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MANNHEIM B-CURVE COUPLES IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we define null Cartan Mannheim and pseudo null Mannheim curves in Minkowski 3-space according to their Bishop frames. We obtain the necessary and the sufficient conditions for pseudo null curves to be Mannheim B-curves in terms of their Bishop curvatures. We prove that there are no null Cartan curves in Minkowski 3-space which are Mannheim B-curves, by considering the cases when their Mannheim B-mate curves are spacelike, timelike, null Cartan and pseudo null curves. Finally, we give some examples of pseudo null Mannheim B-curve pairs.

1. Introduction

The Bishop frame or relatively parallel adapted frame $\{T, N_1, N_2\}$ of a regular curve in Euclidean 3-space is introduced by R.L. Bishop in [1]. It contains the tangential vector field T and two relatively normal vector fields N_1 and N_2 whose derivatives N_1' and N_2' with respect to the arc-length parameter s of the curve are collinear with the tangential vector field T. The Bishop frame is also known as the frame with *minimal rotation property*, since N_1' and N_2' make minimal rotations in the planes $N_1^{\perp}$ and $N_2^{\perp}$ respectively. A new version of the Bishop frame, type-2 Bishop frame in $\mathbb{E}^3$, is introduced in [18]. In Minkowski space-time $\mathbb{E}_1^4$ and Euclidean space $\mathbb{E}^4$, the Bishop frame is studied in [6] and [8]. In Minkowski 3-space $\mathbb{E}_1^3$, the Bishop frame (parallel frame) of the timelike curve and the spacelike curve with non-null principal normal is obtained in [16]. Recently, the Bishop frames of pseudo null curves and null Cartan curves in $\mathbb{E}_1^3$ are derived in [11] and the Bishop frame of a null Cartan curve in $\mathbb{E}_1^4$ is introduced in [12].

It is well known that in the Euclidean space $\mathbb{E}^3$, there are many associated curves (Bertrand mates, Mannheim mates, spherical images, evolutes, the principal-direction curves, etc.) whose Frenet's frame vectors satisfy some extra conditions. *Mannheim curves* in $\mathbb{E}^3$ are defined by the property that their principal normal lines coincide with the binormal lines of their mate

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curves at the corresponding points [5, 13]. Mannheim curves and their partner curves in 3-dimensional space forms are studied in [3]. In the Euclidean 4-space and Minkowski space-time $\mathbb{E}^4_1$, the notion of Mannheim curves is generalized in [7, 10, 14]. It is proved in [9] that the only pseudo null Mannheim curves according to Frenet frame in Minkowski 3-space are the pseudo null circles whose mate curves are pseudo null straight lines. It is also proved in [9] that there are no null Cartan curves in Minkowski 3-space which are Mannheim curves according to their Cartan frame.

In this paper, we define null Cartan Mannheim and pseudo null Mannheim curves according to their Bishop frames in Minkowski 3-space. We call them *null Cartan Mannheim B-curves* and *pseudo null Mannheim B-curves*. We obtain the necessary and the sufficient conditions for pseudo null curves to be Mannheim B-curves in terms of their Bishop curvatures. In the related examples, we show that there are infinity many pairs of pseudo null Mannhem B-curves. We prove that mate curves of pseudo null Mannheim B-curves can not be spacelike, timelike, or null Cartan curves. We also prove that there are no null Cartan curves which are Mannheim B-curves, by considering the cases when their mate curves are the spacelike, the timelike, the null Cartan and the pseudo null curves.

2. Preliminaries

Minkowski space $\mathbb{E}^3_1$ is the real vector space $\mathbb{E}^3$ equipped with the standard indefinite flat metric $\langle\cdot,\cdot\rangle$ given by

$$\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3,$$

for any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{E}^3_1$. Since $\langle\cdot,\cdot\rangle$ is an indefinite metric, an arbitrary vector $x \in \mathbb{E}^3_1$ can have one of three causal characters: it can be a *spacelike*, a *timelike*, or a *null* (*lightlike*), if $\langle x, x\rangle > 0$, $\langle x, x\rangle < 0$, or $\langle x, x\rangle = 0$ and $x \neq 0$ respectively. In particular, the vector $x = 0$ is said to be spacelike. The *norm* (length) of vector $x \in \mathbb{E}^3_1$ is given by $\|x\| = \sqrt{|\langle x, x\rangle|}$. If $\|x\| = 1$, the vector x is called a *unit*. An arbitrary curve $\alpha : I \to \mathbb{E}^3_1$ can be the *spacelike*, the *timelike* or the *null (lightlike)*, if all of its velocity vectors α' are the spacelike, the timelike or the null ([15]).

