}^{w}(F, iF, 2FG)dw.$$

Thus, d-minimal surfaces defined by the Weierstrass data $(-iF, -iG)$ corresponds to the imaginary part of the formulas defined by the Weierstrass data (F, G). For $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, when we consider the d-minimal surface whose Weierstrass data is $(e^{-i\theta}F, e^{-i\theta}G)$, the given immersion is called an *associated family* and, when we denote f_θ, we have the S^1-family of mappings. Moreover, we see

$$\begin{aligned} f_\theta(u,v) &= \mathrm{Re}\int_{w_0}^{w}(e^{-i\theta}F, ie^{-i\theta}F, 2e^{-i\theta}FG)dw \\ &= \cos\theta\left(\mathrm{Re}\int_{w_0}^{w}(F, iF, 2FG)dw\right) + \sin\theta\left(\mathrm{Im}\int_{w_0}^{w}(F, iF, 2FG)dw\right). \end{aligned}$$

In particular, when $\theta = 0, \frac{\pi}{2}$, they correspond to the d-minimal surfaces given by the real part and imaginary part from (F, G) respectively. Moreover, for any $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, since the induced metric of f_θ satisfies

$$((f_\theta)_u, (f_\theta)_u) = ((f_\theta)_v, (f_\theta)_v) = |e^{-i\theta}F|^2 = |F|^2,\ ((f_\theta)_u, (f_\theta)_v) = 0,$$

it gives an isometric deformation between $f = f_0$ and f_θ. We call $f_{\frac{\pi}{2}}$ a *conjugate surface* of f_0.

Example 1.

(0) When $(F, G) = (\alpha, \beta)$ $(\alpha, \beta \in \mathbb{C}, \alpha \neq 0)$, a non-degenerate plane appears.

(1) When $(F, G) = (w, \frac{1}{w})$, we have

$$f_0(u, v) = \left(\frac{1}{2}(u^2 - v^2), -uv, 2u\right), \quad f_{\frac{\pi}{2}}(u, v) = \left(uv, \frac{1}{2}(u^2 - v^2), 2v\right).$$

These are surfaces which have self-intersections and both have singularities called as cross-caps at $(u, v) = (0, 0)$ (refer to (a) of Figure 2).

(2) When $(F, G) = (e^w, e^{-w})$, we have

$$f_0(u, v) = (e^u \cos v, -e^u \sin v, 2u), \quad f_{\frac{\pi}{2}}(u, v) = (e^u \sin v, e^u \cos v, 2v).$$

f_0 is the d-minimal rotational surface given by Proposition 4.5 (3), and $f_{\frac{\pi}{2}}$ is the elliptic helicoid of the second kind (refer to Figure 1).

(3) When $(F, G) = (1, w)$, we have

$$f_0(u, v) = \left(u, -v, u^2 - v^2\right), \quad f_{\frac{\pi}{2}}(u, v) = (u, v, 2uv).$$

These both are minimal hyperbolic paraboloids (refer to Figure 1).

Remark 5. The above Weierstrass-type representation formula contains the ones known in [1] or [12]. However, the formulas stated in [1] or [12] do not give singularities on surfaces. In this sense, Theorem 4.2 is more complete. On the other hand, we can see isotropic minimal surfaces which have isolated singularities in [16].

Here, we recall some Weierstrass(-type) representation formulas [2].

- Case of $\mathbb{R}^3 = (\mathbb{R}^3, dx^2 + dy^2 + dz^2)$, i.e. minimal surfaces.

$$f_{\mathbb{R}^3} = \mathrm{Re} \int_w (F(1 - G^2), iF(1 + G^2), 2FG)dw.$$

- Case of $\mathbb{R}^3_1 = (\mathbb{R}^3, dx^2 + dy^2 - dz^2)$, i.e. maximal surfaces.

$$f_{\mathbb{R}^3_1} = \mathrm{Re} \int_w (F(1 + G^2), iF(1 - G^2), 2FG)dw.$$

Thus, among minimal surfaces $f_{\mathbb{R}^3}$, maximal surfaces $f_{\mathbb{R}^3_1}$ and d-minimal surfaces $f_{\mathbb{R}^{0,2,1}}$, we obtain the relation

$$f_{\mathbb{R}^{0,2,1}} = \frac{1}{2}\left(f_{\mathbb{R}^3} + f_{\mathbb{R}^3_1}\right),$$

where $f_{\mathbb{R}^{0,2,1}}$ is the mapping given by Theorem 4.2.

