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COMPLEX-PARAMETER INTEGRAL ITERATIONS OF CARATHEODORY MAPS

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Abstract. Recent studies in the class of Bazilevič maps as a whole has compelled the development, in this work, of certain complex-parameter integral iterations of Caratheodory maps. The iterations are employed in a similar manner as in [1] to study a certain subfamily of those Bazilevič maps.

1. Background

Recently, Babalola in [2] devised a new approach to the study, as a whole, of the well known schlicht Bazilevič maps defined as

$$f(z)=\left\{\frac{\alpha}{1+\beta^2}\int_0^z [p(t)-i\beta]\,t^{-\left(1+\frac{i\alpha\beta}{1+\beta^2}\right)}g(t)^{\left(\frac{\alpha}{1+\beta^2}\right)}dt\right\}^{\frac{1+i\beta}{\alpha}} \tag{1.1}$$

in the sense that the parameter β is no longer assumed zero. In the representation (1.1), z lies in the unit disk $E=\{z:|z|<1\}$, $f(z)$ is a regular function of the form $f(z)=z+a_2z^2+\cdots$ normalized by $f(0)=0$ and $f'(0)=1$ whose class is designated A. The subclass of A that contains schlicht maps only is denoted by S. The map $g\in S$ is starlike (that is $\operatorname{Re} zg'(z)/g(z)>0$). The map $p(z)=1+c_1z+\cdots$ is Caratheodory (that is $p(0)=1$ and $\operatorname{Re} p(z)>0$) and its family is denoted by P. The parameters α and β are real with $\alpha>0$ and all powers meaning principal determinations only.

The method of analysis employed by Babalola in [2] involved the modification of the normalization of the Caratheodory maps as $h(z)=p(z)+i\mu/\eta=1+i\mu/\eta+c_1z+\cdots$ where μ and η are real with $\eta>0$, $\lambda=\eta+i\mu$ and $p\in P$; and denoted the class of $h(z)$ by P_λ, which for convenience we shall replace by H_λ in this paper. He thus defined a new class $B(\lambda,g)$ of maps satisfying

$$\frac{z\left(f(z)^{\lambda}\right)'}{\eta z^{i\mu}g(z)^{\eta}}\in H_\lambda. \tag{1.2}$$

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The new definition therefore includes the Bazilevič maps as the case $\lambda = \alpha/(1+i\beta)$.

Using the Salagean derivative D^n defined as

$$D^n f(z) = D(D^{n-1} f(z)) = z[D^{n-1} f(z)]'$$

with $D^0 f(z) = f(z)$ [7], Babalola in [3] further generalized (1.2) as $B_n(\lambda, g)$ consisting of maps $f(z)$ which satisfy

$$\frac{D^n f(z)^\lambda}{\eta\lambda^{n-1} z^{i\mu} g(z)^\eta} \in H_\lambda \tag{1.3}$$

and then chose $g(z) = z$ for which he obtained a number of properties of $B_n(\lambda) \equiv B_n(\lambda, z)$.

In the present paper, we shall study (1.3) also for $g(z) = z$ and $h(z) = p(z) + i\mu/\eta$ with Re $p(z) > \gamma$, $0 \le \gamma < 1$ and $H_\lambda(\gamma)$ is designated the class of such functions. Hence for some $h \in H_\lambda(\gamma)$, we have

$$\frac{D^n f(z)^\lambda}{\eta\lambda^{n-1} z^\lambda} = 1 + i\frac{\mu}{\eta} + (1-\gamma)\sum_{k=1}^{\infty} c_k z^k \tag{1.4}$$

so that

$$D^n f(z)^\lambda = \lambda^n z^\lambda + (1-\gamma)\sum_{k=1}^{\infty} \eta\lambda^{n-1} c_k z^{\lambda+k}.$$

Applying the Salagean integral, I_n, also defined in [7] as

$$I_n = I(I_{n-1} f(z)) = \int_0^z [(I_{n-1} f(t))/t]\,dt$$

with $I_0 f(z) = f(z)$, and with some computation, we obtain

$$\frac{\lambda f(z)^\lambda}{\eta z^\lambda} - i\frac{\mu}{\eta} = 1 + (1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^n c_k z^k.$$