A spacelike curve $\alpha : I \to \mathbb{E}^3_1$ is called a *pseudo null curve*, if its principal normal vector field N and binormal vector filed B are null vector fields satisfying the condition $\langle N, B\rangle = 1$. The Frenet formulae of a non-geodesic pseudo null curve α have the form ([17])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & 0 & -\tau \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.1}$$

where the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in arc-length parameter s of α.

A curve $\beta : I \to \mathbb{E}_1^3$ is called a *null curve*, if its tangent vector $\beta' = T$ is a null vector. A null curve β is called a *null Cartan curve*, if it is parameterized by the pseudo-arc function s defined by ([2])

$$s(t) = \int_0^t \sqrt{||\beta''(u)||}\, du. \tag{2.2}$$

There exists a unique Cartan frame $\{T, N, B\}$ along a non-geodesic null Cartan curve β satisfying the Cartan equations ([4])

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\tau & 0 & \kappa \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \tag{2.3}$$

where the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$ is an arbitrary function in pseudo-arc parameter s.

The Frenet equations of a timelike curve, or a spacelike curve with non-null principal normal in $\mathbb{E}_1^3$ according to Bishop frame (parallel transport frame) $\{T_1, N_1, N_2\}$ have the form ([16])

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & -\epsilon_1 k_1 & -\epsilon_2 k_2 \\ \epsilon_0 k_1 & 0 & 0 \\ \epsilon_0 k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix}, \tag{2.4}$$

where T_1, N_1, N_2 are mutually orthogonal vectors satisfying the conditions $\langle T_1, T_1\rangle = \epsilon_0$, $\langle N_1, N_1\rangle = \epsilon_1$, $\langle N_2, N_2\rangle = \epsilon_2$ and $\epsilon_0, \epsilon_1, \epsilon_2 \in \{-1, 1\}$. In particular, it holds $\epsilon_0\epsilon_1\epsilon_2 = -1$. The functions $k_1(s)$ and $k_2(s)$ are called the *first* and the *second Bishop curvature* of the curve, respectively.

The Bishop frames of pseudo null and null Cartan curves are obtained in [11]. If $\{T_1, N_1, N_2\}$ is the Bishop frame of a pseudo null curve, then it satisfies the conditions

$$\begin{aligned} &\langle T_1, T_1\rangle = 1, && \langle N_1, N_1\rangle = \langle N_2, N_2\rangle = 0, \\ &\langle T_1, N_1\rangle = \langle T_1, N_2\rangle = 0, && \langle N_1, N_2\rangle = 1. \end{aligned} \tag{2.5}$$

Analogously, if $\{T_1, N_1, N_2\}$ is the Bishop frame of null Cartan curve, then it satisfies the conditions ([11])

$$\begin{aligned} &\langle T_1, T_1\rangle = \langle N_2, N_2\rangle = 0, && \langle N_1, N_1\rangle = 1, \\ &\langle T_1, N_2\rangle = -1, && \langle T_1, N_1\rangle = \langle N_1, N_2\rangle = 0. \end{aligned} \tag{2.6}$$

Also, the next two theorems are proved in [11].

Theorem 1. *Let α be a pseudo null curve in $\mathbb{E}_1^3$ parameterized by arc-length parameter s with the curvature $\kappa(s) = 1$ and the torsion $\tau(s)$. Then the Bishop frame $\{T_1, N_1, N_2\}$ and the Frenet frame $\{T, N, B\}$ of α are related by:*

(i)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{k_2} & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.7}$$

and the Frenet equations of α *according to the Bishop frame read*

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_2 & k_1 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} \tag{2.8}$$

where $k_1(s) = 0$ *and* $k_2(s) = c_0 e^{\int \tau(s) ds}$, $c_0 \in \mathbb{R}_0^+$ *are the first and the second Bishop curvature;*

(ii)

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -k_1 \\ 0 & -\frac{1}{k_1} & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.9}$$

and the Frenet equations of α *according to the Bishop frame read*

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_2 & k_1 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} \tag{2.10}$$

where $k_1(s) = c_0 e^{\int \tau(s) ds}$, $c_0 \in \mathbb{R}_0^-$ *and* $k_2(s) = 0$ *are the first and the second Bishop curvature.*

Theorem 2. *Let* α *be a null Cartan curve in* $\mathbb{E}_1^3$ *parametrized by pseudo-arc* s *with the curvature* $\kappa(s) = 1$ *and the torsion* $\tau(s)$*. Then the Bishop frame* $\{T_1, N_1, N_2\}$ *and the Cartan frame* $\{T, N, B\}$ *of* α *are related by:*

$$\begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -k_2 & 1 & 0 \\ \frac{k_2^2}{2} & -k_2 & 1 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \tag{2.11}$$

and the Cartan equations of α *according to the Bishop frame read*

$$\begin{bmatrix} T_1' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} k_2 & k_1 & 0 \\ 0 & 0 & k_1 \\ 0 & 0 & -k_2 \end{bmatrix} \begin{bmatrix} T_1 \\ N_1 \\ N_2 \end{bmatrix} \tag{2.12}$$

where the first Bishop curvature $k_1(s) = 1$ *and and the second Bishop curvature satisfies Riccati differential equation* $k_2'(s) = -\frac{1}{2} k_2^2(s) - \tau(s)$.

