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POINTWISE APPROXIMATION OF MODIFIED CONJUGATE FUNCTIONS BY MATRIX OPERATORS OF THEIR FOURIER SERIES WITH THE USE OF SOME PARAMETERS

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Abstract. We extend and generalize the results of Xh. Z. Krasniqi [Acta Comment. Univ. Tartu. Math. 17 (2013), 89-101] and the authors [Acta Comment. Univ. Tartu. Math. 13 (2009), 11-24], [Proc. Estonian Acad. Sci. 2018, 67, 1, 50–60] as well as the joint paper with M. Kubiak [Journal of Inequalities and Applications (2018) 2018:92]. We consider the modified conjugate function $\widetilde{f}_r$ for $2\pi/\rho$–periodic function f. Moreover, the measure of approximations depends on ρ - differences of the entries of matrices defined the method of summability.

1. Introduction

Let $L^p_{2\pi/\rho}$ $(1 \le p < \infty)$ be the class of all $2\pi/\rho$–periodic real–valued functions, integrable in the Lebesgue sense with p–th power over $Q_\rho = [-\pi/\rho, \pi/\rho]$ with the norm

$$\| f(\cdot)\|_{L^p_{2\pi/\rho}} := \left(\int_{Q_\rho} |f(t)|^p \, dt \right)^{1/p},$$

where $\rho \in \mathbb{N}$. It is clear that $L^p_{2\pi/\rho} \subseteq L^p_{2\pi/1} = L^p_{2\pi}$ and for $f \in L^p_{2\pi/\rho}$

$$\| f(\cdot)\|_{L^p_{2\pi}} = \rho^{1/p} \| f(\cdot)\|_{L^p_{2\pi/\rho}}.$$

Taking into account the above relations, we will consider, for $f \in L^1_{2\pi/\rho}$, the trigonometric Fourier series as such a series of $f \in L^1_{2\pi}$ in the following form:

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

with the partial sums $S_k f$ and the conjugate one

$$\widetilde{S} f(x) := \sum_{\nu=1}^{\infty} (a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x)$$

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with the partial sums $\widetilde{S}_k f$. We also know that if $f \in L^1_{2\pi}$, then

$$\widetilde{f}(x) := -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt = \lim_{\epsilon\to 0^+}\widetilde{f}(x,\epsilon) = \lim_{\epsilon\to 0^+}\widetilde{f}_r(x,\epsilon),$$

where, for $r \in \mathbb{N}$,

$$\widetilde{f}_r(x,\epsilon) := \begin{cases} -\frac{1}{\pi}\left(\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\epsilon}^{\frac{2(m+1)\pi}{r}-\epsilon} + \int_{\frac{2[r/2]\pi}{r}+\epsilon}^{\frac{(2[r/2]+1)\pi}{r}}\right)\psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt & \text{for an odd } r, \\ -\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\epsilon}^{\frac{2(m+1)\pi}{r}-\epsilon}\psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt & \text{for an even } r, \end{cases}$$

and

$$\widetilde{f}(x,\epsilon) = \widetilde{f}_1(x,\epsilon) := -\frac{1}{\pi}\int_{\epsilon}^{\pi}\psi_x(t)\frac{1}{2}\cot\frac{t}{2}dt,$$

with

$$\psi_x(t) := f(x+t) - f(x-t),$$

exist for almost all x (cf. [7, Th.(3.1)IV]).

Let $A := (a_{n,k})$ be an infinite matrix of real numbers such that

$$a_{n,k} \geq 0 \text{ when } k,n = 0,1,2,\ldots, \quad \lim_{n\to\infty} a_{n,k} = 0 \text{ and } \sum_{k=0}^{\infty} a_{n,k} = 1,$$

but $A^\circ := (a_{n,k})_{k=0}^{n}$, where

$$a_{n,k} = 0 \text{ when } k > n.$$

We will use the notations

$$A_{n,r} := \sum_{k=0}^{\infty}\left|a_{n,k} - a_{n,k+r}\right|, \; A^\circ_{n,r} := \sum_{k=0}^{n}\left|a_{n,k} - a_{n,k+r}\right|$$

for $r \in \mathbb{N}$ and

$$\widetilde{T}_{n,A}f(x) := \sum_{k=0}^{\infty} a_{n,k}\widetilde{S}_k f(x) \quad (n = 0,1,2,\ldots).$$

for the A–transformation of $\widetilde{S}f$.

In this paper, we will study the estimate of $\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}_r(x,\epsilon)\right|$ by the function of modulus of continuity type, i.e. nondecreasing continuous function $\widetilde{\omega}$ having the following properties: $\widetilde{\omega}(0) = 0$, $\widetilde{\omega}(\delta_1+\delta_2) \leq \widetilde{\omega}(\delta_1) + \widetilde{\omega}(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1+\delta_2 \leq 2\pi$. We will also consider functions from the following subclass $L^p_{2\pi/\rho}(\widetilde{\omega})_\beta$ of $L^p_{2\pi/\rho}$:

$$L^p_{2\pi/\rho}(\widetilde{\omega})_\beta = \left\{f \in L^p_{2\pi/\rho} : \widetilde{\omega}_\beta(f,\delta)_{L^p_{2\pi/\rho}} = O(\widetilde{\omega}(\delta)) \text{ when } \delta \in [0,2\pi] \text{ and } \beta \geq 0\right\},$$

where

$$\widetilde{\omega}_\beta f(\delta)_{L^p_{2\pi/\rho}} = \sup_{0\leq|t|\leq\delta}\left\{\left|\sin\frac{\rho t}{2}\right|^{\beta}\|\psi_\cdot(t)\|_{L^p_{2\pi/\rho}}\right\}.$$