The right-hand side of the above equation is

$$p_{\lambda,n}(z) = \frac{\lambda}{z^\lambda}\int_0^z t^{\lambda-1} p_{\lambda,n-1}(t)\,dt, \tag{1.5}$$

with $p_{\lambda,0}(z) = p(z)$ is a Caratheodory map of order γ, and it is the justification for our study. We call $p_{\lambda,n}(z)$ the complex-parameter n-th integral iterations of $p \in P(\gamma)$, and its class is designated by $P_{\lambda,n}(\gamma)$. Throughout this paper, the word 'iterations' shall imply the phrase 'n-th complex-parameter integral iterations'.

2. The iterations of $P(\gamma)$

First, we observe that the new iterations of Caratheodory maps of order γ agrees in structure with the iterations in [1] and fortunately the complex parameter λ has the desired positive real part. Therefore since $p_0(z) = p \in P(\gamma)$, then the iterations $p_{\lambda,n}(z)$ is analytic and $p_{\lambda,n}(0) = 1$. It follows from (1.5) that if $p \in P(\gamma)$, then

$$p_{\lambda,n}(z) = 1 + \sum_{k=1}^{\infty} c_{n,k} z^k \text{ where } c_{n,k} = (1-\gamma)\left(\frac{\lambda}{\lambda+k}\right)^n c_k \tag{2.1}$$

and therefore

$$|c_{n,k}| \leq 2(1-\gamma)\left|\frac{\lambda}{\lambda+k}\right|^n.$$

With $p_0(z) = L_0(z) = [1+(1-2\gamma)z]/(1-z)$, then the iterations, $L_{\lambda,n}(z)$, of the Moebius map is given as

$$L_{\lambda,n}(z) = \frac{\lambda}{z^\lambda}\int_0^z t^{\lambda-1} L_{\lambda,n-1}(t)\,dt = 1 + 2(1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^n z^k.$$

The iterations $L_{\lambda,n}(z)$ plays a central role with respect to the extremal problem.

As noted in [1], we remark that for complex parameters λ and v both having positive real parts, if we denote $p_{\lambda,n}(z)$ by $\chi_n^{(\lambda)}(p(z))$, then for any $p \in P(\gamma)$ and $m, n \in N_0$, we have

$$\chi_m^{(v)}(\chi_n^{(\lambda)}(p(z))) = \chi_n^{(\lambda)}(\chi_m^{(v)}(p(z)))$$

and for $\lambda = v$, it yields

$$\chi_n^{(\lambda)}(\chi_m^{(v)}(p(z))) = \chi_m^{(v)}(\chi_n^{(\lambda)}(p(z))) = \chi_{n+m}^{(\lambda)}(p(z)).$$

We note here that the case $\mu = 0$, $\eta > 0$ and $\gamma = 0$ yields $P_{\eta,n}(0) \equiv P_n$, the iterated integral transforms of Caratheodory maps defined in [1].

Next, we characterize the new iterations, $p_{\lambda,n}(z)$. The proofs follow *mutatis mutandis* as in [1]. However we again give the proofs for completeness and clarity of this paper, noting that Re $\lambda > 0$ where appropriate.

Theorem 1. *Let $\gamma \neq 1$ be a non negative real number. Then for $n \in N$*

$$Re\, p_{\lambda,n-1}(z) > \gamma \text{ implies } Re\, p_{\lambda,n}(z) > \gamma, \text{ for } 0 \leq \gamma < 1$$

and

$$Re\, p_{\lambda,n-1}(z) < \gamma \text{ implies } Re\, p_{\lambda,n}(z) < \gamma, \text{ for } \gamma > 1.$$

Proof. On differentiation of (1.5), we have

$$\lambda\, z^{\lambda-1}\, p_{\lambda,n}(z) + z^{\lambda}\, p'_{\lambda,n}(z) = \lambda\, z^{\lambda-1} p_{n-1}(z)$$

which yields

$$p_{\lambda,n}(z) + \frac{z\, p'_{\lambda,n}(z)}{\lambda} = p_{n-1}(z). \tag{2.2}$$