3. Null Cartan and pseudo null Mannheim B-curves

In this section we define null Cartan Mannheim and pseudo null Mannheim curves according to their Bishop frames in Minkowski 3-space and call them *Mannheim B-curves*. We obtain the necessary and sufficient conditions for pseudo null curves to be Mannheim B-curves and provide the related examples. We prove that mate curves of pseudo null Mannheim B-curves can not be spacelike, timelike or null Cartan curves. We also prove that there are no null Cartan curves which are Mannheim B-curves, by considering the cases when their Mannheim B-mate curves are the spacelike, timelike, null Cartan and pseudo null curves.

Definition 1. Let β be a pseudo null curve or a null Cartan curve in $\mathbb{E}_1^3$ with the Bishop frame $\{T_1, N_1, N_2\}$ and $\beta^\star$ an arbitrary curve with the Bishop frame $\{T_1^\star, N_1^\star, N_2^\star\}$. If the Bishop vector N_1 is collinear with the Bishop vector $N_2^\star$ at the corresponding points of the curves β and $\beta^\star$, then β is called the *Mannheim B-curve*, $\beta^\star$ is called *Mannheim B-mate curve* of β and $(\beta, \beta^\star)$ curve couple is called *Mannheim B-pair*.

In the first theorem, we give the necessary and the sufficient conditions for pseudo null curve couple $(\beta, \beta^\star)$ to be Mannheim B-pair of curves with non-zero Bishop curvatures κ_2 and $\kappa_1^\star$.

Theorem 3. *Let β and $\beta^\star$ be two pseudo null curves in $\mathbb{E}_1^3$ parameterized by arc-length parameters s and $s^\star$ respectively with the Bishop curvatures $\kappa_1 = \kappa_2^\star = 0$, κ_2 and $\kappa_1^\star$. Then $(\beta, \beta^\star)$ curve couple is a Mannheim B-pair if and only if*

$$\kappa_2 + \lambda'' = c\kappa_1^\star,$$

where $\lambda \neq 0$ is an arbitrary differentiable function and $c \in \mathbb{R}_0$.

Proof. Assume that $(\beta, \beta^\star)$ is Mannheim B-pair of curves. Then we can write the curve $\beta^\star$ as

$$\beta^\star(s^\star) = \beta^\star(f(s)) = \beta(s) + \lambda(s)N_1(s), \tag{3.1}$$

where $\lambda(s) \neq 0$ is some differentiable function. Differentiating (3.1) with respect to s, we find

$$T^\star f' = T + \lambda' N_1 + \lambda N_1'.$$

By using (2.7), (2.8) and (2.9), the previous relation becomes

$$-T_1^\star f' = T_1 + \lambda' N_1. \tag{3.2}$$

By taking the scalar product of (3.2) with $-T_1^\star f'$ and using (2.5), we obtain

$$\langle T_1^\star f', T_1^\star f' \rangle = f'^2 = 1. \tag{3.3}$$

Let us take $f' = 1$. Then we have

$$T_1^\star = -T_1 - \lambda' N_1. \tag{3.4}$$

Differentiating (3.4) with respect to s and using (2.8) and (2.10), we get

$$\kappa_1^\star N_2^\star = -\left(\kappa_2 + \lambda''\right) N_1 \tag{3.5}$$

which implies

$$N_2^\star = -\frac{\kappa_2 + \lambda''}{\kappa_1^\star} N_1.$$

Differentiating the last equation with respect to s and using (2.8) and (2.10), we find

$$\kappa_2 + \lambda'' = c\kappa_1^\star,$$

where $c \in \mathbb{R}_0$.