It is easy to see that $\widetilde{\omega}_0 f(\cdot)_{L^p_{2\pi/\rho}} = \widetilde{\omega} f(\cdot)_{L^p_{2\pi/\rho}}$ is the classical modulus of continuity. Moreover, it is clear that for $\beta \geq \alpha \geq 0$

$$\widetilde{\omega}_\beta f(\delta)_{L^p_{2\pi/\rho}} \leq \widetilde{\omega}_\alpha f(\delta)_{L^p_{2\pi/\rho}}$$

and consequently

$$L^p_{2\pi/\rho}(\widetilde{\omega})_\alpha \subseteq L^p_{2\pi/\rho}(\widetilde{\omega})_\beta .$$

The deviation $\widetilde{T}_{n,A} f(x) - \widetilde{f}_r(x,\epsilon)$ was estimated with $r=1$ in the paper [3] and generalized in [1] as follows:

Theorem A([1, Theorem 8, p. 95])**.** *If* $f \in L^p_{2\pi}(\widetilde{\omega})_\beta$ *with* $1<p<\infty$ and $0 \leq \beta < 1-\frac{1}{p}$, *where* $\widetilde{\omega}$ *satisfies the conditions*

$$\left\{ \int_{\frac{\pi}{n+1}}^{\pi} \left(\frac{t^{-\gamma} \left|\psi_x(t)\right|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x\left((n+1)^\gamma\right) \tag{1}$$

with $0<\gamma<\beta+\frac{1}{p}$ *and*

$$\left\{ \int_0^{\pi/(n+1)} \left(\frac{t \left|\psi_x(t)\right|}{\widetilde{\omega}(t)} \right)^p \sin^{\beta p} \frac{t}{2} dt \right\}^{1/p} = O_x\left((n+1)^{-1}\right), \tag{2}$$

then

$$\left| \widetilde{T}_{n,A^\circ} f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) \right| = O_x\left((n+1)^{\beta+\frac{1}{p}+1} A^\circ_{n,1} \widetilde{\omega}\left(\frac{\pi}{n+1}\right) \right).$$

The next essential generalizations and improvements in [4, Theorem 1] were given. In these results $\widetilde{f}_r(x,\epsilon)$ and $A_{n,r}$ with $r \in \mathbb{N}$ instead of $\widetilde{f}_1(x,\epsilon) = \widetilde{f}(x,\epsilon)$ and $A^\circ_{n,1}$, respectively, were taken. We can formulate them as follows:

Theorem B. [4, Theorem 1] *If* $f \in L^p_{2\pi}$, $1<p<\infty$, $0 \leq \beta < 1-\frac{1}{p}$ *and a function of modulus of continuity type* $\widetilde{\omega}$ *satisfy the conditions:*

$$\left\{ \int_0^{\frac{\pi}{r(n+1)}} \left(\frac{t \left|\psi_x(t)\right| \left|\sin\frac{rt}{2}\right|^\beta}{\widetilde{\omega}(t)} \right)^p dt \right\}^{1/p} = O_x\left((n+1)^{-1}\right), \tag{3}$$

for $r \in \mathbb{N}$,

$$\left\{ \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}} \left(\frac{\left|\psi_x(t)\right| \left|\sin\frac{rt}{2}\right|^\beta}{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)} \right)^p dt \right\}^{1/p} = O_x(1), \tag{4}$$

for a natural $r \geq 3$, *where* $m \in \left\{1,\dots\left[\frac{r}{2}\right]\right\}$ *when* r *is an odd or* $m \in \left\{1,\dots\left[\frac{r}{2}\right]-1\right\}$ *when* r *is an even natural number, and*

$$\left\{ \int_{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}^{\frac{2m\pi}{r}+\frac{\pi}{r}} \left(\frac{\left|\psi_x(t)\right| \left|\sin\frac{rt}{2}\right|^\beta}{\widetilde{\omega}(t)\left(t-\frac{2m\pi}{r}\right)^\gamma} \right)^p dt \right\}^{1/p} = O_x\left((n+1)^\gamma\right), \tag{5}$$

for $r\in\mathbb{N}$ *with* $0<\gamma<\beta+\frac{1}{p}$, *where* $m\in\{0,\dots[\frac{r}{2}]\}$ *when* r *is an odd or* $m\in\{0,\dots[\frac{r}{2}]-1\}$ *when* r *is an even natural number. Moreover, let* $\widetilde{\omega}$ *satisfy, for a natural* $r\geq 2$, *the conditions:*

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{r}-t\right)}\right)^p dt\right\}^{1/p}=O_x(1), \tag{6}$$

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{r(n+1)}}\left(\frac{|\psi_x(t)|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}(t)\left(\frac{2(m+1)\pi}{r}-t\right)^{\gamma}}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{\gamma}\right), \tag{7}$$

with $0<\gamma<\beta+\frac{1}{p}$, *where* $m\in\{0,\dots[\frac{r}{2}]-1\}$. *If a matrix* A *is such that*

$$\left[\sum_{l=0}^{n}\sum_{k=l}^{r+l-1}a_{n,k}\right]^{-1}=O(1), \tag{8}$$

for $r\in\mathbb{N}$ *and*

$$\sum_{k=0}^{\infty}(k+1)^2 a_{n,k}=O\left((n+1)^2\right) \tag{9}$$

are true, then

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\right|=O_x\left((n+1)^{\beta+\frac{1}{p}+1}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right).$$