5 Applications

Theorem 5.1. *Let (M, g) be a connected, two-dimensional complete Riemannian manifold, and let $f : (M, g) \to \mathbb{R}^{0,2,1}$ be an isometric immersion. Then, (M, g) is isometric to the canonical two-dimensional Euclidean space $\mathbb{R}^2$, and the image of f corresponds to an entire graph*

$$\{(u, v, F(u, v)) \in \mathbb{R}^{0,2,1} \mid (u, v) \in \mathbb{R}^2\},$$

where F is a C^∞-function on $\mathbb{R}^2$.

Proof. We define C^∞-functions α, β, γ on M as

$$f(x) = (\alpha(x), \beta(x), \gamma(x)) \quad (x \in M).$$

We assume that $\mathbb{R}^2$ is the canonical Euclidean space which treats (u, v) as the coordinates, and define a C^∞-map $f_0 : (M, g) \to \mathbb{R}^2$ as

$$f_0(x) := (\alpha(x), \beta(x)) \quad (x \in M).$$

f_0 is an isometric immersion. We prove that f_0 is an isometric diffeomorphism. We remark that $\dim M = \dim \mathbb{R}^2 = 2$ and, from the inverse function theorem, f_0 is a local diffeomorphism. Thus, in order to prove that f_0 is an isometric diffeomorphism, it is sufficient to prove that f_0 is bijective.

For the surjectivity, since f_0 is a locally homeomorphism, f_0 is an open mapping. Thus, $\mathrm{Im} f_0$ is an open subset of $\mathbb{R}^2$. Next, since isometric mappings preserve the geodesic completeness, from Hopf-Rinow's theorem, $(\mathrm{Im} f_0, du^2 + dv^2) \subset \mathbb{R}^2$ is complete, where we consider $\mathrm{Im} f_0$ as the metric subspace of $\mathbb{R}^2$ naturally. Thus, $\mathrm{Im} f_0$ is a closed subset of $\mathbb{R}^2$. Therefore, since $\mathrm{Im} f_0$ is an open and closed subset of $\mathbb{R}^2$, it holds $\mathrm{Im} f_0 = \mathbb{R}^2$, i.e. $f_0 : M \to \mathbb{R}^2$ is surjective.

For the injectivity, we denote the Riemannian distance with respect to the metric g by d_M. For arbitrary points $x, y \in M$ which are distinct, since (M, g) is complete, there exists a shortest geodesic $\delta : [0, 1] \to M$ such that $\delta(0) = x, \delta(1) = y$. Moreover, since f_0 is isometric, $f_0 \circ \delta : [0, 1] \to \mathbb{R}^2$ is a geodesic in $\mathbb{R}^2$ which connects $f_0(x)$ and $f_0(y)$. For a curve c, when we denote the length of c by $L(c)$, we see

$$0 < d_M(x, y) = L(\delta) = L(f_0 \circ \delta) = |f_0(x) - f_0(y)|_{\mathbb{R}^2}.$$

This implies $f_0(x) \neq f_0(y)$, i.e. $f_0 : M \to \mathbb{R}^2$ is injective. As a remark, we use the fact that geodesics in $\mathbb{R}^2$ are straight lines for the last equation above.

In summary, since we obtain that $f_0 : M \to \mathbb{R}^2$ is a locally isometric diffeomorphism and bijection, it is an isometric diffeomorphism, that is, (M, g) is isometric to the canonical two-dimensional Euclidean space $\mathbb{R}^2$. We denote the inverse of f_0 by $\phi : \mathbb{R}^2 \to M$. For any $(u, v) \in \mathbb{R}^2$, we have

$$\begin{aligned} f(\phi(u,v)) &= (\alpha(\phi(u,v)), \beta(\phi(u,v)), \gamma(\phi(u,v))) \\ &= ((f_0 \circ \phi)(u,v), (\gamma \circ \phi)(u,v)) = (u, v, F(u,v)), \end{aligned}$$

where $F := \gamma \circ \phi$ is a C^∞-function on $\mathbb{R}^2$. Therefore, the image of f is the entire graph expressed by a function F on $\mathbb{R}^2$. □

