COMPLEX-PARAMETER INTEGRAL ITERATIONS OF CARATHEODORY MAPS

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Abstract. Recent studies in the class of Bazilević maps as a whole has compelled the development, in this work, of certain complex-parameter integral iterations of Carathéodory maps. The iterations are employed in a similar manner as in [1] to study a certain subfamily of those Bazilević maps.

1. Background

Recently, Babalola in [2] devised a new approach to the study, as a whole, of the well known schlicht Bazilević maps defined as

$$f(z) = \left\{ \frac{\alpha}{1 + \beta^2} \int_0^z [p(t) - i\beta] t^{\frac{1}{1 + \beta^2}} g(t) \left( \frac{\alpha}{1 + \beta^2} \right)^{1 + i\beta} dt \right\}^{1 + i\beta}$$  (1.1)

in the sense that the parameter $\beta$ is no longer assumed zero. In the representation (1.1), $z$ lies in the unit disk $E = \{ z : |z| < 1 \}$, $f(z)$ is a regular function of the form $f(z) = z + a_2 z^2 + \cdots$ normalized by $f(0) = 0$ and $f'(0) = 1$ whose class is designated $A$. The subclass of $A$ that contains schlicht maps only is denoted by $S$. The map $g \in S$ is starlike (that is $\Re z g'(z)/g(z) > 0$). The map $p(z) = 1 + c_1 z + \cdots$ is Carathéodory (that is $p(0) = 1$ and $\Re p(z) > 0$) and its family is denoted by $P$. The parameters $\alpha$ and $\beta$ are real with $\alpha > 0$ and all powers meaning principal determinations only.

The method of analysis employed by Babalola in [2] involved the modification of the normalization of the Carathéodory maps as $h(z) = p(z) + i\mu/\eta = 1 + i\mu/\eta + c_1 z + \cdots$ where $\mu$ and $\eta$ are real with $\eta > 0$, $\lambda = \eta + i\mu$ and $p \in P$; and denoted the class of $h(z)$ by $P_\lambda$, which for convenience we shall replace by $H_\lambda$ in this paper. He thus defined a new class $B(\lambda, g)$ of maps satisfying

$$\frac{z (f(z)^\lambda)'}{\eta z^i \mu g(z)^\eta} \in H_\lambda.$$  (1.2)
The new definition therefore includes the Bazilević maps as the case \( \lambda = \alpha / (1 + i \beta) \).

Using the Salagean derivative \( D^n \) defined as

\[
D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'
\]

with \( D^0 f(z) = f(z) \) [7], Babalola in [3] further generalized (1.2) as \( B_n(\lambda, g) \) consisting of maps \( f(z) \) which satisfy

\[
\frac{D^n f(z) \lambda}{\eta \lambda^{n-1} z^\mu g(z) \eta} \in H_\lambda
\]

and then chose \( g(z) = z \) for which he obtained a number of properties of \( B_n(\lambda) \equiv B_n(\lambda, z) \).

In the present paper, we shall study (1.3) also for \( g(z) = z \) and \( h(z) = p(z) + i \mu/\eta \) with \( \Re p(z) > \gamma \), \( 0 \leq \gamma < 1 \) and \( H_\lambda(\gamma) \) is designated the class of such functions. Hence for some \( h \in H_\lambda(\gamma) \), we have

\[
\frac{D^n f(z) \lambda}{\eta \lambda^{n-1} z^\mu g(z) \eta} = 1 + i \frac{\mu}{\eta} + (1 - \gamma) \sum_{k=1}^{\infty} c_k z^k
\]

so that

\[
D^n f(z) \lambda = \lambda^n z^\lambda + (1 - \gamma) \sum_{k=1}^{\infty} \eta \lambda^{n-1} c_k z^{\lambda+k}.
\]

Applying the Salagean integral, \( I_n \), also defined in [7] as

\[
I_n = I(I_{n-1} f(z)) = \int_0^z [(I_{n-1} f(t)) / t] dt
\]

with \( I_0 f(z) = f(z) \), and with some computation, we obtain

\[
\frac{\lambda f(z) \lambda}{\eta z^\lambda} - i \frac{\mu}{\eta} = 1 + (1 - \gamma) \sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda + k} \right)^n c_k z^k.
\]