Theorem C. [4, Theorem 2] *Let* $f\in L^p_{2\pi}$, $1<p<\infty$, $0\leq\beta<1-\frac{1}{p}$ *and a function of modulus of continuity type* $\widetilde{\omega}$ *satisfy, for* $r\in\mathbb{N}$, *the conditions:*

$$\left\{\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{r(n+1)}}\left(\frac{|\psi_x(t)|\left|\sin\frac{rt}{2}\right|^{\beta}}{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)}\right)^p dt\right\}^{1/p}=O_x(1), \tag{10}$$

and (5) *with* $0<\gamma<\beta+\frac{1}{p}$, *where* $m\in\{0,\dots[\frac{r}{2}]\}$ *when* r *is an odd or* $m\in\{0,\dots[\frac{r}{2}]-1\}$ *when* r *is an even natural number. Moreover, let* $\widetilde{\omega}$ *satisfy, for natural* $r\geq 2$, *the conditions:* (6) *and* (7) *with* $0<\gamma<\beta+\frac{1}{p}$, *where* $m\in\{0,\dots[\frac{r}{2}]-1\}$. *If a matrix* A *is such that* (8) *for* $r\in\mathbb{N}$ *and*

$$\sum_{k=0}^{\infty}(k+1)a_{n,k}=O(n+1), \tag{11}$$

are true, then

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{r(n+1)}\right)\right|=O_x\left((n+1)^{\beta+\frac{1}{p}+1}A_{n,r}\widetilde{\omega}\left(\frac{\pi}{n+1}\right)\right). \tag{12}$$

The another generalization in [2, Theorem 1] was given. In this paper the same estimate as in (12) but with $f\in L^p_{2\pi/r}$ and with simpler assumptions on a function $\widetilde{\omega}$ was obtained.

Now, we will generalize the above results using different parameters on the left and right hand sides of the estimation (12). Namely, we will estimate the deviations $\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{s(n+1)}\right)\right|$ but as a measure of such approximation will be use the quantity $A_{n,\rho}$, where ρ is not necessary equal s or r.

In the paper $\sum_{k=a}^{b}=0$ when $a>b$.

2. Statement of the results

At the beginning we will present the estimate of the quantity $\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{s(n+1)}\right)\right|$. Finally, we will formulate some remarks and corollaries.

Theorem 1. *Let $\rho,r,s\in\mathbb{N}$, $s\geq\rho\geq r$ and $r\mid\rho$. Suppose that $f\in L^p_{2\pi/\rho}$, $1<p<\infty$, $0\leq\beta<1-\frac{1}{p}$ and a function of the modulus of continuity type $\widetilde{\omega}$ satisfies the conditions:*

$$\left\{\int_0^{\frac{\pi}{s(n+1)}}\left(\frac{|\psi_x(t)|\left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)}\right)^p dt\right\}^{1/p}=O_x(1) \tag{13}$$

and

$$\left\{\int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}}\left(\frac{|\psi_x(t)|\left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)\,t^{\gamma}}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{\gamma}\right), \tag{14}$$

with $0<\gamma<\beta+\frac{1}{p}$. If a matrix A is such that (11) and

$$\left[\sum_{l=0}^{n}\sum_{k=l}^{\rho+l-1}a_{n,k}\right]^{-1}=O(1), \tag{15}$$

for $\rho\in\mathbb{N}$ are true, then

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{s(n+1)}\right)\right|=O_x\left((n+1)^{\beta+\frac{1}{p}+1}A_{n,\rho}\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right)\right).$$

Theorem 2. *Let $\rho,r,s\in\mathbb{N}$, $s\geq\rho\geq r$ and $r\mid\rho$. Suppose that $f\in L^p_{2\pi/\rho}$, $1<p<\infty$, $0\leq\beta<1-\frac{1}{p}$ and a function of the modulus of continuity type $\widetilde{\omega}$ satisfies the conditions:*

$$\left\{\int_0^{\frac{\pi}{s(n+1)}}\left(\frac{t\,|\psi_x(t)|\left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{-1}\right) \tag{16}$$

when $\rho=1$ or (13) when $\rho\geq 2$ and (14) with $0<\gamma<\beta+\frac{1}{p}$. If a matrix A is such that (9) and (15) for $\rho\in\mathbb{N}$ are true, then

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{s(n+1)}\right)\right|=O_x\left((n+1)^{\beta+\frac{1}{p}+1}A_{n,\rho}\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right)\right).$$

Remark 1. We note that our extra conditions (15), (9) and (11) for a lower triangular infinite matrix A° always hold. Additionally, we can observe that the Hölder inequality gives

$$\sum_{k=0}^{\infty}(k+1)\,a_{n,k}=\sum_{k=0}^{\infty}(k+1)\,a_{n,k}^{1/2}a_{n,k}^{1/2}\leq\left[\sum_{k=0}^{\infty}(k+1)^2\,a_{n,k}\right]^{1/2}\left[\sum_{k=0}^{\infty}a_{n,k}\right]^{1/2}$$
$$=\left[\sum_{k=0}^{\infty}(k+1)^2\,a_{n,k}\right]^{1/2}$$

and thus the condition (9) implies (11) but the condition (13) implies (16). Therefore Theorems 1 and 2 are not comparable.

Corollary 1. *Taking $r=s=\rho=1$ the conditions* (16) *and* (14) *in Theorem 2 reduce to* (2) *and* (1)*, respectively. Thus we obtain the results from [3] and Theorem A [1, Theorem 8, p. 95], but in case [4] we reduce the assumptions. Taking only $r=s=\rho$ we obtain the results from [2].*

Next, using more natural conditions we can formulate without proof the following theorem.