Corollary 5.2. *Let $f : M^2 \to \mathbb{R}^{0,2,1}$ be a connected, complete d-minimal surface. Then, the image of f corresponds to the entire graph*

$$\{(u, v, \psi(u,v)) \in \mathbb{R}^{0,2,1} \mid (u,v) \in \mathbb{R}^2\},$$

where ψ is a harmonic function on $\mathbb{R}^2$.

Proof. From Proposition 4.6 (2), it follows immediately. □

Corollary 5.3. *Let M be a connected, compact two-dimensional manifold, i.e. a connected closed surface. Then, there exists no non-degenerate immersion $f : M \to \mathbb{R}^{0,2,1}$.*

Proof. We prove the corollary by contradiction. We assume that there exists a non-degenerate immersion $f : M \to \mathbb{R}^{0,2,1}$. When we denote the induced metric by g, (M, g) is a connected, compact Riemannian manifold. In particular, it is complete. From Theorem 5.1, as we have a homeomorphism $M \cong \mathbb{R}^2$, this contradicts the compactness of M. □

Let $f : M \to \mathbb{R}^{0,2,1}$ be a non-degenerate immersion, and let h be its second fundamental form. Then, we recall that the Gauss-Codazzi equation of the non-degenerate immersion is given by the formula (3.2), i.e.

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) \quad (X, Y, Z \in \Gamma(TM)).$$

By using the flat local coordinates (u, v), the formula (3.2) is equivalent to

$$(h_{11})_v = (h_{12})_u, \quad (h_{22})_u = (h_{12})_v, \tag{5.1}$$

where h_{ij} are coefficients of h.

Theorem 5.4 (The fundamental theorem of non-degenerate surfaces, [17], Theorem 8.8). *Let $U \subset \mathbb{R}^2$ be a simply-connected domain, (u, v) be coordinates on U, and h_{11}, h_{12} and h_{22} be C^∞-functions on U. Then, there exists, up to affine isometry, a non-degenerate immersion whose induced metric and second fundamental form are*

$$du^2 + dv^2 \quad \text{and} \quad h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2$$

respectively, if and only if the functions h_{ij} satisfy the Gauss-Codazzi equation (5.1) of the non-degenerate surface.

From now on, we consider four-dimensional Minkowski space $\mathbb{R}^4_1$ equipped with the Lorentzian metric

$$\langle \cdot, \cdot \rangle_1 := -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is the canonical coordinates of $\mathbb{R}^4$. We deal with spacelike surfaces only, i.e. we require that the induced metric of surfaces is positive definite.

A surface M is called *zero mean curvature* if it holds $\vec{H} \equiv 0$, where $\vec{H}$ is the mean curvature vector field of M, and a surface M is called *flat* if it holds $K \equiv 0$, where K is the Gaussian curvature of M. We abbreviate zero mean curvature to ZMC.

Remark 6. We give one of the motivations of studying flat and zero mean curvature surfaces. We firstly remark that flat minimal submanifolds in n-dimensional Euclidean space $\mathbb{R}^n$ and spacelike flat ZMC surfaces in three dimensonal Minkowski space $\mathbb{R}^3_1$ are totally geodesic. On the other hand, there exist timelike flat ZMC surfaces in $\mathbb{R}^3_1$ [19]. Thus, we are interested in the question of whether spacelike flat ZMC surfaces should be trivial, that is, totally geodesic surfaces.

Next, spacelike flat ZMC surfaces in four-dimensional Minkowski space $\mathbb{R}^4_1$ are not always planes. In particular, we also remark that spacelike flat ZMC surfaces in four-dimensional semi-Euclidean space $\mathbb{R}^4_2$ equipped with the neutral metric are totally geodesic again [11].