Conversely, assume that $\kappa_2 + \lambda'' = c\kappa_1^\star$, where $\lambda \neq 0$ is an arbitrary differentiable function and $c \in \mathbb{R}_0$. Define the curve $\beta^\star$ by

$$\beta^\star(s) = \beta(s) + \lambda(s) N_1(s). \tag{3.6}$$

Differentiating (3.6) with respect to s and using (2.7) and (2.8), we find

$$\frac{d\beta^\star}{ds} = T_1 + \lambda' N_1, \tag{3.7}$$

which together with relation (2.5) leads to

$$\langle \frac{d\beta^\star}{ds}, \frac{d\beta^\star}{ds} \rangle = 1.$$

Therefore, the curve $\beta^\star$ is parameterized by arc-length parameter s. Then from (2.9) and (3.7) we have

$$T^\star = -T_1^\star = T_1 + \lambda' N_1. \tag{3.8}$$

Differentiating (3.8) with respect to s and using (2.8) and (2.10), we obtain

$$-\frac{dT_1^\star}{ds} = -\kappa_1^\star N_2^\star = \left(\kappa_2 + \lambda''\right) N_1. \tag{3.9}$$

By using the assumption, we get

$$N_2^\star = -\frac{\kappa_2 + \lambda''}{\kappa_1^\star} = -cN_1.$$

Hence $(\beta, \beta^\star)$ is Mannheim B-pair of curves. □

Example 1. Let us consider a pseudo null curve β in $\mathbb{E}_1^3$ with parameter equation

$$\beta(s) = \left(\frac{s^3}{3} + \frac{s^2}{2}, \frac{s^3}{3} + \frac{s^2}{2}, s\right)$$

and the Frenet frame

$$\begin{aligned} T(s) &= \left(s^2 + s, s^2 + s, 1\right), \\ N(s) &= (2s+1)(1,1,0), \\ B(s) &= \left(-\frac{(s^2+s)^2+1}{2(2s+1)}, \frac{1-\left(s^2+s\right)^2}{2(2s+1)}, -\frac{s^2+s}{2s+1}\right). \end{aligned}$$

A straightforward calculation shows that Frenet curvatures of β read $\kappa(s) = 1$, $\tau(s) = \frac{2}{2s+1}$. According to statement (i) of Theorem 1, the Bishop curvatures of β are given by $\kappa_1(s) = 0$ and $\kappa_2(s) = c_0(2s+1)$, $c_0 \in \mathbb{R}_0^+$. In particular, the Bishop frame of β has the form

$$\begin{aligned} T_1(s) &= \left(s^2 + s, s^2 + s, 1\right), \\ N_1(s) &= \frac{1}{c_0}(1,1,0), \\ N_2(s) &= c_0\left(-\frac{\left(s^2+s\right)^2+1}{2}, \frac{1-\left(s^2+s\right)^2}{2}, -s^2 - s\right). \end{aligned}$$

Let us take $\lambda(s) = -\frac{c_0 s^3}{3}$. Define the curve $\beta^\star$ by

$$\beta^\star(s) = \beta(s) + \lambda(s) N_1(s) = \left(\frac{s^2}{2}, \frac{s^2}{2}, s\right).$$

Therefore, $\beta^\star$ is pseudo null circle with Frenet curvatures $\kappa^\star(s) = 1$, $\tau^\star(s) = 0$. By using statement (ii) of Theorem 1, the Bishop curvatures of $\beta^\star$ are given by

$$\kappa_1^\star(s) = c_1, \quad \kappa_2^\star(s) = 0,$$

and the Bishop frame of $\beta^\star$ reads

$$\begin{aligned} T_1^\star(s) &= -(s, s, 1), \\ N_1^\star(s) &= -c_1\left(\frac{-1-s^2}{2}, \frac{1-s^2}{2}, -s\right), \\ N_2^\star(s) &= -\frac{1}{c_1}(1,1,0), \end{aligned}$$

where $c_1 \in \mathbb{R}_0^-$. Since N_1 and $N_2^\star$ are collinear, $(\beta, \beta^\star)$ is Mannheim B-pair of curves (Figure 1). It can be easily verified that the equation $\kappa_2 + \lambda'' = \frac{c_0}{c_1}\kappa_1^\star$ holds.

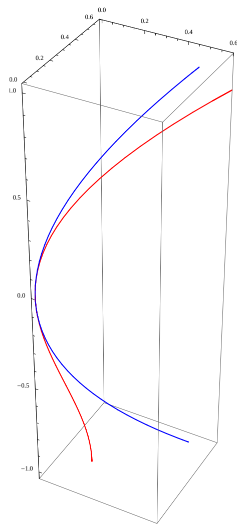

Figure 1: Mannheim B-pair of curves $(\beta, \beta^{\star})$.

In the next theorem, we give the necessary and the sufficient conditions for pseudo null curve couple $(\beta, \beta^{\star})$ to be Mannheim B-pair of curves with non-zero Bishop curvatures κ_1 and $\kappa_2^{\star}$.

