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A RESULT ON INVARIANT APPROXIMATION

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Abstract. In this paper an extension of a theorem of Singh on invariant approximation is given. An interesting corollary for Opial spaces is also derived.

The study of fixed points for multivalued contraction and nonexpansive maps using the Hausdorff metric was initiated independently by Markin [5] and Nadler [7]. Later, an interesting and rich fixed point theory for such maps has been developed. For references and a survey of the subject see [10].

Application of fixed point theorems to approximation theory are well known. Among others Brosowski [1], Meinardus [6], and Singh [9] used fixed point theory for single valued maps to prove some interesting results in approximation theory.

In this paper, applying a classical fixed point result of Nadler for multivalued contraction maps we prove a result on invariant approximation for multivalued nonexpansive maps. The result herein extends a well known result of Singh for single valued nonexpansive maps. A corollary in the setting of Opial spaces is also obtained.

We recall the following notions and results needed in the sequel.

Let X be a normed linear space and let K be any nonempty subset of X. For $S = X$ or $S = K$, we denote by 2^s, $\mathcal{CB}(S)$ and $\mathcal{K}(S)$ the collections of all nonempty, nonempty closed bounded, nonempty compact subsets of S, respectively. Let H be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the norm of X, i.e.,

$$H(A,B) = \max\{\sup\{D(a,B): \ a \in A\}, \ \sup\{D(A,b): \ b \in B\}\},$$

for all A, B in $\mathcal{CB}(X)$; here we write $D(x,K) = \inf\{||x-y|| : y \in K\}$, the distance from the point $x \in X$ to the set $K \in 2^X$.

A multivalued map $T : K \subseteq X \to \mathcal{CB}(X)$ is called a *contraction* iff for a fixed constant $h \in (0,1)$ and for each $x, y \in K$,

$$H(T(x),T(y)) \le h||x-y||.$$

Further, if T satisfies

$$H(T(x),T(y)) \le ||x-y||,$$

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then T is said to be *nonexpansive*. An element $x \in K$ is called a fixed point of a multivalued map $T : K \subseteq X \to 2^X$ iff $x \in T(x)$. We denote by $F(T)$ the set of fixed points of T.

A multivalued map T of $K \subseteq X$ into 2^X is said to be *demiclosed* if for every sequence $\{x_n\} \subset K$ and any $y_n \in T(x_n)$, $n = 1, 2, \ldots$, such that $x_n \xrightarrow{w} x$ and $y_n \to y$, we have $x \in K$ and $y \in T(x)$. Here and throughout the paper $\to$ and $\xrightarrow{w}$ denote strong and weak convergence respectively.

A Banach space X is said to satisfy *Opial's property* [8] if for each given sequence $\{x_n\}$ in X with $x_n \xrightarrow{w} x$, we have

$$\liminf_{n\to\infty} ||x_n - x|| < \liminf_{n\to\infty} ||x_n - y||, \qquad \text{for all } y \in X \setminus \{x\}.$$

A Banach space which satisfies Opial's property is called an *Opial space*. It is well known that all Hilbert spaces and $l^p (1 < p < \infty)$ are Opial spaces while L^p spaces $(p \neq 2)$ are not Opial spaces [3,8].

The following useful result, proved originally by LamiDozo [3] (also, see [10], Theorem 1.89), is a special case of Lemma 3.1 in [4].

Lemma 1. *Let K be a nonempty weakly compact subset of an Opial space X. Let $T : K \to \mathcal{K}(K)$ be a nonexpansive multivalued map. Then $I - T$ is demiclosed on K. (Where $I : K \to K$ is an identity map.)*

We also need the following definitions from the approximation theory.

For each $x \in X$ and $K \in 2^X$, we denoted by $P_K(x)$ the set $\{y \in K : ||x - y|| = D(x, K)\}$, which is usually known as the set of best K-approximants to x. The set $P_K(x)$ is closed and bounded, and is convex if K is convex. We also recall that K is *Chebyshev* if for every $x \in X$, there is a unique element $u \in K$ such that $||x - u|| = D(x, K)$. A subset K is said to be *starshaped* with respect to a point $p \in K$ if, for all $x \in K$, $\{tx + (1 - t)p : 0 \leq t \leq 1\} \subset K$. The point p is called a starcenter for K. Clearly, each convex set is starshaped with repsect to each of its points. It is well known [2] that, in the strictly convex Banach space, if $P_k(x)$ is nonempty and starshaped, then it contains at most one element.

If T is a singlevalued or multivalued map defined on X with $T(K) \subseteq K$, then K is called *T-invariant* subset of X.

In [9], Singh has extended the well known result of Brosowski [1], relaxing the condition of linearity of the function and the convexity of the set, proved the following:

Theorem 2. *Let X be a Banach space and let $T : X \to X$ be a single valued nonexpansive map. Let K be a T-invariant subset of X and let $x_0 \in F(T)$. If $P_K(x_0)$ is nonempty, compact, and starshaped, then $P_K(x_0) \cap F(T) \neq \emptyset$.*

In the following we extend Theorem 2 to multivalued nonexpansive maps.