It follows by the proof of Theorem 3.1 in [1] that

$$\mathrm{Re}\left(p_{\lambda,n}(z) + \frac{z\, p'_{\lambda,n}(z)}{\lambda}\right) = \mathrm{Re}\; p_{n-1}(z) > \gamma.$$

This implies that Re $p_{\lambda,n}(z) > \gamma$ for $0 \le \gamma < 1$ and

$$\mathrm{Re}\left(p_{\lambda,n}(z) + \frac{z\, p'_{\lambda,n}(z)}{\lambda}\right) = \mathrm{Re}\; p_{n-1}(z) < \gamma.$$

This implies that Re $p_{\lambda,n}(z) < \gamma$ for $\gamma > 1$. □

Corollary 1.

$$P_{\lambda,n}(\gamma) \subset P(\gamma),\ n \in N.$$

Proof. Since $p_0(z) = p(z) \in P(\gamma)$, then Re $p(z) > \gamma$. Hence by Theorem 1, we have, Re $p_{\lambda,1}(z) > \gamma$. Also $p_1(z) \in P(\gamma)$, thus Re $p_{\lambda,2}(z) > \gamma$ and so on for all $n \in N$. Therefore $P_{\lambda,n}(\gamma) \subset P(\gamma)$. □

Theorem 2.

$$P_{\lambda,n+1}(\gamma) \subset P_{\lambda,n}(\gamma),\ n \in N.$$

Proof. Let $p_{\lambda,n+1} \in P_{\lambda,n+1}(\gamma)$, hence, there exist $p \in P(\gamma)$ such that

$$p_{\lambda,n+1}(z) = \chi^{(\lambda)}_{n+1}(p(z))$$

as earlier noted, then $p_{\lambda,n+1}(z) = \chi^{(\lambda)}_n(\chi^{(\lambda)}_1(p(z)))$ and using Corollary 1, we have $\chi^{(\lambda)}_1(p(z)) \in P(\gamma)$. Hence $p_{\lambda,n+1}$ is the nth integral transform of a function in $P(\gamma)$, that is, $p_{\lambda,n+1}$ belongs to $P_{\lambda,n}$. Therefore $P_{\lambda,n+1}(\gamma) \subset P_{\lambda,n}(\gamma)$. □

Corollary 2. *Let $p \in P(\gamma)$ and Re $\lambda + c > 0$. Then*

$$q(z) = 1 + (1-\gamma)(\lambda + c)\sum_{k=1}^{\infty} \frac{c_k}{\lambda + c + k} z^k, \quad z \in E$$

is also in $P(\gamma)$.

Proof. Let $q(z) \in P_{\lambda+c,1}(\gamma)$, then we have

$$q(z) = p_{\lambda+c,1}(z) = \frac{\lambda+c}{z^{\lambda+c}} \int_0^z t^{\lambda+c-1} p_0(t)\,dt.$$

Hence

$$q(z) = 1 + (1-\gamma)(\lambda+c) \sum_{k=1}^{\infty} \frac{c_k}{\lambda+c+k} z^k$$

is also in $P(\gamma)$. □

In the next result we shall proof the surbordination $p_{\lambda,n} \prec L_{\lambda,n}$ using the technique of Briot-Bouquet differential subordination. A function $p \in P$ is said to satisfy the Briot-Bouquet differential subordination if

$$p(z) + \frac{zp'(z)}{\eta p(z) + \gamma} \prec h(z), \quad z \in E \tag{2.3}$$

where η and γ are complex constants and $h(z)$ a complex function satisfying $h(0) = 1$, and $Re(\eta h(z) + \gamma) > 0$ in E. It is well known that if $p \in P$ satisfies the Briot-Bouquet differential subordination, then $p(z) \prec h(z)$ [5].

A univalent function $q(z)$ is said to be a dominant of (2.3) if $p(z) \prec q(z)$ for all $p(z)$ satisfying (2.3). If $\tilde{q}(z)$ is a dominant of (2.3) and $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (2.3), then $\tilde{q}(z)$ is said to be the best dominant of (2.3).

Theorem 3. $p_{\lambda,n} \prec L_{\lambda,n}$.