The right-hand side of the above equation is

\[
p_{\lambda,n}(z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} p_{\lambda,n-1}(t) dt,
\]

with \( p_{\lambda,0}(z) = p(z) \) is a Caratheodory map of order \( \gamma \), and it is the justification for our study. We call \( p_{\lambda,n}(z) \) the complex-parameter \( n \)-th integral iterations of \( p \in P(\gamma) \), and its class is designated by \( P_{\lambda,n}(\gamma) \). Throughout this paper, the word ‘iterations’ shall imply the phrase ‘\( n \)-th complex-parameter integral iterations’.
2. The iterations of $P(\gamma)$

First, we observe that the new iterations of Carathéodory maps of order $\gamma$ agrees in structure with the iterations in [1] and fortunately the complex parameter $\lambda$ has the desired positive real part. Therefore since $p_0(z) = p \in P(\gamma)$, then the iterations $p_{\lambda,n}(z)$ is analytic and $p_{\lambda,n}(0) = 1$. It follows from (1.5) that if $p \in P(\gamma)$, then

$$p_{\lambda,n}(z) = 1 + \sum_{k=1}^{\infty} c_{n,k} z^k$$

where $c_{n,k} = (1 - \gamma) \left( \frac{\lambda}{\lambda + k} \right)^n c_k$ (2.1)

and therefore

$$|c_{n,k}| \leq 2(1 - \gamma) \left| \frac{\lambda}{\lambda + k} \right|^n.$$

With $p_0(z) = L_0(z) = [1 + (1 - 2\gamma)z]/(1 - z)$, then the iterations, $L_{\lambda,n}(z)$, of the Moebius map is given as

$$L_{\lambda,n}(z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} L_{\lambda,n-1}(t) \, dt = 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda + k} \right)^n z^k.$$

The iterations $L_{\lambda,n}(z)$ plays a central role with respect to the extremal problem.

As noted in [1], we remark that for complex parameters $\lambda$ and $\nu$ both having positive real parts, if we denote $p_{\lambda,n}(z)$ by $\chi_n^{(\lambda)}(p(z))$, then for any $p \in P(\gamma)$ and $m, n \in N_0$, we have

$$\chi_m^{(\nu)}(\chi_n^{(\lambda)}(p(z))) = \chi_n^{(\lambda)}(\chi_m^{(\nu)}(p(z)))$$

and for $\lambda = \nu$, it yields

$$\chi_n^{(\lambda)}(\chi_m^{(\lambda)}(p(z))) = \chi_m^{(\lambda)}(\chi_n^{(\lambda)}(p(z))) = \chi_{n+m}^{(\lambda)}(p(z)).$$

We note here that the case $\mu = 0$, $\eta > 0$ and $\gamma = 0$ yields $P_{\eta,n}(0) = P_n$, the iterated integral transforms of Carathéodory maps defined in [1].

Next, we characterize the new iterations, $p_{\lambda,n}(z)$. The proofs follow mutatis mutandis as in [1]. However we again give the proofs for completeness and clarity of this paper, noting that Re $\lambda > 0$ where appropriate.

**Theorem 1.** Let $\gamma \neq 1$ be a non negative real number. Then for $n \in N$

$$\text{Re } p_{\lambda,n-1}(z) > \gamma \text{ implies } \text{Re } p_{\lambda,n}(z) > \gamma, \text{ for } 0 \leq \gamma < 1$$

and

$$\text{Re } p_{\lambda,n-1}(z) < \gamma \text{ implies } \text{Re } p_{\lambda,n}(z) < \gamma, \text{ for } \gamma > 1.$$
Proof. On differentiation of (1.5), we have
\[
\lambda z^{\lambda - 1} p_{\lambda,n}(z) + z^{\lambda} p'_{\lambda,n}(z) = \lambda z^{\lambda - 1} p_{n-1}(z)
\]
which yields
\[
p_{\lambda,n}(z) + \frac{z p'_{\lambda,n}(z)}{\lambda} = p_{n-1}(z).
\]

(2.2)

It follows by the proof of Theorem 3.1 in [1] that
\[
\text{Re} \left( p_{\lambda,n}(z) + \frac{z p'_{\lambda,n}(z)}{\lambda} \right) = \text{Re} p_{n-1}(z) > \gamma.
\]
This implies that \( \text{Re} p_{\lambda,n}(z) > \gamma \) for \( 0 \leq \gamma < 1 \) and
\[
\text{Re} \left( p_{\lambda,n}(z) + \frac{z p'_{\lambda,n}(z)}{\lambda} \right) = \text{Re} p_{n-1}(z) < \gamma.
\]
This implies that \( \text{Re} p_{\lambda,n}(z) < \gamma \) for \( \gamma > 1 \). □

Corollary 1. \( P_{\lambda,n}(\gamma) \subset P(\gamma), \ n \in \mathbb{N} \).