Theorem 3. *Let X be a Banach space, and let $T : X \to \mathcal{CB}(X)$ be a nonexpansive map such that $T(x_0) = \{x_0\}$ for some $x_0 \in X$. Let K be a nonempty T-invariant subset* *of X. Assume that $P_K(x_0)$ is nonempty, weakly compact and starshaped. If $I-T$ is demiclosed on $P_K(x_0)$, then $P_K(x_0)\cap F(T)\neq\emptyset$.*

Proof. Put $M=P_K(x_0)$ and let $u\in M$. Then $u\in K$ and $||x_0-u||=D(x_0,K)$. Let $v\in T(u)\subset K$, then since

$$||v-x_0||\leq H(T(u),T(x_0))\leq ||u-x_0||=D(x_0,K),$$

we have $v\in M$ and thus $T(u)\subset M$. Therefore T carries M into $\mathcal{CB}(M)$. Now, let q be the starcenter of M. Then, for $x\in M$ and $\lambda(0<\lambda<1)$,

$$(1-\lambda)q+\lambda x\in M.$$

Take a sequence $\{\lambda_n\}$ of real numbers such that $0<\lambda_n<1$ and $\lambda_n\to 1$ as $n\to\infty$. Now, for each n define a multivalued map T_n by setting

$$T_n(x)=(1-\lambda_n)q+\lambda_n T(x),\qquad \text{for all } x\in M.$$

Clearly, each T_n is a map from M into $\mathcal{CB}(M)$. Furthermore, for any x, $y\in M$ we have

$$H(T_n(x)),T_n(y))=\lambda_n H(T(x),T(y))\leq\lambda_n||x-y||,$$

which proves that each T_n is a contraction map. Also, since M is a complete metric space, therefore it follows from Nadler's result ([7], Theorem 5), that for each $n\geq 1$, there exists $x_n\in M$ such that $x_n\in T_n(x_n)$. This implies that there is a $y_n\in T(x_n)$ such that

$$x_n=(1-\lambda_n)q+\lambda_n y_n.$$

M being weakly compact, for a convenient subsequence still denoted by $\{x_n\}$, we have $x_n\xrightarrow{w} z\in M$. Now

$$||x_n-y_n||=(1-\lambda_n)||q-y_n||.$$

Put $z_n=x_n-y_n\in(I-T)x_n$, then $z_n\to 0$ as $n\to\infty$. Since $I-T$ is demiclosed on M, it follows that $0\in(I-T)z$, that is, $z\in F(T)$.

By virtue of Lemma 1, we have the following result on invariant approximation for Opial spaces.

Corollary 4. *Let X be an Opial space, and let $T:X\to\mathcal{K}(X)$ be a nonexpansive map such that $T(x_0)=\{x_0\}$ for some $x_0\in X$. Let K be a nonempty T-invariant subset of X. If $P_K(x_0)$ is nonempty, weakly compact and starshaped, then $P_K(x_0)\cap F(T)\neq\emptyset$.*

Proof. Following the proof of Theorem 3, we observe that T maps $M=P_K(x_0)$ into $\mathcal{K}(M)$. Thus by Lemma 1, $I-T$ is demiclosed on M. Now the result follows from Theorem 3.

Remark 5. (1)The classical result of Hicks and Humphries ([2], Theorem 1) is a particular case of Corollary 4.

(2) If X is strictly convex Banach space and $P_K(x_0)$ is nonempty and starshaped in Theorem 3, then we observe that $P_K(x_0) \subset F(T)$.

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References

[1] B. Brosowski, *Fixpunktsatze in der approximations-theorie*, Mathematica (Cluj), **11**(1969), 195-220.
[2] T. L. Hicks and M. D. Humphries, *A note on fixed-point theorems*, J. Approx. Theory, **34**(1982), 221-225.
[3] E. Lami Dozo, *Multivalued nonexpansive mappings and Opial's condition*, Proc. Amer. Math. Soc., **38**(1973), 286-292.
[4] A. Latif and I. Tweddle, *Some results on coincidence points*, Bull. Austral. Math. Soc., **59**(1999), 111-117.
[5] J. T. Markin, *A fixed point theorem for set-valued mappings*, Bull. Amer. Math. Soc., **74**(1968), 639-640.
[6] G. Meinardus, *Invarianze bei linearen approximationen*, Arch. Rational Mech. Anal., **14**(1963), 301-303.
[7] S. B. Nadler, *Multivalued contraction mappings*, Pacific J. Math., **30**(1969), 475-488.
[8] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc., **73**(1967), 591-597.
[9] S. P. Singh, *An application of a fixed-point theorem to approximation theory*, J. Approx. Theory, **25**(1979), 89-90.
[10] S. Singh, B. Watson and P. Srivastava, Fixed Point Theory and Best Approximation: The KKM-map Principle, Kluwer Academic Publishers, 1997.

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