Theorem 4. *Let β and $\beta^{\star}$ be two pseudo null curves in $\mathbb{E}_1^3$ parameterized by arc-length parameters s and $s^{\star}$ respectively with the Bishop curvatures κ_1, $\kappa_2^{\star}$ and $\kappa_2 = \kappa_1^{\star} = 0$. Then $(\beta, \beta^{\star})$ curve couple is a Mannheim B-pair if and only if*

$$\kappa_2^{\star} = -\frac{\kappa_1 (\lambda')^2}{c(1+\lambda\kappa_1)^3} \tag{3.10}$$

where c is a non-zero real number and $\lambda(s)$ is differentiable function satisfying differential equation

$$\lambda'(s) \int \kappa_1(s) ds + 2(1 + \lambda(s)\kappa_1(s)) = 0. \tag{3.11}$$

Proof. Assume that $(\beta, \beta^{\star})$ is Mannheim B-pair of curves. Then we can write the curve $\beta^{\star}$ as

$$\beta^{\star}(s^{\star}) = \beta^{\star}(f(s)) = \beta(s) + \lambda(s) N_1(s), \tag{3.12}$$

where $\lambda(s)$ is some differentiable function. Differentiating (3.12) with respect to s and using (2.7), (2.8) and (2.9), we get

$$T_1^{\star} f' = (-1 - \lambda\kappa_1) T_1 + \lambda' N_1. \tag{3.13}$$

By taking the scalar product of (3.13) with $T_1^{\star} f'$ and using (2.5), we obtain

$$f'^2 = (1 + \lambda\kappa_1)^2. \tag{3.14}$$

Let us take $f' = 1 + \lambda\kappa_1$. Then $1 + \lambda\kappa_1 \neq 0$. Putting $a = \lambda'/f'$ in (3.13), we find

$$T_1^\star = -T_1 + aN_1. \tag{3.15}$$

Differentiating (3.15) with respect to s and using (2.8) and (2.10), we obtain

$$\kappa_2^\star f' N_1^\star = -a\kappa_1 T_1 + a' N_1 - \kappa_1 N_2. \tag{3.16}$$

Since $N_1^\star$ is a null vector, relations (2.5) and (3.16) imply

$$-2a'\kappa_1 + a^2\kappa_1^2 = 0.$$

Integrating the previous equation, we obtain

$$a(s) = -\frac{2}{\int \kappa_1(s)ds}.$$

Substituting $a = \lambda'/f'$ in the last equation, it follows that λ satisfies differential equation (3.11).

By taking the scalar product of (3.16) with $N_1 = \mu N_2^\star$, where $\mu(s) \neq 0$ is some differentiable function and using (2.5), we find

$$\mu\kappa_2^\star f' = -\kappa_1. \tag{3.17}$$

Substituting (3.17) in (3.16), we get

$$N_1^\star = a\mu T_1 - \frac{\mu a'}{\kappa_1} N_1 + \mu N_2. \tag{3.18}$$

Differentiating (3.18) with respect to s and using (2.8) and (2.10), we find

$$(a\mu' + 2\mu a')\, T_1 - \left(\frac{\mu a'}{\kappa_1}\right)' N_1 + (\mu' + a\mu\kappa_1)\, N_2 = 0,$$

which implies that

$$a\mu' + 2\mu a' = 0, \quad \frac{\mu a'}{\kappa_1} = \text{constant}, \quad \mu' + a\mu\kappa_1 = 0. \tag{3.19}$$

From the first equation of (3.19) we find

$$\mu = \frac{c}{a^2}, \tag{3.20}$$

where $c \in \mathbb{R}\setminus\{0\}$. The other two equations of (3.19) hold automatically. Substituting relations $f' = 1 + \lambda\kappa_1$ and (3.20) in (3.17), we get that Bishop curvature $\kappa_2^\star$ satisfies (3.10).

Conversely, assume relation (3.10) holds and that λ satisfies differential equation (3.11). Define a curve $\beta^\star$ by

$$\beta^\star(s^\star) = \beta^\star(f(s)) = \beta(s) + \lambda N_1(s). \tag{3.21}$$

Differentiating (3.21) with respect to s and using (2.7), (2.8) and (2.9), we obtain

$$T_1^\star f' = (-1-\lambda\kappa_1)\, T_1 + \lambda' N_1. \tag{3.22}$$

By taking the scalar product of (3.22) with $T_1^\star f'$ and using (2.5), we find $f'^2 = (1+\lambda\kappa_1)^2$. Thus we may take $f' = 1+\lambda\kappa_1$. Then from (3.22) we have

$$T_1^\star = -T_1 + aN_1, \tag{3.23}$$

where $a = \lambda'/f'$. Differentiating (3.23) with respect to s and using (2.8) and (2.10), we find

$$\kappa_2^\star f' N_1^\star = -a\kappa_1 T_1 + a' N_1 - \kappa_1 N_2. \tag{3.24}$$