Proof. Since \( p_0(z) = p(z) \in P(\gamma) \), then \( \text{Re} p(z) > \gamma \). Hence by Theorem 1, we have, \( \text{Re} p_{\lambda,1}(z) > \gamma \). Also \( p_1(z) \in P(\gamma) \), thus \( \text{Re} p_{\lambda,2}(z) > \gamma \) and so on for all \( n \in \mathbb{N} \). Therefore \( P_{\lambda,n}(\gamma) \subset P(\gamma) \). □

Theorem 2. \( P_{\lambda,n+1}(\gamma) \subset P_{\lambda,n}(\gamma), \ n \in \mathbb{N} \).

Proof. Let \( p_{\lambda,n+1} \in P_{\lambda,n+1}(\gamma) \), hence, there exist \( p \in P(\gamma) \) such that
\[
p_{\lambda,n+1}(z) = \chi_{n+1}^{(\lambda)}(p(z))
\]
as earlier noted, then \( p_{\lambda,n+1}(z) = \chi_{n}^{(\lambda)}(\chi_{1}^{(\lambda)}(p(z))) \) and using Corollary 1, we have \( \chi_{1}^{(\lambda)}(p(z)) \in P(\gamma) \). Hence \( p_{\lambda,n+1} \) is the \( n \)th integral transform of a function in \( P(\gamma) \), that is, \( p_{\lambda,n+1} \) belongs to \( P_{\lambda,n} \). Therefore \( P_{\lambda,n+1}(\gamma) \subset P_{\lambda,n}(\gamma) \). □

Corollary 2. Let \( p \in P(\gamma) \) and \( \text{Re} \lambda + c > 0 \). Then
\[
q(z) = 1 + (1 - \gamma)(\lambda + c) \sum_{k=1}^{\infty} \frac{c_k}{\lambda + c + k} z^k, \quad z \in E
\]
is also in \( P(\gamma) \).
Proof. Let \( q(z) \in P_{\lambda+c,1}(\gamma) \), then we have
\[
q(z) = p_{\lambda+c,1}(z) = \frac{\lambda + c}{z^{\lambda+c}} \int_0^z t^{\lambda+c-1} p_0(t) \, dt.
\]
Hence
\[
q(z) = 1 + (1 - \gamma)(\lambda + c) \sum_{k=1}^{\infty} \frac{c_k}{\lambda + c + k} z^k
\]
is also in \( P(\gamma) \).

In the next result we shall proof the surbordination \( p_{\lambda,n} \prec L_{\lambda,n} \) using the technique of Briot-Bouquet differential subordination. A function \( p \in P \) is said to satisfy the Briot-Bouquet differential subordination if
\[
p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} < h(z), \quad z \in E
\]
where \( \eta \) and \( \gamma \) are complex constants and \( h(z) \) a complex function satisfying \( h(0) = 1 \), and \( \Re(\eta h(z) + \gamma) > 0 \) in \( E \). It is well known that if \( p \in P \) satisfies the Briot-Bouquet differential subordination, then \( p(z) < h(z) \) [5].

A univalent function \( q(z) \) is said to be a dominant of (2.3) if \( p(z) < q(z) \) for all \( p(z) \) satisfying (2.3). If \( \tilde{q}(z) \) is a dominant of (2.3) and \( \tilde{q}(z) < q(z) \) for all dominants \( q(z) \) of (2.3), then \( \tilde{q}(z) \) is said to be the best dominant of (2.3).