From (2.2) we have

$$p_{\lambda,1}(z) + \frac{z\,p'_{\lambda,1}(z)}{\lambda} = p(z).$$

Now since $p \in P$, then $p \prec L_0(z) = (1+(1-2\gamma)z)/(1-z)$ so that

$$p_{\lambda,1}(z) + \frac{z\,p'_{\lambda,1}(z)}{\lambda} \prec L_0(z).$$

However the differential equation

$$q(z) + \frac{zq'(z)}{\lambda} = L_0(z)$$

has univalent solution

$$q_{\gamma}(z) = \frac{\lambda}{z^{\lambda}} \int_0^z t^{\lambda-1} L_0(t)\,dt = L_{\lambda,1}(z)$$

which, by the principle of the Briot-Bouquet differential subordination, is the best dominant. Similarly, from (2.2),

$$p_{\lambda,2}(z) + \frac{z\,p'_{\lambda,2}(z)}{\lambda} \prec L_{\lambda,1}(z),$$

while the differential equation

$$q(z)+\frac{zq'(z)}{\lambda}=L_{\lambda,1}(z)$$

also has univalent solution

$$q_\gamma(z)=\frac{\lambda}{z^\lambda}\int_0^z t^{\lambda-1}L_{\lambda,1}(t)\,dt=L_{\lambda,2}(z)$$

which again, by the principle of the Briot-Bouquet differential subordination, is the best dominant. Continuing, we have that

$$p_{\lambda,n}(z)+\frac{z\,p'_{\lambda,n}(z)}{\lambda}\prec L_{\lambda,n-1}(z),$$

with the differential equation

$$q(z)+\frac{zq'(z)}{\lambda}=L_{\lambda,n-1}(z)$$

having univalent solution

$$q_\gamma(z)=\frac{\lambda}{z^\lambda}\int_0^z t^{\lambda-1}L_{\lambda,n-1}(t)\,dt=L_{\lambda,n}(z).$$

This implies that $p_{\lambda,n}\prec L_{\lambda,n}$ which completes the proof.

Theorem 4. *$P_{\lambda,n}(\gamma)$ is a convex set.*

Proof. Let $p_{\lambda,n}(\gamma), q_{\lambda,n}(\gamma)\in P_{\lambda,n}(\gamma)$. Then for non negative real numbers τ_1 and τ_2 with $\tau_1+\tau_2=1$, we have

$$\tau_1 p_{\lambda,n}(z)+\tau_2 q_{\lambda,n}(z)=\frac{\lambda}{z^\lambda}\int_0^z t^{\lambda-1}\left[\tau_1 p_{n-1}+\tau_2 q_{n-1}\right](t)\,dt$$

with $\tau_1 p_{\lambda,0}(z)+\tau_2 q_{\lambda,0}(z)=p(z)\in P(\gamma)$.
Now for $n=1$, it yields

$$\tau_1 p_{\lambda,1}(z)+\tau_2 q_{\lambda,1}(z)=1+(1-\gamma)\sum_{k=1}^{\infty}\frac{\lambda}{\lambda+k}c_k z^k\in P_{\lambda,n}(\gamma).$$

Suppose, it is true for $n=v-1$, thus

$$\tau_1 p_{\lambda,v-1}(z)+\tau_2 q_{\lambda,v-1}(z)=1+(1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^{v-1}c_k z^k\in P_{\lambda,n}(\gamma).$$

Now for $n=v$

$$\begin{aligned}\tau_1 p_{\lambda,v}(z)+\tau_2 q_{\lambda,v}(z)&=\frac{\lambda}{z^\lambda}\int_0^z t^{\lambda-1}\left[1+(1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^{v-1}c_k t^k\right]dt\\&=1+(1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^{v}c_k z^k\in P_{\lambda,n}(\gamma).\end{aligned}$$

Therefore, $P_{\lambda,n}(\gamma)$ is a convex set. □

The lower bound in the next theorem is not the best possible. We provide the proof of our estimate and follow that with our expectation.