Theorem 3. *Let $f\in L^p_{2\pi/\rho}$, where $\rho,r,s\in\mathbb{N}$, $s\geq\rho\geq r$ such that $r\mid\rho$, $1<p<\infty$ and $0<\beta<1-\frac{1}{p}$. Suppose that a function of the modulus of continuity type $\widetilde{\omega}$ satisfies the conditions:*

$$\left\{\int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}}\left(\frac{t^{-\gamma}\left|\psi_x(t)\right|\left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{\gamma-\frac{1}{p}}\right),\tag{17}$$

for $\gamma\in\left(\frac{1}{p},\frac{1}{p}+\beta\right)$ (instead of (14)*), and*

$$\left\{\int_0^{\frac{\pi}{s(n+1)}}\left(\frac{\left|\psi_x(t)\right|\left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{-\frac{1}{p}}\right),\tag{18}$$

(instead of (13)*). If a matrix A is such that* (11) *and* (15) *for $\rho\in\mathbb{N}$ are true, then*

$$\left|\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{s(n+1)}\right)\right|=O_x\left((n+1)^{\beta+1}A_{n,\rho}\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right)\right).\tag{19}$$

Moreover, if a function of the modulus of continuity type $\widetilde{\omega}$ and a matrix A satisfy the following conditions: (17) *with $\gamma\in\left(\frac{1}{p},\frac{1}{p}+\beta\right)$,*

$$\left\{\int_0^{\frac{\pi}{s(n+1)}}\left(\frac{t\left|\psi_x(t)\right|\left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)}\right)^p dt\right\}^{1/p}=O_x\left((n+1)^{-1-\frac{1}{p}}\right)$$

when $\rho=1$ or (18) *when $\rho\geq2$,* (8) *with $\rho\in\mathbb{N}$ and* (9)*, then the estimate* (19) *is also true.*

From Theorem 3 we can obtain the following norm estimate:

Theorem 4. *Let* $f \in L^p_{2\pi/\rho}(\widetilde{\omega})_\beta$, *where* $\rho, r, s \in \mathbb{N}$, $s \geq \rho \geq r$ *such that* $r \mid \rho$, $1 < p < \infty$ *and* $0 < \beta < 1 - \frac{1}{p}$. *If a matrix* A *is such that* (15) *for* $\rho \in \mathbb{N}$ *and* (11) *are true, then*

$$\left\| \widetilde{T}_{n,A} f(\cdot) - \widetilde{f}_r\left(\cdot, \frac{\pi}{s(n+1)}\right) \right\|_{L^p_{2\pi/\rho}} = O_x\left((n+1)^{\beta+1} A_{n,\rho} \widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right)\right).$$

Remark 2. We can observe that the order of the norm approximation in Theorem 4 is essentially better than the order of the norm approximation obtained by all earlier results.

Remark 3. It is clear that for $\rho, r \in \mathbb{N}$, where $\rho \geq r$ and $r \mid \rho$,

$$A_{n,\rho} \leq \frac{\rho}{r} A_{n,r}. \tag{20}$$

Namely, let $m \in \mathbb{N}$ and $\rho = mr$. Then

$$A_{n,\rho} = \sum_{k=0}^{\infty} \left| a_{n,k} - a_{n,k+\rho} \right| \leq \sum_{k=0}^{\infty} \left| \sum_{l=0}^{m-1} \left(a_{n,k+lr} - a_{n,k+(l+1)r} \right) \right|$$
$$\sum_{l=0}^{m-1} \sum_{k=lr}^{\infty} \left| a_{n,k} - a_{n,k+r} \right| \leq m \sum_{k=0}^{\infty} \left| a_{n,k} - a_{n,k+r} \right| = \frac{\rho}{r} A_{n,r}.$$

Under the above Remarks and the above inequality our results also improve and generalize the mentioned result of Xh. Z. Krasniqi [1].

Remark 4. We note that instead of $L^p_{2\pi/\rho}(\widetilde{\omega})_\beta$ one can consider an another subclasses of $L^p_{2\pi/\rho}$ generated by any function of modulus continuity type e. g. $\widetilde{\omega}_x$ such that

$$\widetilde{\omega}_x(f, \delta) = \sup_{|t| \leq \delta} \left| \psi_x(t) \right| \leq \widetilde{\omega}_x(\delta)$$

and

$$\widetilde{\omega}_x(f, \delta) = \frac{1}{\delta} \int_0^\delta \left| \psi_x(t) \right| dt \leq \widetilde{\omega}_x(\delta).$$

3. Auxiliary results

We begin this section by some notations from [6] and [7, Section 5 of Chapter II]. Let for $r = 1, 2, \ldots$

$$D^\circ_{k,r}(t) = \frac{\sin \frac{(2k+r)t}{2}}{2 \sin \frac{rt}{2}}, \qquad \widetilde{D^\circ}_{k,r}(t) = \frac{\cos \frac{(2k+r)t}{2}}{2 \sin \frac{rt}{2}}$$

and

$$\widetilde{D}_{k,r}(t) = \frac{\cos \frac{rt}{2} - \cos \frac{(2k+r)t}{2}}{2 \sin \frac{rt}{2}} = \frac{\cos \frac{rt}{2}}{2 \sin \frac{rt}{2}} - \widetilde{D^\circ}_{k,r}(t).$$

It is clear by [7] that

$$\widetilde{S}_k f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\,\widetilde{D}_{k,1}(t)\,dt$$

and

$$\widetilde{T}_{n,A} f(x) = -\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\sum_{k=0}^{\infty} a_{n,k}\widetilde{D}_{k,1}(t)\,dt.$$

Now, we present very useful property of the modulus of continuity.

Lemma 1 ([7]). *A function $\widetilde{\omega}$ of modulus of continuity type on the interval $[0,2\pi]$ satisfies the following condition*

$$\delta_2^{-1}\widetilde{\omega}(\delta_2) \le 2\delta_1^{-1}\widetilde{\omega}(\delta_1) \text{ for } \delta_2 \ge \delta_1 > 0.$$

Next, we give some the known estimates:

Lemma 2 ([7]). *If $0 < |t| \le \pi$ then*

$$\left|\widetilde{D^{\circ}}_{k,1}(t)\right| \le \frac{\pi}{2|t|},\ \left|\widetilde{D}_{k,1}(t)\right| \le \frac{\pi}{|t|}$$

and, for any real t, we have

$$\left|D^{\circ}_{k,1}(t)\right| \le k+\frac{1}{2},\ \left|\widetilde{D}_{k,1}(t)\right| \le \frac{1}{2}k(k+1)|t|,\ \left|\widetilde{D}_{k,1}(t)\right| \le k+1.$$