**Theorem 3.** \( p_{\lambda,n} \prec L_{\lambda,n} \).

From (2.2) we have
\[
p_{\lambda,1}(z) + \frac{zp'_{\lambda,1}(z)}{\lambda} = p(z).
\]
Now since \( p \in P \), then \( p < L_0(z) = (1 + (1 - 2\gamma)z)/(1 - z) \) so that
\[
p_{\lambda,1}(z) + \frac{zp'_{\lambda,1}(z)}{\lambda} < L_0(z).
\]
However the differential equation
\[
q(z) + \frac{zq'(z)}{\lambda} = L_0(z)
\]
has univalent solution
\[
q_{\gamma}(z) = \frac{\lambda}{z^{\lambda}} \int_0^z t^{\lambda-1} L_0(t) \, dt = L_{\lambda,1}(z)
\]
which, by the principle of the Briot-Bouquet differential subordination, is the best dominant. Similarly, from (2.2),
\[
p_{\lambda,2}(z) + \frac{zp'_{\lambda,2}(z)}{\lambda} < L_{\lambda,1}(z),
\]
while the differential equation
\[ q(z) + \frac{z q'(z)}{\lambda} = L_{\lambda,1}(z) \]
also has univalent solution
\[ q_\gamma(z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} L_{\lambda,1}(t) \, dt = L_{\lambda,2}(z) \]
which again, by the principle of the Briot-Bouquet differential subordination, is the best dominant. Continuing, we have that
\[ p_{\lambda,n}(z) + \frac{z p'_{\lambda,n}(z)}{\lambda} < L_{\lambda,n-1}(z), \]
with the differential equation
\[ q(z) + \frac{z q'(z)}{\lambda} = L_{\lambda,n-1}(z) \]
having univalent solution
\[ q_\gamma(z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} L_{\lambda,n-1}(t) \, dt = L_{\lambda,n}(z). \]
This implies that \( p_{\lambda,n} \prec L_{\lambda,n} \) which completes the proof.

**Theorem 4.** \( P_{\lambda,n}(\gamma) \) is a convex set.

**Proof.** Let \( p_{\lambda,n}(\gamma), q_{\lambda,n}(\gamma) \in P_{\lambda,n}(\gamma) \). Then for non negative real numbers \( \tau_1 \) and \( \tau_2 \) with \( \tau_1 + \tau_2 = 1 \), we have
\[ \tau_1 p_{\lambda,n}(z) + \tau_2 q_{\lambda,n}(z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} \left[ \tau_1 p_{n-1} + \tau_2 q_{n-1} \right] (t) \, dt \]
with \( \tau_1 p_{\lambda,0}(z) + \tau_2 q_{\lambda,0}(z) = p(z) \in P(\gamma) \).
Now for \( n = 1 \), it yields
\[ \tau_1 p_{\lambda,1}(z) + \tau_2 q_{\lambda,1}(z) = 1 + (1 - \gamma) \sum_{k=1}^\infty \frac{\lambda}{\lambda + k} c_k z^k \in P_{\lambda,n}(\gamma). \]
Suppose, it is true for \( n = v - 1 \), thus
\[ \tau_1 p_{\lambda,v-1}(z) + \tau_2 q_{\lambda,v-1}(z) = 1 + (1 - \gamma) \sum_{k=1}^\infty \left( \frac{\lambda}{\lambda + k} \right)^{v-1} c_k z^k \in P_{\lambda,n}(\gamma). \]
Now for \( n = v \)
\[ \tau_1 p_{\lambda,v}(z) + \tau_2 q_{\lambda,v}(z) = \frac{\lambda}{z^\lambda} \int_0^z t^{\lambda-1} \left[ 1 + (1 - \gamma) \sum_{k=1}^\infty \left( \frac{\lambda}{\lambda + k} \right)^{v-1} c_k t^k \right] \, dt \]
\[ = 1 + (1 - \gamma) \sum_{k=1}^\infty \left( \frac{\lambda}{\lambda + k} \right)^v c_k z^k \in P_{\lambda,n}(\gamma). \]
Therefore, $P_{\lambda,n}(\gamma)$ is a convex set.

The lower bound in the next theorem is not the best possible. We provide the proof of our estimate and follow that with our expectation.