Theorem 5. *Let* $p_{\lambda,n} \in P_{\lambda,n}(\gamma)$. *Then*

$$|p_{\lambda,n}(z)| \le 1 + 2(1-\gamma)\sum_{k=1}^{\infty}\left|\frac{\lambda}{\lambda+k}\right|^{n} r^{k},\ |z| = r$$

and

$$Re\, p_{\lambda,n}(z) \ge 1 - 2(1-\gamma)\sum_{k=1}^{\infty}\left|\frac{\lambda}{\lambda+k}\right|^{n} r^{k},\ |z| = r.$$

Proof. For the lower bound, then from (2.1), we have

$$|p_{\lambda,n}(z) - 1| \le 2(1-\gamma)\sum_{k=1}^{\infty}\left|\frac{\lambda}{\lambda+k}\right|^{n} r^{k}.$$

Hence

$$\text{Re}\left(p_{\lambda,n}(z) - 1\right) \ge -2(1-\gamma)\sum_{k=1}^{\infty}\left|\frac{\lambda}{\lambda+k}\right|^{n} r^{k}$$

so that

$$\text{Re}\ p_{\lambda,n}(z) \ge 1 - 2(1-\gamma)\sum_{k=1}^{\infty}\left|\frac{\lambda}{\lambda+k}\right|^{n} r^{k}.$$

□

Conjecture: Let $p_{\lambda,n} \in P_{\lambda,n}(\gamma)$. Then

$$\text{Re}\ p_{\lambda,n}(z) \ge 1 + 2(1-\gamma)\sum_{k=1}^{\infty}\left|\frac{\lambda}{\lambda+k}\right|^{n} (-r)^{k},\ |z| = r.$$

3. Some applications of $P_{\lambda,n}(\gamma)$

Here we present several different applications of the iterations $p_{\lambda,n}$ of Caratheodory maps to the study of a new subclass $B_n^{\lambda}(\gamma)$ of Bazilevič maps. We begin with

Definition 3.1. A regular map $f \in A$ belongs to the class $B_n^{\lambda}(\gamma)$ if and only if

$$\frac{D^n f(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}} \in P_{\lambda}(\gamma)$$

where all parameters have their usual definitions.

Remark 1. (i) If $\gamma = 0$, we recovers the class $B_n(\lambda)$ studied in [3].
(ii) If $\mu = 0$, we then deduce the class $B_n^{\eta}(\gamma)$ which is equivalent to class $T_n^{\eta}(\gamma)$ studied in [1, 6] with change in notation as a matter of convenience. (iii) For all $f \in B_n^{\lambda}(\gamma)$, Equations (1.4) and (1.5) hold for some $p \in P_{\lambda}(\gamma)$. Hence we have the following relation between $P_{\lambda,n}(\gamma)$ and $B_n^{\lambda}(\gamma)$ which will be used in our characterization of the class $B_n^{\lambda}(\gamma)$:

Lemma 1. *Let $f \in A$, then the following are equivalent:*

(i) $f \in B_n^{\lambda}(\gamma)$,

(ii) $\frac{D^n f(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}} \in P_{\lambda}(\gamma)$,

(iii) $\frac{\lambda f(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} \in P_{\lambda,n}(\gamma)$.

Proof. If $f \in A$, then by Definition 3.1 it is clear that (i)$\Leftrightarrow$(ii). Now (ii) is true $\Leftrightarrow$ there exist $h \in P(\gamma)$ such that

$$\frac{D^n f(z)^{\lambda}}{\eta\lambda^{n-1}z^{\lambda}} = h(z) = p(z) + \frac{i\mu}{\eta} = 1 + \frac{i\mu}{\eta} + (\gamma)\sum_{k=1}^{\infty} c_k z^k$$

for some $p \in P(\gamma)$, so that

$$D^n f(z)^{\lambda} = \lambda^n z^{\lambda} + (1-\gamma)\sum_{k=1}^{\infty} \eta\lambda^{n-1} c_k z^{\lambda+k}. \tag{3.1}$$

Applying on (3.1) the integral operator, I_n, defined by Salagean (1983) as

$$I_n f(z) = I(I_{n-1}f(z)) = \int_0^z \frac{I_{n-1}f(t)}{t}dt$$

with $I_0 f(z) = f(z)$ and with some computation we have

$$\frac{\lambda f(z)^{\lambda}}{\eta z^{\lambda}} = 1 + i\frac{\mu}{\eta} + (1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^n c_k z^k.$$

Therefore

$$\frac{\lambda f(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} = 1 + (1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^n c_k z^k \in P_{\lambda,n}(\gamma).$$

This proves the Lemma. □

Now we discuss properties of the class $B_n^{\lambda}(\gamma)$.