Theorem 5.5. *Let $f : M^2 \to \mathbb{R}^4_1$ be an immersion which gives a non-totally geodesic, connected spacelike flat ZMC surface, and let h be the second fundamental form of M. We define a subset E of M as*

$$E := \{x \in M \mid h_x = 0\}.$$

Then, it holds the following assertions:

(1) $M \setminus E$ is an open dense subset of M, and it is connected.

(2) The normal bundle of M is flat, i.e. the normal curvature $R^\perp \equiv 0$.

(3) M is, by an isometry of $\mathbb{R}^4_1$, immersed in $\mathbb{R}^{0,2,1} \subset \mathbb{R}^4_1$, and it is a d-minimal surface.

Proof. For the claim (1), it is easily proved that E is a closed subset of M. Let U be the flat coordinate neighborhood of M. We define a $\mathbb{C}^4$-valued mapping $\varphi = \varphi(w)$ for a complex variable $w = u + iv$ $((u,v) \in U)$ as

$$\varphi(w) := f_{uu}(u,v) - i f_{uv}(u,v). \tag{5.2}$$

Then, by using that f is smooth and harmonic, we compute

$$\frac{\partial \varphi}{\partial \bar{w}} = \frac{1}{2}(f_{uuu} + f_{uvv}) + \frac{i}{2}(f_{uuv} - f_{uvu}) = 0,$$

and φ is a holomorphic mapping on U. Since M is not totally geodesic, we obtain the interior of E is empty. Moreover, we see that zero points of φ correspond to elements of E. Therefore, since the set of zero points for a holomorphic function is discrete, the set E is a discrete subset of M which is made of isolated points. Since M is connected and E is discrete, it is proved for $M \setminus E$ to be connected.

For (2), see Corollary 1.2 in [1]. The claim (3) is proved by using Proposition 4.6 and Proposition 3.5 in [1] □

Remark 7. The set E is a discrete subset of M consisted of isolated points. As an example which satisfies $E \neq \emptyset$, when we define a C^∞-immersion $f : \mathbb{R}^2 \to \mathbb{R}^{0,2,1} \subset \mathbb{R}^4_1$ as

$$f(u,v) := (u^3 - 3uv^2, u, v, u^3 - 3uv^2),$$

it is a spacelike flat ZMC surface which satisfies $h = 0$ at the origin $(0,0)$ only.

Let $f : M \to \mathbb{R}^{0,2,1}$ be a d-minimal surface. Then, by the isometric embedding ι given in Eq. (4.1), we see that M is a spacelike flat ZMC surface in $\mathbb{R}^4_1$. M is a spacelike flat surface since ι is an isometric embedding. To show that M is ZMC, we directly calculate the mean curvature vector field of M. By using a harmonic function φ, since we can locally express f by

$$f(u,v) = (u, v, \varphi(u,v)),$$

from the composition of ι, we have

$$(\iota \circ f)(u,v) = (\varphi(u,v), u, v, \varphi(u,v)).$$

Thus, we compute that the mean curvature vector field $\vec{H}$ is

$$2\vec{H} = (\iota \circ f)_{uu} + (\iota \circ f)_{vv} = (\varphi_{uu} + \varphi_{vv})(1,0,0,1) \equiv 0.$$

Therefore, we obtain the following corollary.

Corollary 5.6. *Let X be the set of the classes of congruent spacelike flat ZMC surfaces in $\mathbb{R}^4_1$, and let Y be the set of equivalence classes of d-minimal surfaces in $\mathbb{R}^{0,2,1}$ by a subgroup*

$$K := \left\{ \left(\begin{array}{ccc} \multicolumn{2}{c}{\multirow{2}{*}{T}} & 0 \\ & & 0 \\ 0 & 0 & c \end{array} \right) \middle| \; c \neq 0, \; T \in O(2) \right\} \ltimes \mathbb{R}^3 \subset Aut(\mathbb{R}^{0,2,1}, d).$$

Then, except for planes, we have that X and Y are in one-to-one correspondence.

Proof. It is obvious as long as we remark that this subgroup K corresponds to the subgroup of isometries of $\mathbb{R}^4_1$ which preserves the degenerate subspace $\mathbb{R}^{0,2,1} \subset \mathbb{R}^4_1$. □

Notice that all 1-parameter subgroups of simply isotropic isometries have been already described by [17] and [22].