**Theorem 5.** Let $p_{\lambda,n} \in P_{\lambda,n}(\gamma)$. Then

$$|p_{\lambda,n}(z)| \leq 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \left| \frac{\lambda}{\lambda + k} \right|^n r^k, \quad |z| = r$$

and

$$\text{Re} \, p_{\lambda,n}(z) \geq 1 - 2(1 - \gamma) \sum_{k=1}^{\infty} \left| \frac{\lambda}{\lambda + k} \right|^n r^k, \quad |z| = r.$$

**Proof.** For the lower bound, then from (2.1), we have

$$|p_{\lambda,n}(z) - 1| \leq 2(1 - \gamma) \sum_{k=1}^{\infty} \left| \frac{\lambda}{\lambda + k} \right|^n r^k.$$

Hence

$$\text{Re} \left( p_{\lambda,n}(z) - 1 \right) \geq -2(1 - \gamma) \sum_{k=1}^{\infty} \left| \frac{\lambda}{\lambda + k} \right|^n r^k$$

so that

$$\text{Re} \, p_{\lambda,n}(z) \geq 1 - 2(1 - \gamma) \sum_{k=1}^{\infty} \left| \frac{\lambda}{\lambda + k} \right|^n r^k.$$

**Conjecture:** Let $p_{\lambda,n} \in P_{\lambda,n}(\gamma)$. Then

$$\text{Re} \, p_{\lambda,n}(z) \geq 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \left| \frac{\lambda}{\lambda + k} \right|^n (-r)^k, \quad |z| = r.$$

**3. Some applications of $P_{\lambda,n}(\gamma)$**

Here we present several different applications of the iterations $p_{\lambda,n}$ of Carathéodory maps to the study of a new subclass $B^\lambda_n(\gamma)$ of Bazilevič maps. We begin with

**Definition 3.1.** A regular map $f \in A$ belongs to the class $B^\lambda_n(\gamma)$ if and only if

$$\frac{D^n f(z)^{\lambda}}{\eta \lambda^{n-1} z^\lambda} \in P_{\lambda}(\gamma)$$

where all parameters have their usual definitions.
Remark 1. (i) If \( \gamma = 0 \), we recover the class \( B_n(\lambda) \) studied in [3].

(ii) If \( \mu = 0 \), we then deduce the class \( B_n^\eta(\gamma) \) which is equivalent to class \( T_n^\eta(\gamma) \) studied in [1, 6] with change in notation as a matter of convenience. (iii) For all \( f \in B_n^\lambda(\gamma) \), Equations (1.4) and (1.5) hold for some \( p \in P_\lambda(\gamma) \). Hence we have the following relation between \( P_\lambda, n(\gamma) \) and \( B_n^\lambda(\gamma) \) which will be used in our characterization of the class \( B_n^\lambda(\gamma) \):

Lemma 1. Let \( f \in A \), then the following are equivalent:

(i) \( f \in B_n^\lambda(\gamma) \),

(ii) \( \frac{D_n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} \in P_\lambda(\gamma) \),

(iii) \( \frac{\lambda f(z)^\lambda}{\eta z^\lambda} - i \frac{\mu}{\eta} \in P_{\lambda, n}(\gamma) \).

Proof. If \( f \in A \), then by Definition 3.1 it is clear that (i) \( \Leftrightarrow \) (ii). Now (ii) is true \( \Leftrightarrow \) there exist \( h \in P(\gamma) \) such that

\[
\frac{D_n f(z)^\lambda}{\eta \lambda^{n-1} z^\lambda} = h(z) = p(z) + \frac{i \mu}{\eta} = 1 + \frac{i \mu}{\eta} + (\gamma) \sum_{k=1}^{\infty} c_k z^k
\]

for some \( p \in P(\gamma) \), so that

\[
D_n f(z)^\lambda = \lambda^n z^\lambda + (1 - \gamma) \sum_{k=1}^{\infty} \eta \lambda^{n-1} c_k z^{\lambda+k}.
\] (3.1)

Applying on (3.1) the integral operator, \( I_n \), defined by Salagean (1983) as

\[
I_n f(z) = I(I_{n-1} f(z)) = \int_0^z \frac{I_{n-1} f(t)}{t} \, dt
\]

with \( I_0 f(z) = f(z) \) and with some computation we have

\[
\frac{\lambda f(z)^\lambda}{\eta z^\lambda} = 1 + \frac{i \mu}{\eta} + (1 - \gamma) \sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda + k} \right)^n c_k z^k.
\]

Therefore

\[
\frac{\lambda f(z)^\lambda}{\eta z^\lambda} - i \frac{\mu}{\eta} = 1 + (1 - \gamma) \sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda + k} \right)^n c_k z^k \in P_{\lambda, n}(\gamma).
\]

This proves the Lemma. \( \square \)

Now we discuss properties of the class \( B_n^\lambda(\gamma) \).