Theorem 6.

$$B_{n+1}^{\lambda}(\gamma) \subset B_n^{\lambda}(\gamma).$$

Proof. Let $f \in B^{\lambda}_{n+1}(\gamma)$. Then by Lemma 1 we have

$$\frac{\lambda f(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} = 1 + (1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^{n+1} c_k z^k$$

which implies

$$\frac{\lambda f(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} \in P_{\lambda,n+1}(\gamma).$$

Then by Theorem 2

$$\frac{\lambda f(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} \in P_{\lambda,n}(\gamma).$$

Hence by Lemma 1 again, $f \in B^{\lambda}_{n}(\gamma)$. □

Corollary 3. *For $n \geq 1$, $B^{\lambda}_{n}(\gamma)$ consists of schlicht maps only.*

Proof. For $n \geq 1$, by Theorem 7 and Remark 3 we have $B^{\lambda}_{n+1}(\gamma) \subset B^{\lambda}_{n}(\gamma)$ and $B^{\lambda}_{n}(\gamma) \subset B_n(\lambda)$, which is known to consist only of schlicht maps for $n \geq 1$ (see Corollary 1 in [1]). Therefore the assertion follows. □

Theorem 7. *Let $f \in B^{\lambda}_{n}(\gamma)$. Then the integral*

$$F(z)^{\lambda} = \frac{\lambda+c}{z^c}\int_0^z t^{c-1} f(t)^{\lambda}\,dt \tag{3.2}$$

is also in $B^{\lambda}_{n}(\gamma)$.

Proof. From (3.2) we have

$$\frac{\lambda F(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} = \frac{\lambda+c}{z^{\lambda+c}}\int_0^z t^{\lambda+c-1}\left(\frac{\lambda f(t)^{\lambda}}{\eta t^{\lambda}} - i\frac{\mu}{\eta}\right)dt.$$

Now suppose $f \in B^{\lambda}_{n}(\gamma)$ and let $v = \lambda + c$ be a complex number, then we have

$$\frac{\lambda F(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} = \frac{u}{z^{u}}\int_0^z t^{u-1}\left(\frac{\lambda f(t)^{\lambda}}{\eta t^{\lambda}} - i\frac{\mu}{\eta}\right)dt.$$

Hence, we have

$$\frac{F(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} = \chi_1^{(v)}(\chi_n^{(\lambda)}(p(z))) = \chi_n^{(\lambda)}(\chi_1^{(v)}(p(z))).$$

With Corollary 1, $\chi_1^{(v)}(p(z)) \in P(\gamma)$, thus $\chi_n^{(\lambda)}(\chi_1^{(v)}(p(z))) \in P_{\lambda,n}(\gamma)$, which implies that $\frac{\lambda F(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} \in P_{\lambda,n}(\gamma)$. Therefore by Lemma 1, $F \in B^{\lambda}_{n}(\gamma)$. □

Theorem 8. *Let $f \in B_n^\lambda(\gamma)$ and define*

$$M_T(n,\lambda^*,\gamma,r) = 1+2(1-\gamma)\sum_{k=1}^{\infty}\lambda_k^* r^k$$

and

$$m_T(n,\lambda^*,\gamma,r) = 1-2(1-\gamma)\sum_{k=1}^{\infty}\lambda_k^* r^k$$

where $\lambda^* = \eta|\lambda|^{n-1}/|\lambda+k|^n$. Then

$$m_T(n,\lambda^*,\gamma,r) \le \left|\left(\frac{f(z)}{z}\right)^\lambda\right| \le M_T(n,\lambda^*,\gamma,r).$$

The inequality on the right side is sharp while the one on the left is not.