Regarding minimal surfaces in $\mathbb{R}^3$, maximal surfaces in $\mathbb{R}^3_1$ and d-minimal surfaces in $\mathbb{R}^{0,2,1}$, we have

$$\left\{ \begin{array}{c} \text{minimal, maximal,} \\ \text{and } d\text{-minimal surfaces} \end{array} \right\} \subset \{\text{spacelike ZMC surfaces in } \mathbb{R}^4_1\}.$$

In fact, for the spaces $\mathbb{R}^3$ and $\mathbb{R}^3_1$, there exist isometric embeddings defined by

$$\mathbb{R}^3 \ni (x, y, z) \mapsto (0, x, y, z) \in \mathbb{R}^4_1, \tag{5.3}$$

$$\mathbb{R}^3_1 \ni (x, y, z) \mapsto (x, y, z, 0) \in \mathbb{R}^4_1 \tag{5.4}$$

respectively. Since minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{R}^3_1$ are ZMC surfaces in $\mathbb{R}^4_1$ via the above embeddings, we see that there quite fruitfully exist ZMC surfaces in $\mathbb{R}^4_1$. On the other hand, there exist spacelike ZMC surfaces in $\mathbb{R}^4_1$ which are neither minimal, maximal nor d-minimal. For example, see Section 4 in [1].

In general, singularity points appear in d-minimal surfaces. Refer to the figures from (a) to (d) in Figure 2 as such examples. From the Whitney's criterion, a cross-cap appears in (a), and from the Saji's criterion [18], a D_4^--type singularity appears in (c). Other singularities have been not identified and classified. In summary, we give Table 1 which compares properties among each surfaces. We assume the connectedness of surfaces;

At the end of this paper, we point out that there may exist the relation among minimal surfaces in $\mathbb{R}^3$, maximal surfaces in $\mathbb{R}^3_1$ and d-minimal surfaces in $\mathbb{R}^{0,2,1}$.

Theorem 5.7 (F. J. M. Estudillo and A. Romero (1991), [8])**.** *Let $n \geq 3$, and (M, g) be a two-dimensional oriented Riemannian manifold, and $f : (M, g) \to \mathbb{R}^n_1 = (\mathbb{R}^n, \; dx_1^2 + \cdots + dx_{n-1}^2 - dx_n^2)$ be a ZMC isometric immersion. Then, f is locally expressed by the following:*

$$f = Re \int_w (\phi_1, \cdots, \phi_{n-1}, \phi_n) dw,$$

where $\phi_1, \cdots, \phi_n$ *are holomorphic functions which satisfy*

$$\phi_1^2 + \cdots + \phi_{n-1}^2 - \phi_n^2 = 0, \quad |\phi_1|^2 + \cdots + |\phi_{n-1}|^2 - |\phi_n|^2 > 0.$$

From Theorem 5.7, we can define an immersion with S^1-parameter as

$$\tilde{f}_\theta(u, v) = \text{Re} \int_w (F(1 - \cos 2\theta G^2), iF(1 + \cos 2\theta G^2), 2\cos\theta FG, 2\sin\theta FG) dw.$$

$\tilde{f}_\theta$ implies a spacelike ZMC surface in four-dimensional Minkowski space for arbitrary $\theta \in S^1$. In particular, through embeddings of formulas (4.1), (5.3) and (5.4), $\tilde{f}_0$, $\tilde{f}_{\frac{\pi}{2}}$ and $\tilde{f}_{\frac{\pi}{4}}$ coincide with the Weierstrass representation formulas of minimal, maximal and d-minimal surfaces in $\mathbb{R}^3$, $\mathbb{R}^3_1$ and $\mathbb{R}^{0,2,1}$ respectively. See also Remark 5. As a remark, we compute the induced metric g_θ of $\tilde{f}_\theta$ as

$$g_\theta = (1 + \cos 2\theta |G|^2)^2 |F|^2 (du^2 + dv^2).$$

Thus, we should note that this deformation of surfaces is not isometric. However, there may be applications in the study of singularities of d-minimal surfaces.