Theorem 6.

\[
B_{n+1}^\lambda(\gamma) \subset B_n^\lambda(\gamma).
\]
\textbf{Proof.} Let }f \in B_{n+1}^A(\gamma).\text{ Then by Lemma 1 we have}
\[
\frac{\lambda f(z)^A}{\eta z^A} - i \frac{\mu}{\eta} = 1 + (1 - \gamma) \sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda + k} \right)^{n+1} c_k z^k
\]
which implies
\[
\frac{\lambda f(z)^A}{\eta z^A} - i \frac{\mu}{\eta} \in P_{\lambda,n+1}(\gamma).
\]
Then by Theorem 2
\[
\frac{\lambda f(z)^A}{\eta z^A} - i \frac{\mu}{\eta} \in P_{\lambda,n}(\gamma).
\]
Hence by Lemma 1 again, }f \in B_{n+1}^A(\gamma).\]

\textbf{Corollary 3.} \textit{For }n \geq 1, B_{n+1}^A(\gamma) \text{ consists of schlicht maps only.}

\textbf{Proof.} For }n \geq 1, \text{ by Theorem 7 and Remark 3 we have }B_{n+1}^A(\gamma) \subset B_n^A(\gamma) \text{ and } B_n^A(\gamma) \subset B_n(\lambda),\text{ which is known to consist only of schlicht maps for }n \geq 1 \text{ (see Corollary 1 in [1]). Therefore the assertion follows.}

\textbf{Theorem 7.} \textit{Let }f \in B_n^A(\gamma). \text{ Then the integral}
\[
F(z)^A = \frac{\lambda + c}{z^c} \int_0^z t^{c-1} f(t)^A \, dt \tag{3.2}
\]
is also in }B_n^A(\gamma).\]

\textbf{Proof.} From (3.2) we have
\[
\frac{\lambda F(z)^A}{\eta z^A} - i \frac{\mu}{\eta} = \frac{\lambda + c}{z^{\lambda+c}} \int_0^z t^{\lambda+c-1} \left( \frac{\lambda f(t)^A}{\eta t^A} - i \frac{\mu}{\eta} \right) dt.
\]
Now suppose }f \in B_n^A(\gamma) \text{ and let } v = \lambda + c \text{ be a complex number, then we have}
\[
\frac{\lambda F(z)^A}{\eta z^A} - i \frac{\mu}{\eta} = \frac{u}{z^u} \int_0^z t^{u-1} \left( \frac{\lambda f(t)^A}{\eta t^A} - i \frac{\mu}{\eta} \right) dt.
\]
Hence, we have
\[
F(z)^A = \frac{\lambda}{\eta z^A} - i \frac{\mu}{\eta} = \chi_1^{(v)}(\chi_n^{(\lambda)}(p(z))) = \chi_n^{(\lambda)}(\chi_1^{(v)}(p(z))).
\]
With Corollary 1, }\chi_1^{(v)}(p(z)) \in P(\gamma), \text{ thus } \chi_n^{(\lambda)}(\chi_1^{(v)}(p(z))) \in P_{\lambda,n}(\gamma), \text{ which implies that } \frac{\lambda F(z)^A}{\eta z^A} - i \frac{\mu}{\eta} \in P_{\lambda,n}(\gamma). \text{ Therefore by Lemma 1, } F \in B_n^A(\gamma).\]
Theorem 8. Let \( f \in B^1_n(\gamma) \) and define

\[
M_T(n, \lambda^*, \gamma, r) = 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \lambda_k^* r^k
\]

and

\[
m_T(n, \lambda^*, \gamma, r) = 1 - 2(1 - \gamma) \sum_{k=1}^{\infty} \lambda_k^* r^k
\]

where \( \lambda^* = \eta|\lambda|^{n-1}/|\lambda + k|^n \). Then

\[
m_T(n, \lambda^*, \gamma, r) \leq \left| \left( \frac{f(z)}{z} \right)^\lambda \right| \leq M_T(n, \lambda^*, \gamma, r).
\]

The inequality on the right side is sharp while the one on the left is not.