Proof. Taking $p_{\lambda,n}(z) = \frac{\lambda f(z)^\lambda}{\eta z^\lambda} - i\frac{\mu}{\eta}$ in Theorem 6, we have

$$\frac{\lambda f(z)^\lambda}{\eta z^\lambda} - i\frac{\mu}{\eta} = 1+(1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^n c_k z^k.$$

Then

$$\left(\frac{f(z)}{z}\right)^\lambda = 1+(1-\gamma)\eta\lambda^{n-1}\sum_{k=1}^{\infty}\frac{c_k}{(\lambda+k)^n}z^k.$$

Using triangle inequality, then we have

$$\begin{aligned}\left|\left(\frac{f(z)}{z}\right)^\lambda\right| &\le 1+2(1-\gamma)\eta|\lambda|^{n-1}\sum_{k=1}^{\infty}\frac{r^k}{|\lambda+k|^n}\\ &\le 1+2(1-\gamma)\sum_{k=1}^{\infty}\lambda_k^* r^k.\end{aligned}$$

For lower bound, we have

$$\operatorname{Re}\left(\frac{f(z)^\lambda}{z^\lambda}-1\right) \ge -2(1-\gamma)\eta|\lambda|^{n-1}\sum_{k=1}^{\infty}\frac{r^k}{|\lambda+k|^n}.$$

Hence

$$\operatorname{Re}\left(\frac{f(z)}{z}\right)^\lambda \ge 1-2(1-\gamma)\sum_{k=1}^{\infty}\lambda_k^* r^k$$

which concludes the proof. □

Theorem 9. *Let $f \in B_n^\lambda(\gamma)$ and define*

$$M_T^*(n,\lambda^{**},\gamma,r) = 1+2(1-\gamma)\sum_{k=1}^{\infty}\lambda_k^{**}\, r^k$$

and

$$m_T^*(n,\lambda^{**},\gamma,r) = 1-2(1-\gamma)\sum_{k=1}^{\infty}\lambda_k^{**}\, r^k$$

where $\lambda^{**} = \eta|\lambda|^{n-2}/|\lambda+k|^{n-1}$. *Then*

$$m_T^*(n,\lambda^{**},\gamma,r) \le \left|\frac{f(z)^{\lambda-1}f'(z)}{z^{\lambda-1}}\right| \le M_T^*(n,\lambda^{**},\gamma,r).$$

The inequality on the right side is sharp while the one on the left side is not.

Proof. Since $f \in B_n^{\lambda}(\gamma)$, then by Lemma 1 there exist $p_{\lambda,n} \in P_{\lambda,n}(\gamma)$ such that

$$\frac{\lambda f(z)^{\lambda}}{\eta z^{\lambda}} - i\frac{\mu}{\eta} = p_{\lambda,n}(z).$$

Then we differentiate to get

$$\frac{\lambda f(z)^{\lambda-1}f'(z)}{\eta z^{\lambda-1}} - i\frac{\mu}{\eta} = p_{\lambda,n}(z) + \frac{zp'_{\lambda,n}(z)}{\lambda}. \tag{3.3}$$

Hence from (1.5) we have

$$p_{\lambda,n}(z) + \frac{zp'_{\lambda,n}(z)}{\lambda} = p_{\lambda,n-1}(z). \tag{3.4}$$

Now substitute (3.4) in (3.3), then (3.3) now becomes

$$\frac{\lambda f(z)^{\lambda-1}f'(z)}{\eta z^{\lambda-1}} - i\frac{\mu}{\eta} = p_{\lambda,n-1}(z).$$

with

$$p_{\lambda,n-1}(z) = 1+(1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^{n-1} c_k z^k.$$

Hence by triangular inequality, we have

$$\begin{aligned}\left|\frac{f(z)^{\lambda-1}f'(z)}{z^{\lambda-1}}\right| &\le 1+2(1-\gamma)\eta|\lambda|^{n-2}\sum_{k=1}^{\infty}\frac{r^k}{|\lambda+k|^{n-1}}\\ &\le 1+2(1-\gamma)\sum_{k=1}^{\infty}\lambda_k^{**}\, r^k.\end{aligned}$$