	min.	max.	d-min.
Compact	$\nexists$	$\nexists$	$\nexists$ (Cor. 5.3)
Entire graph	Planes only	Planes only	$\exists$ (Prop. 4.5)
Singularity	$\nexists$	$\exists$ ([10])	$\exists$
Complete	$\exists$	Planes only	$\exists$ (Thm. 5.1)
Gaussian curvature	≤ 0	≥ 0	$\equiv 0$

Table 1: In terms of singularity, the symbol $\exists$ expresses that singularities appear, and in terms of otherwise, $\exists$ expresses that there exist such surfaces which are not planes. In addition, the abbreviations min., max. and d-min. are minimal surfaces in $\mathbb{R}^3$, maximal surfaces in $\mathbb{R}^3_1$ and d-minimal surfaces in $\mathbb{R}^{0,2,1}$ respectively.

Acknowledgment

The author would like to express his deepest gratitude to his advisor, Professor Takashi Sakai. The author is also very grateful to the referee, Shintaro Akamine and Luiz Carlos Barbosa da Silva for their comments and valuable advices.

References

[1] L. J. Alı́as, B. Palmer, *Curvature properties of zero mean curvature surfaces in four-dimensional Lorentzian space forms*, Math. Proc. Camb. Phil. Soc. **124** (1998), 315–327.

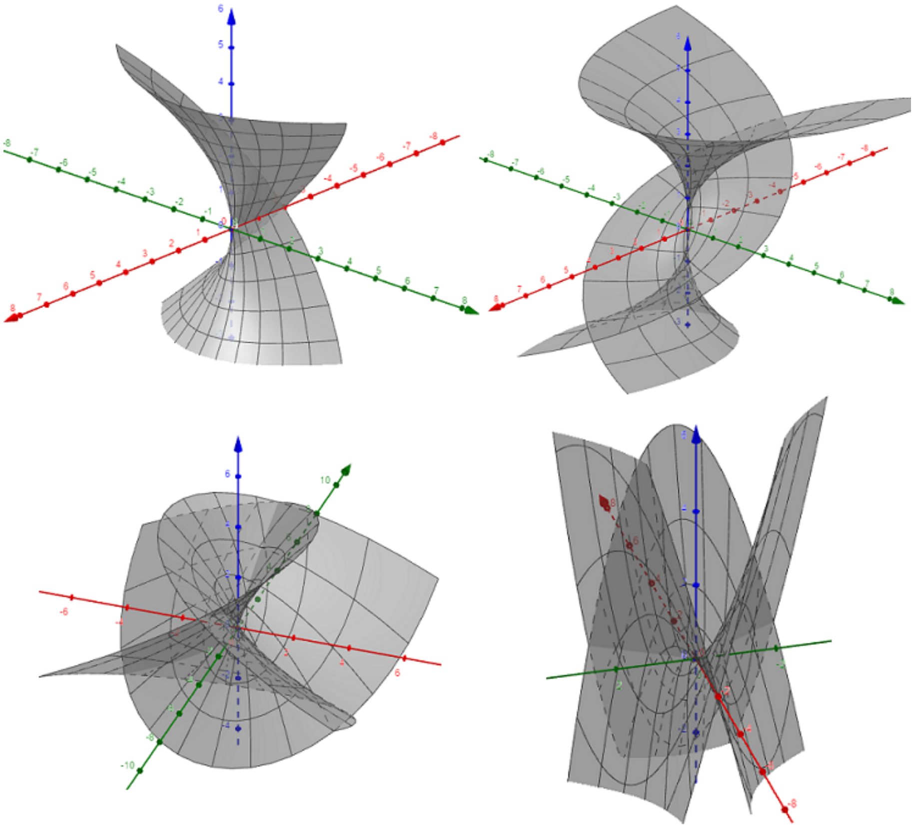

Figure 2: (a) Upper-left: $(F, G) = (w, \frac{1}{w})$. (b) Upper-right: $(F, G) = (w^2, \frac{1}{w^2})$. (c) Lower-left: $(F, G) = (w^2, \frac{1}{w})$. (d) Lower-right: $(F, G) = (w, w)$. Singularities appear at the origin $w = 0$. The rank of the Jacobi matrix is one for upper figures and that is zero for lower figures. And, the red, green and blue axis correspond to x-axis, y-axis and z-axis respectively.