Therefore,

$$N_1^\star = -\frac{c}{a} T_1 + \frac{ca'}{a^2\kappa_1} N_1 + \frac{c}{a^2} N_2. \tag{3.25}$$

By using the conditions $\langle N_2^\star, N_2^\star\rangle = \langle N_2^\star, T_1^\star\rangle = 0$, $\langle N_1^\star, N_2^\star\rangle = 1$ and relations (3.23) and (3.25), we find

$$N_2^\star = \frac{a^2}{c} N_1 = \frac{1}{\mu} N_1.$$

Hence $(\beta, \beta^\star)$ is Mannheim B-pair of curves. □

Example 2. Consider a unit speed pseudo null curve in $\mathbb{E}_1^3$ with parameter equation

$$\beta(s) = \left(s^3, s^3, s\right).$$

The Frenet frame of β reads

$$\begin{aligned} T(s) &= \left(3s^2, 3s^2, 1\right), \\ N(s) &= 6s\,(1,1,0), \\ B(s) &= (-\frac{9s^4+1}{12s}, \frac{1-9s^4}{12s}, -\frac{s}{2}), \end{aligned}$$

and the Frenet curvatures of β are given by $\kappa(s) = 1$, $\tau(s) = \frac{1}{s}$. According to statement (ii) of Theorem 1, the Bishop curvatures of β read

$$\kappa_1(s) = c_0 s, \quad \kappa_2(s) = 0, \quad c_0 \in \mathbb{R}_0^-.$$

Hence the Bishop frame of β has the form

$$\begin{aligned} T_1(s) &= -\left(3s^2, 3s^2, 1\right), \\ N_1(s) &= c_0\left(\frac{1+9s^4}{12}, \frac{9s^4-1}{12}, \frac{s^2}{2}\right), \end{aligned}$$

$$N_2(s) = -\frac{6}{c_0}(1,1,0).$$

Substituting $\kappa_1(s) = c_0 s$ in (3.11), we get

$$\lambda(s) = -\frac{4}{3c_0 s}.$$

Let us define pseudo null curve $\beta^\star$ by

$$\beta^\star(s) = \beta(s) + \lambda(s)N_1(s).$$

Then $\beta^\star$ has parameter equation

$$\beta^\star(s) = (-\frac{1}{9s}, \frac{1}{9s}, \frac{s}{3}).$$

In particular, the arc-length parameter of β^* is given by $s^\star = \frac{s}{3}$. Therefore, the Frenet frame of $\beta^\star$ reads

$$\begin{aligned} T^\star(s) &= (\frac{1}{27s^{\star 2}}, -\frac{1}{27s^{\star 2}}, 1), \\ N^\star(s) &= (-\frac{2}{27s^{\star 3}}, \frac{2}{27s^{\star 3}}, 0), \\ B^\star(s) &= (\frac{27}{4}s^{\star 3} + \frac{1}{108s^\star}, \frac{27}{4}s^{\star 3} - \frac{1}{108s^\star}, \frac{s^\star}{2}), \end{aligned}$$

and Frenet curvatures of $\beta^\star$ have the form $\kappa^\star(s) = 1$, $\tau^\star(s) = -\frac{3}{s^\star}$. Hence statement (i) of Theorem 1 implies that the Bishop curvatures of $\beta^\star$ read

$$\kappa_1^\star(s) = 0, \quad \kappa_2^\star(s) = \frac{c_1}{s^{\star 3}}, \quad c_1 \in \mathbb{R}_0^+.$$

Also, according to statement (i) of Theorem 1, the Bishop frame of $\beta^\star$ reads

$$\begin{aligned} T_1^\star(s) &= T^\star, \\ N_1^\star(s) &= \frac{s^{\star 3}}{c_1}(-\frac{2}{27s^{\star 3}}, \frac{2}{27s^{\star 3}}, 0), \\ N_2^\star(s) &= \frac{c_1}{s^{\star 3}}(\frac{27}{4}s^{\star 3} + \frac{1}{108s^\star}, \frac{27}{4}s^{\star 3} - \frac{1}{108s^\star}, \frac{s^\star}{2}). \end{aligned}$$

It can be easily verified that

$$N_1 = \frac{9c_0 s^{\star 4}}{c_1} N_2^\star.$$

Consequently, $(\beta, \beta^\star)$ is Mannheim B-pair of curves (Figure 2).