Proof. Taking \( p_{\lambda,n}(z) = \frac{\lambda f(z)^{\lambda}}{\eta z^\lambda} - i \frac{\mu}{\eta} \) in Theorem 6, we have

\[
\frac{\lambda f(z)^{\lambda}}{\eta z^\lambda} - i \frac{\mu}{\eta} = 1 + (1 - \gamma) \sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda + k} \right)^n c_k z^k.
\]

Then

\[
\left( \frac{f(z)}{z} \right)^\lambda = 1 + (1 - \gamma) \eta \lambda^{n-1} \sum_{k=1}^{\infty} \frac{c_k}{(\lambda + k)^n} z^k.
\]

Using triangle inequality, then we have

\[
\left| \left( \frac{f(z)}{z} \right)^\lambda \right| \leq 1 + 2(1 - \gamma) \eta|\lambda|^{n-1} \sum_{k=1}^{\infty} \frac{r^k}{|\lambda + k|^n}
\]

\[
\leq 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \lambda_k^* r^k.
\]

For lower bound, we have

\[
\text{Re} \left( \frac{f(z)^\lambda}{z^\lambda} - 1 \right) \geq -2(1 - \gamma) \eta|\lambda|^{n-1} \sum_{k=1}^{\infty} \frac{r^k}{|\lambda + k|^n}.
\]

Hence

\[
\text{Re} \left( \frac{f(z)}{z} \right)^\lambda \geq 1 - 2(1 - \gamma) \sum_{k=1}^{\infty} \lambda_k^* r^k
\]

which concludes the proof. \( \square \)

Theorem 9. Let \( f \in B^1_n(\gamma) \) and define

\[
M^*_T(n, \lambda^{**}, \gamma, r) = 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \lambda_k^{**} r^k
\]

and
\[ m_T^*(n, \lambda^{**}, \gamma, r) = 1 - 2(1 - \gamma) \sum_{k=1}^{\infty} \lambda_k^{**} r^k \]

where \( \lambda^{**} = \eta|\lambda|^{n-2}/|\lambda + k|^{n-1} \). Then
\[
m_T^*(n, \lambda^{**}, \gamma, r) \leq \left| \frac{f(z)^{\lambda-1} f'(z)}{z^{\lambda-1}} \right| \leq M_T^*(n, \lambda^{**}, \gamma, r).
\]
The inequality on the right side is sharp while the one on the left side is not.

**Proof.** Since \( f \in B^\lambda_h(\gamma) \), then by Lemma 1 there exist \( p_{\lambda, n} \in P_{\lambda, n}(\gamma) \) such that
\[
\frac{\lambda f(z)^\lambda}{\eta z^\lambda} - i \frac{\mu}{\eta} = p_{\lambda, n}(z).
\]
Then we differentiate to get
\[
\frac{\lambda f(z)^{\lambda-1} f'(z)}{\eta z^{\lambda-1}} - i \frac{\mu}{\eta} = p_{\lambda, n}(z) + \frac{zp'_{\lambda, n}(z)}{\lambda}.
\]
Hence from (1.5) we have
\[
p_{\lambda, n}(z) + \frac{zp'_{\lambda, n}(z)}{\lambda} = p_{\lambda, n-1}(z).
\]
Now substitute (3.4) in (3.3), then (3.3) now becomes
\[
\frac{\lambda f(z)^{\lambda-1} f'(z)}{\eta z^{\lambda-1}} - i \frac{\mu}{\eta} = p_{\lambda, n-1}(z).
\]
with
\[
p_{\lambda, n-1}(z) = 1 + (1 - \gamma) \sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda + k} \right)^{n-1} c_k z^k.
\]
Hence by triangular inequality, we have
\[
\left| \frac{f(z)^{\lambda-1} f'(z)}{z^{\lambda-1}} \right| \leq 1 + 2(1 - \gamma) \eta|\lambda|^{n-2} \sum_{k=1}^{\infty} \frac{r^k}{|\lambda + k|^{n-1}}
\]
\[
\leq 1 + 2(1 - \gamma) \sum_{k=1}^{\infty} \lambda_k^{**} r^k.
\]
This bound is sharp. For the lower bound, we have
\[
\frac{\lambda f(z)^{\lambda-1} f'(z)}{\eta z^{\lambda-1}} - i \frac{\mu}{\eta} - 1 = (1 - \gamma) \sum_{k=1}^{\infty} \left( \frac{\lambda}{\lambda + k} \right)^{n-1} c_k z^k.
\]
\[
\text{Re}\left( \frac{f(z)^{\lambda-1} f'(z)}{z^{\lambda-1}} - i \frac{\mu}{\eta} - 1 \right) \geq -2(1 - \gamma) \eta|\lambda|^{n-2} \sum_{k=1}^{\infty} \frac{r^k}{|\lambda + k|^{n-1}}
\]
\[
\geq -2(1 - \gamma) \sum_{k=1}^{\infty} \lambda_k^{**} r^k.
\]
Therefore
\[
\text{Re} \left( \frac{f(z)^{\lambda-1} f'(z)}{z^{\lambda-1}} \right) \geq 1 - 2(1 - \gamma) \sum_{k=1}^{\infty} \lambda_k^{**} r^k.
\]
The bound is not the best possible. \(\Box\)