Lemma 3 ([5], [6]). *Let $r\in\mathbb{N}$, $l\in\mathbb{Z}$ and $(a_n)\subset\mathbb{C}$. If $t\neq\frac{2l\pi}{r}$ and $m,n\in\mathbb{N}$, then for every $m\ge n$*

$$\sum_{k=n}^{m} a_k \sin kt = -\sum_{k=n}^{m}(a_k - a_{k+r})\widetilde{D^{\circ}}_{k,r}(t) + \sum_{k=m+1}^{m+r} a_k\widetilde{D^{\circ}}_{k,-r}(t) - \sum_{k=n}^{n+r-1} a_k\widetilde{D^{\circ}}_{k,-r}(t),$$
$$\sum_{k=n}^{m} a_k \cos kt = \sum_{k=n}^{m}(a_k - a_{k+r})D^{\circ}_{k,r}(t) - \sum_{k=m+1}^{m+r} a_k D^{\circ}_{k,-r}(t) + \sum_{k=n}^{n+r-1} a_k D^{\circ}_{k,-r}(t).$$

We additionally prove the following estimate as a consequence of Lemma 3.

Lemma 4. *Let $r\in\mathbb{N}$, $l\in\mathbb{Z}$ and $(a_{n,k})\subset\mathbb{R}_0^+$ for $n,k\in\mathbb{N}_0$. If $t\neq\frac{2l\pi}{r}$ and $\lim_{k\to\infty} a_{n,k}=0$ for all $n\in\mathbb{N}$, then*

$$\left|\frac{1}{2}\sum_{k=0}^{\infty} a_{n,k}\cos\frac{(2k+1)t}{2}\right| \le \frac{1}{2\left|\sin\frac{rt}{2}\right|}\left(A_{n,r}+\sum_{k=0}^{r-1} a_{n,k}\right) \le \frac{1}{\left|\sin\frac{rt}{2}\right|}A_{n,r}.$$

Proof. By Lemma 3,

$$\begin{aligned}\frac{1}{2}\sum_{k=0}^{\infty} a_{n,k}\cos\frac{(2k+1)t}{2} &= \frac{1}{2}\left(\sum_{k=0}^{\infty} a_{n,k}\cos kt\cos\frac{t}{2} - \sum_{k=0}^{\infty} a_{n,k}\sin kt\sin\frac{t}{2}\right)\\ &= \frac{\cos\frac{t}{2}}{2}\left(\sum_{k=0}^{\infty}(a_{n,k}-a_{n,k+r})D^{\circ}_{k,r}(t) + \sum_{k=0}^{r-1} a_{n,k}D^{\circ}_{k,-r}(t)\right)\end{aligned}$$

$$-\frac{\sin\frac{t}{2}}{2}\left(-\sum_{k=0}^{\infty}\left(a_{n,k}-a_{n,k+r}\right)\widetilde{D^{\circ}}_{k,r}(t)-\sum_{k=0}^{r-1}a_{n,k}\widetilde{D^{\circ}}_{k,-r}(t)\right)$$

and our inequalities follow. □

We also prove some special inequalities which follow from the conditions mentioned early.

Lemma 5. *Let $\rho, r, s \in \mathbb{N}$, $s \ge \rho \ge r \ge 2$ and $r \mid \rho$. Suppose that $f \in L^p_{2\pi/r}$, where $1 \le p < \infty$. If the condition (13) holds with any continuous and positive function $\widetilde{\omega}$, and $\beta \ge 0$, then*

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{r}-t\right)}\right)^p\left|\sin\frac{\rho t}{2}\right|^{\beta p}dt\right\}^{\frac{1}{p}}=O_x(1),$$

where $m \in \{0,\ldots[\frac{r}{2}]-1\}$.

Proof. By substitution $t=\frac{2(m+1)\pi}{r}-u$, we obtain

$$\left\{\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{r}-t\right)}\right)^p\left|\sin\frac{\rho t}{2}\right|^{\beta p}dt\right\}^{1/p}$$
$$=\left\{\int_0^{\frac{\pi}{s(n+1)}}\left(\frac{\left|\psi_x\left(\frac{2(m+1)\pi}{r}-u\right)\right|}{\widetilde{\omega}(u)}\left|\sin\left[\frac{\rho}{2}\left(\frac{2(m+1)\pi}{r}-u\right)\right]\right|^{\beta}\right)^p du\right\}^{1/p}$$
$$=\left\{\int_0^{\frac{\pi}{s(n+1)}}\left(\frac{|\psi_x(u)|}{\widetilde{\omega}(u)}\left|\sin\frac{\rho u}{2}\right|^{\beta}\right)^p du\right\}^{1/p}.$$

Hence, by (13) our estimate follows. □

Lemma 6. *Let $\rho, r, s \in \mathbb{N}$, $s \ge \rho \ge r$ and $r \mid \rho$. Suppose that $f \in L^p_{2\pi/r}$, where $1 \le p < \infty$. If the condition (13) holds with any continuous and positive function $\widetilde{\omega}$, and $\beta \ge 0$, then*

$$\left\{\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)}\right)^p\left|\sin\frac{\rho t}{2}\right|^{\beta p}dt\right\}^{\frac{1}{p}}=O_x(1),$$

where $m \in \{0,\ldots[\frac{r}{2}]\}$.