Moreover, it can be easily checked that the equation (3.10) is satisfied.

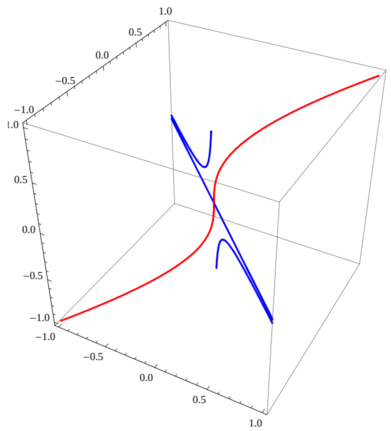

Figure 2: Mannheim B-pair of curves $(\beta, \beta^\star)$.

Remark 1. Note that to any pseudo null curve in $\mathbb{E}^3_1$ with Bishop curvatures $k_1 \neq 0$ and $k_2 = 0$, we may assign function λ as the solution of differential equation (3.11). The function λ determines Mannheim mate B-curve $\beta^\star = \beta + \lambda N_1$ of β. This means that there are infinity many pseudo null Mannheim B-curve couples.

The next theorem can be proved analogously, so we omit its proof.

Theorem 5. *There are no Mannheim B-pair of curves* $(\beta, \beta^\star)$ *in* $\mathbb{E}^3_1$*, where* β *and* $\beta^\star$ *are pseudo null curves with Bishop curvatures* $\kappa_1 = \kappa_1^\star$ *and* $\kappa_2 = \kappa_2^\star$.

Theorem 6. *There are no Mannheim B-pair of curves* $(\beta, \beta^\star)$ *in* $\mathbb{E}^3_1$*, where* β *is a pseudo null curve and* $\beta^\star$ *is a null Cartan curve.*

Proof. Let β be a pseudo null Mannheim B-curve parametrized by arc-length s with the Bishop curvatures κ_1 and κ_2 and Bishop frame $\{T_1, N_1, N_2\}$. Assume that there exists Mannheim B-pair of curves $(\beta, \beta^\star)$, where $\beta^\star$ is a null Cartan curve with Bishop frame $\{T_1^\star, N_1^\star, N_2^\star\}$. Then we can write the curve $\beta^\star$ as

$$\beta^\star(s^\star) = \beta^\star(f(s)) = \beta(s) + \lambda(s) N_1(s), \tag{3.26}$$

where $\lambda(s)$ is some differentiable function. Now we may distinguish two possibilities: (i) $\kappa_1 = 0$, $\kappa_2 \neq 0$ and (ii) $\kappa_1 \neq 0$, $\kappa_2 = 0$.

(i) If $\kappa_1 = 0$ and $\kappa_2 \neq 0$, differentiating (3.26) with respect to s and using (2.7), (2.8) and (2.11), we get

$$T_1^\star f' = T_1 + \lambda' N_1. \tag{3.27}$$

By taking the scalar product of (3.27) with $T_1^\star f'$ and using (2.5) and (2.6), we obtain a contradiction.

(ii) If $\kappa_1 \neq 0$ and $\kappa_2 = 0$, differentiating (3.26) with respect to s and using (2.9), (2.10) and (2.11), we obtain

$$T_1^\star f' = (-1 - \lambda\kappa_1)\,T_1 + \lambda' N_1. \tag{3.28}$$

By taking the scalar product of (3.28) with $N_1 = \mu N_2^\star$ where $\mu(s) \neq 0$ is some differentiable function and using (2.5) and (2.6), we get $f'\mu = 0$ which is a contradiction again. □

The proof of the following theorem follows from the fact that a null Bishop vector N_1 of β can be collinear with non-null Bishop vector $N_2^\star$ of $\beta^\star$.

Theorem 7. *There are no Mannheim B-pair of curves $(\beta, \beta^\star)$ in $\mathbb{E}_1^3$, where β is a pseudo null curve and $\beta^\star$ is a timelike or spacelike curve with non-null Bishop vector $N_2^\star$.*

Theorem 8. *There are no Mannheim B-pair of curves $(\beta, \beta^\star)$ in $\mathbb{E}_1^3$, where β is a null Cartan curve and $\beta^\star$ is a timelike curve, or a spacelike curve with a non-null Bishop vector $N_2^\star$.*