**Theorem 10.** The set of points \(\frac{\lambda f(z)^{\lambda}}{\eta z} - \frac{i \mu}{\eta}\) for each \(f \in B_{n}^{\lambda}(\gamma), z \in E\) is convex.

**Proof.** Since for each \(f \in B_{n}^{\lambda}(\gamma),\) the point \(\frac{\lambda f(z)^{\lambda}}{\eta z} - \frac{i \mu}{\eta} \in P_{\lambda,n}(\gamma),\) then by Theorem 5, the result follows. \(\Box\)

### 4. Consequences

We state here the analogues of Theorems 8 to 11 for class \(B(\alpha, \beta) \equiv B(\alpha, \beta, \gamma)\) of Bazilevic maps. This is achieved by setting \(\lambda = \alpha/(1 + i \beta)\) so that \(\eta = \alpha/(1 + \beta^2), \mu = -\alpha \beta/(1 + \beta^2);\) and taking \(n = 1\) and \(\gamma = 0\) in the theorems, we have respectively:

**Corollary 4.** Let \(f \in B(\alpha, \beta).\) Then the function \(F(z)\) defined by
\[
F(z)^{\frac{\alpha}{1+i\beta}} = \frac{\alpha + c(1 + i \beta)}{(1 + i \beta)z^c} \int_{0}^{z} t^{c-1} f(t)^{\frac{\alpha}{1+i\beta}} dt
\]
is also in \(B(\alpha, \beta).\)

**Corollary 5.** Let \(f \in B(\alpha, \beta)\) and define
\[
M_T(\alpha, \beta, r) = 1 + 2 \sum_{k=1}^{\infty} \frac{\alpha}{\sqrt{(1 + \beta^2)[(\alpha + k)^2 + \beta^2 k^2]}} r^k.
\]
and
\[
m_T(\alpha, \beta, r) = 1 - 2 \sum_{k=1}^{\infty} \frac{\alpha}{\sqrt{(1 + \beta^2)[(\alpha + k)^2 + \beta^2 k^2]}} r^k.
\]
Then \(m_T(\alpha, \beta, r) \leq \left| \left( \frac{f(z)}{z} \right)^{\frac{\alpha}{1+i\beta}} \right| \leq M_T(\alpha, \beta, r).\) The inequality on the right side is sharp while the one on the left is not.

**Corollary 6.** Let \(f \in B(\alpha, \beta)\) and define
\[
M_T^*(\alpha, \beta, r) = 1 + 2 \sum_{k=1}^{\infty} \frac{r^k}{\sqrt{1 + \beta^2}}
\]
and
\[
m_T^*(\alpha, \beta, r) = 1 - 2 \sum_{k=1}^{\infty} \frac{r^k}{\sqrt{1 + \beta^2}}.
\]
Then \(m_T^*(\alpha, \beta, r) \leq \left| \left( \frac{f(z)}{z} \right)^{\frac{\alpha-1-i\beta}{1+i\beta}} f'(z) \right| \leq M_T^*(\alpha, \beta, r).\) The right hand side inequality is sharp while the one on the left is not.

**Corollary 7.** The set of points \((1 - i \beta) \left( \frac{f(z)}{z} \right)^{\frac{\alpha}{1+i\beta}} + i \beta\) for each \(f \in B(\alpha, \beta), z \in E\) is convex.
References


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