Proof. By substitution $t=\frac{2m\pi}{r}+u$, analogously to the above proof, we obtain

$$\left\{\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)}\right)^p\left|\sin\frac{\rho t}{2}\right|^{\beta p}dt\right\}^{1/p}$$

$$= \left\{ \int_0^{\frac{\pi}{s(n+1)}} \left(\frac{\left|\psi_x\left(\frac{2m\pi}{r}+u\right)\right|}{\widetilde{\omega}(u)} \left|\sin\frac{\rho}{2}\left(\frac{2m\pi}{r}+u\right)\right|^{\beta} \right)^p du \right\}^{1/p}$$

$$\leq \left\{ \int_0^{\frac{\pi}{s(n+1)}} \left(\frac{\left|\psi_x(u)\right|}{\widetilde{\omega}(u)} \left|\sin\frac{\rho u}{2}\right|^{\beta} \right)^p dt \right\}^{1/p} = O_x(1)$$

and we have the desired estimate. □

Lemma 7. *Suppose that $f \in L^p_{2\pi/\rho}$, where $1 \leq p < \infty$, $\rho, s \in \mathbb{N}$ and $s \geq \rho \geq 2$. If the condition (14) holds with any continuous, positive and nondecreasing function $\widetilde{\omega}$, and $\gamma, \beta \geq 0$, then*

$$\left\{ \int_{\frac{2(m+1)\pi}{\rho}-\frac{\pi}{\rho}}^{\frac{2(m+1)\pi}{\rho}-\frac{\pi}{s(n+1)}} \left(\frac{\left|\psi_x(t)\right| \left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)\left(\frac{2(m+1)\pi}{\rho}-t\right)^{\gamma}} \right)^p dt \right\}^{1/p} = O_x\left((n+1)^{\gamma}\right),$$

where $m \in \{0, 1, \dots [\frac{\rho}{2}] - 1\}$.

Proof. By substitution $t = \frac{2(m+1)\pi}{\rho} - u$, we obtain

$$\left\{ \int_{\frac{2(m+1)\pi}{\rho}-\frac{\pi}{\rho}}^{\frac{2(m+1)\pi}{\rho}-\frac{\pi}{s(n+1)}} \left(\frac{\left|\psi_x(t)\right| \left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)\left(\frac{2(m+1)\pi}{\rho}-t\right)^{\gamma}} \right)^p dt \right\}^{1/p}$$

$$= \left\{ \int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}} \left(\frac{\left|\psi_x\left(\frac{2(m+1)\pi}{\rho}-u\right)\right|}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{\rho}-u\right)u^{\gamma}} \left|\sin\left[\frac{\rho}{2}\left(\frac{2(m+1)\pi}{\rho}-u\right)\right]\right|^{\beta} \right)^p du \right\}^{1/p}$$

$$= \left\{ \int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}} \left(\frac{\left|\psi_x(u)\right|}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{\rho}-u\right)u^{\gamma}} \left|\sin\frac{\rho u}{2}\right|^{\beta} \right)^p du \right\}^{1/p}.$$

Since $\frac{2(m+1)\pi}{\rho} - u \geq u$

$$\left\{ \int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}} \left(\frac{\left|\psi_x(u)\right|}{\widetilde{\omega}\left(\frac{2(m+1)\pi}{\rho}-u\right)u^{\gamma}} \left|\sin\frac{\rho u}{2}\right|^{\beta} \right)^p du \right\}^{1/p} \leq \left\{ \int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}} \left(\frac{\left|\psi_x(u)\right|}{\widetilde{\omega}(u)u^{\gamma}} \left|\sin\frac{\rho u}{2}\right|^{\beta} \right)^p du \right\}^{1/p}.$$

Hence, by (14) our estimate follows. □

Lemma 8. *Suppose that $f \in L^p_{2\pi/\rho}$, where $1 \leq p < \infty$, $\rho, s \in \mathbb{N}$ and $s \geq \rho$. If the condition (14) holds with any continuous, positive and nondecreasing function $\widetilde{\omega}$, and $\gamma, \beta \geq 0$, then*

$$\left\{ \int_{\frac{2m\pi}{\rho}+\frac{\pi}{s(n+1)}}^{\frac{2m\pi}{\rho}+\frac{\pi}{\rho}} \left(\frac{\left|\psi_x(t)\right| \left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)\left(t-\frac{2m\pi}{\rho}\right)^{\gamma}} \right)^p dt \right\}^{1/p} = O_x\left((n+1)^{\gamma}\right),$$

where $m \in \{0, 1, \dots [\frac{\rho}{2}]\}$.

Proof. By substitution $t=\frac{2m\pi}{\rho}+u$, analogously to the above proof, we obtain

$$\begin{aligned}
&\left\{\int_{\frac{2m\pi}{\rho}+\frac{\pi}{s(n+1)}}^{\frac{2m\pi}{\rho}+\frac{\pi}{\rho}}\left(\frac{\left|\psi_x(t)\right|\left|\sin\frac{\rho t}{2}\right|^{\beta}}{\widetilde{\omega}(t)\left(t-\frac{2m\pi}{\rho}\right)^{\gamma}}\right)^{p}dt\right\}^{1/p}\\
&=\left\{\int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}}\left(\frac{\left|\psi_x\left(\frac{2m\pi}{\rho}+u\right)\right|}{\widetilde{\omega}\left(\frac{2m\pi}{\rho}+u\right)u^{\gamma}}\left|\sin\frac{\rho}{2}\left(\frac{2m\pi}{\rho}+u\right)\right|^{\beta}\right)^{p}du\right\}^{1/p}\\
&=\left\{\int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}}\left(\frac{\left|\psi_x(u)\right|}{\widetilde{\omega}\left(\frac{2m\pi}{\rho}+u\right)u^{\gamma}}\left|\sin\frac{\rho u}{2}\right|^{\beta}\right)^{p}dt\right\}^{1/p}\\
&\le\left\{\int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}}\left(\frac{\left|\psi_x(u)\right|}{\widetilde{\omega}(u)u^{\gamma}}\left|\sin\frac{\rho u}{2}\right|^{\beta}\right)^{p}dt\right\}^{1/p}=O_x\left((n+1)^{\gamma}\right)
\end{aligned}$$