Proof. Assume that there exists Mannheim B-pair of curves $(\beta, \beta^\star)$. Denote by s and $s^\star$ pseudo-arc and arc-length of β and $\beta^\star$ respectively and by $\{T_1, N_1, N_2\}$ and $\left\{T_1^\star, N_1^\star, N_2^\star\right\}$ their Bishop frames. It is sufficient to assume that $N_2^\star$ is a spacelike vector. Otherwise, we easily get a contradiction. We can write the curve β as

$$\beta(s) = \beta(f(s^\star)) = \beta^\star(s^\star) + \lambda(s^\star) N_2^\star(s^\star), \tag{3.29}$$

where $\lambda(s^\star)$ is some differentiable function. Differentiating the relation (3.29) with respect to $s^\star$ and using relations (2.4) and (2.11), we obtain

$$T_1 f' = T_1^\star + \lambda' N_2^\star + \epsilon_0 \lambda \kappa_2^\star T_1^\star.$$

By taking the scalar product of the previous equation with $N_1 = \mu N_2^\star$ where $\mu \neq 0$ is some differentiable function and using (2.6) and the conditions $\langle T_1^\star, N_2^\star\rangle = 0$, $\langle N_2^\star, N_2^\star\rangle = 1$, we get $\lambda' = 0$. Hence

$$T_1 f' = (1 + \epsilon_0 \lambda \kappa_2^\star)\,T_1^\star.$$

This means that a null vector T_1 is collinear with a non-null vector $T_1^\star$, which is a contradiction. □

The proof of the last theorem follows from the fact that a spacelike Bishop vector N_1 of β can not be collinear with a null Bishop vector $N_2^\star$ of $\beta^\star$.

Theorem 9. *There are no Mannheim B-pair of curves $(\beta, \beta^\star)$ in $\mathbb{E}_1^3$, where β is null Cartan curve and $\beta^\star$ is Cartan null or pseudo null curve.*

References

[1] L.R. Bishop, *There is more than one way to frame a curve*, Amer. Math. Monthly 82(3) (1975), 246-251.

[2] W. B. Bonnor, *Null curves in a Minkowski space-time*, Tensor, **20** (1969), 229–242.

[3] J. H. Choi, T. H. Kang and Y. H. Kim, *Mannheim curves in 3-dimensional space forms*, Bull. Korean Math. Soc., **50** (2013), No.4, 1099–1108.

[4] K. L. Duggal and D. H. Jin, Null Curves and Hypersurfaces of Semi Riemann Manifolds, World Scientific, Singapore, 2007.

[5] L. P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces. Dover Edition. Dover Publication, New York, 1960.

[6] M. Erdoğdu, *Parallel frame of non-lightlike curves in Minkowski space-time*, Int. J. Geom. Methods Mod. Phys., **12** (2015), 16 pages.

[7] S. Ersoy, M. Tosun and H. Matsuda, *Generalized Mannheim curves in Minkowski space-time* E_1^4, Hokkaido Math. J., **41** (2012), No.3, 441–461.

[8] F. Gökçelik, Z. Bozkurt, I. Gök, F. N. Ekmekçi and Y. Yaylı, *Parallel transport frame in 4-dimensional Euclidean space* $\mathbb{E}^4$, Caspian J. Math. Sci., **3** (2014), 91–103.

[9] M. Grbović, K. Ilarslan and E. Nešović, *On null and pseudo null Mannheim curves in Minkowski 3-space*, J. Geom., **105** (2014), 177–183.

[10] M. Grbović, K. Ilarslan and E. Nešović, *On generalized null Mannhem curves in Minkowski space-time*, Publ. Inst. Math. (Belgrade), **99** (2016), No.113, 77–98.

[11] M. Grbović and E. Nešović, *On the Bishop frames of pseudo null and null Cartan curves in Minkowski 3-space*, J. Math. Anal. Appl., **461** (2018), 219–233.

[12] K. Ilarslan and E. Nešović, *On Bishop frame of a null Cartan curve in Minkowski space-time*, Int. J. Geom. Meth. Mod. Phys., **15** (2018), No.8, 1850142 (16 pages).

[13] H. Liu and F. Wang, *Mannheim partner curves in* 3*-space*, J. Geom., **88**, (2008), 120–126.

[14] H. Matsuda and S. Yorozu, *On generalized Mannheim curves in Euclidean 4-space*, Nihonkai Math. J., **20** (2009), 33–56.

[15] B. O'Neill, Semi Riemannian Geometry with Applications to Relativity, Academic Press, New York, 1983

[16] M. Özdemir and A. A. Ergin, *Parallel frame of non-lightlike curves*, Missouri J. Math. Sci., **20** (2008), No.2, 127–137.

[17] J. Warave, Curves and surfaces in Minkowski space, Ph.D. Thesis, Leuven University, 1995.

[18] S. Yılmaz and M. Turgut, *A new version of Bishop frame and an application to spherical images*, J. Math. Anal. Appl., **371** (2010), 764–776.

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