[2] H. Anciaux, Minimal submanifolds in pseudo-Riemannian geometry, World Scientific (2011).

[3] M. E. Aydin, *Classification results on surfaces in the isotropic* 3*-space*, AKU J. Sci. Eng. **16** (2016), 239–246.

[4] M. E. Aydin, *Constant curvature surfaces in a pseudo-isotropic space*, Tamkang J. Math. **49** (2018), 221–233.

[5] A. Bejancu and K. L. Duggal, Lightlike submanifolds of semi-Riemannian manifolds and applications, Kluwer Academic Publishers (1996).

[6] B. Y. Chen, *Black holes, marginally trapped surfaces and quasi-minimal surfaces*, Tamkang J. Math. **40** (2009), 313–341.

[7] Z. Erjavec, B. Divjak and D. Horvat, *The general solutions of Frenet's system in the equiform geometry of the Galilean, pseudo-Galilean, simple isotropic and double isotropic space*, Int. Math. Forum **6** (2011), no. 17, 837–856.

[8] F. J. M. Estudillo and A. Romero, *On maximal surfaces in the n-dimensional Lorentz-Minkowski space*, Geom. Dedicata **38** (1991), 167–174.

[9] S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara and K. Yamada, *Entire zero-mean curvature graphs of mixed type in Lorentz-Minkowski 3-space*, Quarterly J. Math. **67** (2016), 801–837.

[10] S. Fujimori, K. Saji, M. Umehara and K. Yamada, *Singularities of maximal surfaces*, Math. Z. **259** (2008), 827–848.

[11] T. Ishihara, *Maximal spacelike submanifolds of a pseudo-Riemannian space of constant curvature*, Mich. Math. J. **35** (1988), 345–352.

[12] X. Ma, C. P. Wang and P. Wang, *Global geometry and topology of spacelike stationary surfaces in the 4-dimensional Lorentz space*, Adv. Math. **249** (2013), 311–347.

[13] K. Nomizu and T. Sasaki, Affine differential geometry, Cambridge University Press (1994).

[14] M. O'Neill, Semi-Riemannian geometry with applications to relativity, Academic Press, London (1983).

[15] A. L. Onishchik and R. Sulanke, Projective and Cayley-Klein geometries, Springer (2006).

[16] H. Pottmann, P. Grohs and N. J. Mitra, *Laguerre minimal surfaces, isotropic geometry and linear elasticity*, Adv. Comput. Math. **31** (2009), 391–419.

[17] H. Sachs, Isotrope Geometrie des Raumes, Vieweg, Braunschewig/Wiesbaden (1990).

[18] K. Saji, *Criteria for D_4 singularities of wave fronts*, Tohoku Math. J. **63** no.1 (2011), 137–147.

[19] Y. Sato, *On the classification of ruled minimal surfaces in pseudo-Euclidean space*, Math. J. Okayama Univ. **61** (2019), 173–186.

[20] L. C. B. da Silva, *Rotation minimizing frames and spherical curves in simply isotropic and pseudo-isotropic* 3*-spaces*, To appear in Tamkang J. Math.

[21] L. C. B. da Silva, *The geometry of Gauss map and shape operator in simply isotropic and pseudo-isotropic spaces*, J. Geom. **110**: 31 (2019).

[22] L. C. B. da Silva, *Differential geometry of invariant surfaces in simply and pseudo isotropic spaces*, To appear in Math. J. Okayama Univ.

[23] O. C. Stoica, *On singular semi-Riemannian manifolds*, Int. J. Geom. Methods Mod. Phys. **11**, (2014), no. 5, 1450041.

[24] M. Umehara and K. Yamada, *Maximal surfaces with singularities in Minkowski space*, Hokkaido Math. J. **35** (2006), 13–40.

[25] W. O. Vogel, *Über lineare Zusammenhänge in singulären Riemannschen Räumen*, Archiv der Mathematik **16** (1965), 106–116.

Yuichiro Sato Department of Mathematical Sciences, Tokyo Metropolitan University, Minami-Osawa 1-1, Hachioji, Tokyo, 192-0397, Japan.

E-mail: satou-yuuichirou@ed.tmu.ac.jp