and we have the desired estimate. □

4. Proofs of Theorems

4.1. Proof of Theorem 1

It is clear that for an odd r

$$\begin{aligned}
&\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{s(n+1)}\right)\\
&=-\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\,dt\\
&\quad+\frac{1}{\pi}\left(\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}+\int_{\frac{2[r/2]\pi}{r}+\frac{\pi}{s(n+1)}}^{\frac{(2[r/2]+1)\pi}{r}}\right)\psi_x(t)\frac{1}{2}\cot\frac{t}{2}\,dt\\
&=-\frac{1}{\pi}\left(\int_0^{\frac{\pi}{s(n+1)}}+\sum_{m=1}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}+\sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\,dt\\
&\quad+\frac{1}{\pi}\left(\sum_{m=0}^{[r/2]}\int_{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}}+\sum_{m=0}^{[r/2]-1}\int_{\frac{(2m+1)\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}\right)\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\,dt\\
&=I_0(x)+I_1(x)+I_2(x)+I_3(x)+I_4(x)
\end{aligned}$$

and for an even r

$$\begin{aligned}
&\widetilde{T}_{n,A}f(x)-\widetilde{f}_r\left(x,\frac{\pi}{s(n+1)}\right)\\
&=-\frac{1}{\pi}\int_0^{\pi}\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\,dt+\frac{1}{\pi}\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}\psi_x(t)\frac{1}{2}\cot\frac{t}{2}\,dt
\end{aligned}$$

$$= -\frac{1}{\pi}\left(\int_0^{\frac{\pi}{s(n+1)}} + \sum_{m=1}^{[r/2]-1}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}} + \sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D}_{k,1}(t)\,dt$$
$$+\frac{1}{\pi}\left(\sum_{m=0}^{[r/2]-1}\int_{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}^{\frac{(2m+1)\pi}{r}} + \sum_{m=0}^{[r/2]-1}\int_{\frac{(2m+1)\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}\right)\psi_x(t)\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\,dt$$
$$= I_0(x) + I_1'(x) + I_2(x) + I_3'(x) + I_4(x),$$

whence

$$\left|\widetilde{T}_{n,A}f(x) - \widetilde{f}_r\left(x,\frac{\pi}{s(n+1)}\right)\right| \le |I_0(x)| + |I_1(x)| + \left|I_1'(x)\right| + |I_2(x)| + |I_3(x)| + \left|I_3'(x)\right| + |I_4(x)|.$$

Next, using Lemma 2, (15), the Hölder inequality with $p>1$ and $q=\frac{p}{p-1}$ and (16),

$$|I_0(x)| = O(n+1)\int_0^{\frac{\pi}{s(n+1)}}\left|\psi_x(t)\right|dt$$
$$\le O(n+1)\left\{\int_0^{\frac{\pi}{s(n+1)}}\left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)}\right)^p \sin^{\beta p}\frac{\rho t}{2}dt\right\}^{\frac{1}{p}}\left\{\int_0^{\frac{\pi}{s(n+1)}}\left(\frac{\widetilde{\omega}(t)}{\sin^{\beta}\frac{\rho t}{2}}\right)^q dt\right\}^{\frac{1}{q}}$$
$$= O_x(1)(n+1)^{\beta+1/p}\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right),$$

for $0\le\beta<1-\frac{1}{p}$. Further, we note that applying condition (15) we have

$$\left[(n+1)A_{n,\rho}\right]^{-1} = \left[\sum_{l=0}^{n}A_{n,\rho}\right]^{-1} \le \left[\sum_{l=0}^{n}\sum_{k=l}^{\infty}\left|a_{n,k}-a_{n,k+\rho}\right|\right]^{-1}$$
$$\le \left[\sum_{l=0}^{n}\left|\sum_{k=l}^{\infty}\left(a_{n,k}-a_{n,k+\rho}\right)\right|\right]^{-1} = \left[\sum_{l=0}^{n}\sum_{k=l}^{\rho+l-1}a_{n,k}\right]^{-1} = O(1),$$

whence

$$|I_0(x)| = O_x\left((n+1)^{1+\beta+\frac{1}{p}}A_{n,s}\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right)\right).$$

Analogously, by Lemma 2

$$|I_1(x)| + \left|I_1'(x)\right| + |I_2(x)| \le \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}} + \sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\frac{|\psi_x(t)|}{t}dt$$
$$\le \frac{1}{\pi}\left(\sum_{m=1}^{[r/2]}\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}} + \sum_{m=0}^{[r/2]-1}\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}}\right)\frac{|\psi_x(t)|}{\pi/r}dt$$

and using the Hölder inequality with $p>1$ and $q=\frac{p}{p-1}$

$$\le O_x(1)\sum_{m=1}^{[r/2]}\left[\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}\left(\frac{|\psi_x(t)|\sin^{\beta}\frac{\rho t}{2}}{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)}\right)^p dt\right]^{\frac{1}{p}}\left[\int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}\left(\frac{\widetilde{\omega}\left(t-\frac{2m\pi}{r}\right)}{\sin^{\beta}\frac{\rho t}{2}}\right)^q dt\right]^{\frac{1}{q}}$$
$$+O_x(1)\sum_{m=1}^{[r/2]-1}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{|\psi_x(t)|\sin^{\beta}\frac{\rho t}{2}}{\widetilde{\omega}\left(t-\frac{2(m+1)\pi}{r}\right)}\right)^p dt\right]^{\frac{1}{p}}\left[\int_{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}^{\frac{2(m+1)\pi}{r}}\left(\frac{\widetilde{\omega}\left(t-\frac{2(m+1)\pi}{r}\right)}{\sin^{\beta}\frac{\rho t}{2}}\right)^q dt\right]^{\frac{1}{q}}.$$

Hence, by Lemmas 5 and 6 with (13) and (15),

$$\begin{aligned}|I_1(x)|+\left|I_1'(x)\right|+|I_2(x)| &= O_x(1)\,\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right)\left[\int_0^{\frac{\pi}{s(n+1)}}\left(\frac{1}{\sin^{\beta}\frac{\rho t}{2}}\right)^q dt\right]^{\frac{1}{q}}\\ &= O_x\left((n+1)^{\beta-\frac{1}{q}}\right)\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right)=O_x\left((n+1)^{\beta+\frac{1}{p}}A_{n,\rho}\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right)\right),\end{aligned}$$

for $0\le\beta<1-\frac{1}{p}$.