This bound is sharp. For the lower bound, we have

$$\frac{\lambda f(z)^{\lambda-1}f'(z)}{\eta z^{\lambda-1}} - i\frac{\mu}{\eta} - 1 = (1-\gamma)\sum_{k=1}^{\infty}\left(\frac{\lambda}{\lambda+k}\right)^{n-1} c_k z^k.$$

$$\begin{aligned}\operatorname{Re}\left(\frac{f(z)^{\lambda-1}f'(z)}{z^{\lambda-1}} - \frac{i\mu}{\eta} - 1\right) &\ge -2(1-\gamma)\eta|\lambda|^{n-2}\sum_{k=1}^{\infty}\frac{r^k}{|\lambda+k|^{n-1}}\\ &\ge -2(1-\gamma)\sum_{k=1}^{\infty}\lambda_k^{**}\, r^k.\end{aligned}$$

Therefore

$$\operatorname{Re}\left(\frac{f(z)^{\lambda-1}f'(z)}{z^{\lambda-1}}\right) \geq 1-2(1-\gamma)\sum_{k=1}^{\infty}\lambda_k^{**}\,r^k.$$

The bound is not the best possible. □

Theorem 10. *The set of points* $\frac{\lambda f(z)^{\lambda}}{\eta z^{\lambda}} - \frac{i\mu}{\eta}$ *for each* $f \in B_n^{\lambda}(\gamma)$, $z \in E$ *is convex.*

Proof. Since for each $f \in B_n^{\lambda}(\gamma)$, the point $\frac{\lambda f(z)^{\lambda}}{\eta z^{\lambda}} - \frac{i\mu}{\eta} \in P_{\lambda,n}(\gamma)$, then by Theorem 5, the result follows. □

4. Consequences

We state here the analogues of Theorems 8 to 11 for class $B(\alpha,\beta) \equiv B(\alpha,\beta,z)$ of Bazilevic maps. This is achieved by setting $\lambda = \alpha/(1+i\beta)$ so that $\eta = \alpha/(1+\beta^2)$, $\mu = -\alpha\beta/(1+\beta^2)$; and taking $n = 1$ and $\gamma = 0$ in the theorems, we have respectively:

Corollary 4. *Let* $f \in B(\alpha,\beta)$. *Then the function* $F(z)$ *defined by*

$$F(z)^{\frac{\alpha}{1+i\beta}} = \frac{\alpha + c(1+i\beta)}{(1+i\beta)z^c}\int_0^z t^{c-1} f(t)^{\frac{\alpha}{1+i\beta}}\,dt$$

is also in $B(\alpha,\beta)$.

Corollary 5. *Let* $f \in B(\alpha,\beta)$ *and define*

$$M_T(\alpha,\beta,r) = 1+2\sum_{k=1}^{\infty}\frac{\alpha}{\sqrt{(1+\beta^2)[(\alpha+k)^2+\beta^2k^2]}}r^k$$

and

$$m_T(\alpha,\beta,r) = 1-2\sum_{k=1}^{\infty}\frac{\alpha}{\sqrt{(1+\beta^2)[(\alpha+k)^2+\beta^2k^2]}}r^k.$$

Then $m_T(\alpha,\beta,r) \leq \left|\left(\frac{f(z)}{z}\right)^{\frac{\alpha}{1+i\beta}}\right| \leq M_T(\alpha,\beta,r)$. *The inequality on the right side is sharp while the one on the left is not.*

Corollary 6. *Let* $f \in B(\alpha,\beta)$ *and define*

$$M_T^*(\alpha,\beta,r) = 1+2\sum_{k=1}^{\infty}\frac{r^k}{\sqrt{1+\beta^2}}$$

and

$$m_T^*(\alpha,\beta,r) = 1-2\sum_{k=1}^{\infty}\frac{r^k}{\sqrt{1+\beta^2}}.$$

Then $m_T^*(\alpha,\beta,r) \leq \left|\left(\frac{f(z)}{z}\right)^{\frac{\alpha-1-i\beta}{1+i\beta}} f'(z)\right| \leq M_T^*(\alpha,\beta,r)$. *The right hand side inequality is sharp while the one on the left is not.*

Corollary 7. *The set of points* $(1-i\beta)\left(\frac{f(z)}{z}\right)^{\frac{\alpha}{1+i\beta}} + i\beta$ *for each* $f \in B(\alpha,\beta)$, $z \in E$ *is convex.*

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