In case of the last integrals, applying Lemmas 2 and 4 we obtain for $s\ge\rho\ge r$

$$\begin{aligned}&|I_3(x)|+\left|I_3'(x)\right|+|I_4(x)|\\ &\le\frac{1}{\pi}\left(\sum_{m=0}^{[r/2]}\int_{\frac{2m\pi}{r}+\frac{\pi}{s(n+1)}}^{\frac{(2m+1)\pi}{r}}+\sum_{m=0}^{[r/2]-1}\int_{\frac{(2m+1)\pi}{r}}^{\frac{2(m+1)\pi}{r}-\frac{\pi}{s(n+1)}}\right)\left|\psi_x(t)\right|\left|\sum_{k=0}^{\infty}a_{n,k}\widetilde{D^{\circ}}_{k,1}(t)\right|dt\\ &\le\frac{1}{\pi}\sum_{m=0}^{[r/2]}\left\{\sum_{\mu=\left[\frac{\rho m}{r}\right]}^{\left[\frac{\rho(m+1)}{r}\right]-1}\int_{\frac{2\mu\pi}{\rho}+\frac{\pi}{s(n+1)}}^{\frac{2(\mu+1)\pi}{\rho}-\frac{\pi}{s(n+1)}}\frac{\left|\psi_x(t)\right|}{\left|\sin\frac{t}{2}\sin\frac{\rho t}{2}\right|}A_{n,\rho}\,dt\right.\\ &\left.+\left(\sum_{\mu=\left[\frac{\rho m}{r}\right]+1}^{\left[\frac{\rho(m+1)}{r}\right]-1}\int_{\frac{2\mu\pi}{\rho}}^{\frac{2\mu\pi}{\rho}+\frac{\pi}{s(n+1)}}+\sum_{\mu=\left[\frac{\rho m}{r}\right]}^{\left[\frac{\rho(m+1)}{r}\right]-2}\int_{\frac{2(\mu+1)\pi}{\rho}-\frac{\pi}{s(n+1)}}^{\frac{2(\mu+1)\pi}{\rho}}\right)\frac{\left|\psi_x(t)\right|}{t}dt\right\}.\end{aligned}$$

Using the estimate $\left|\sin\frac{\rho t}{2}\right|\ge\frac{\rho t}{\pi}-2\mu$ for $t\in\left[\frac{2\mu\pi}{\rho}+\frac{\pi}{s(n+1)},\frac{(2\mu+1)\pi}{\rho}\right]$, and the estimate $\left|\sin\frac{t}{2}\right|\ge\frac{|t|}{\pi}$ and $\left|\sin\frac{\rho t}{2}\right|\ge 2(\mu+1)-\frac{\rho t}{\pi}$ for $t\in\left[\frac{(2\mu+1)\pi}{\rho},\frac{(2\mu+1)\pi}{\rho}-\frac{\pi}{s(n+1)}\right]$, where $\mu\in\left\{\left[\frac{\rho m}{r}\right],\dots,\left[\frac{\rho(m+1)}{r}\right]-1\right\}$ and $m\in\{0,\dots,[r/2]\}$ and $\sin\frac{t}{2}\ge\frac{t}{\pi}$ such that $t\in[0,\pi]$, we obtain

$$\begin{aligned}&|I_3(x)|+\left|I_3'(x)\right|+|I_4(x)|\\ &\le\sum_{m=0}^{[r/2]}\left\{A_{n,\rho}\sum_{\mu=\left[\frac{\rho m}{r}\right]}^{\left[\frac{\rho(m+1)}{r}\right]-1}\int_{\frac{2\mu\pi}{\rho}+\frac{\pi}{s(n+1)}}^{\frac{(2\mu+1)\pi}{\rho}}\frac{\left|\psi_x(t)\right|}{t\left(\frac{\rho t}{\pi}-2\mu\right)}dt\right.\\ &+A_{n,\rho}\sum_{\mu=\left[\frac{\rho m}{r}\right]}^{\left[\frac{\rho(m+1)}{r}\right]-1}\int_{\frac{(2\mu+1)\pi}{\rho}}^{\frac{2(\mu+1)\pi}{\rho}-\frac{\pi}{s(n+1)}}\frac{\left|\psi_x(t)\right|}{t\left(2(\mu+1)-\frac{\rho t}{\pi}\right)}dt\\ &\left.+\left(\sum_{\mu=\left[\frac{\rho m}{r}\right]+1}^{\left[\frac{\rho(m+1)}{r}\right]-1}\int_{\frac{2\mu\pi}{\rho}}^{\frac{2\mu\pi}{\rho}+\frac{\pi}{s(n+1)}}+\sum_{\mu=\left[\frac{\rho m}{r}\right]}^{\left[\frac{\rho(m+1)}{r}\right]-2}\int_{\frac{2(\mu+1)\pi}{\rho}-\frac{\pi}{s(n+1)}}^{\frac{2(\mu+1)\pi}{\rho}}\right)\frac{\left|\psi_x(t)\right|}{t}dt\right\}\\ &\le\sum_{m=0}^{[r/2]}\left\{A_{n,\rho}\sum_{\mu=\left[\frac{\rho m}{r}\right]}^{\left[\frac{\rho(m+1)}{r}\right]-1}\left[\int_{\frac{2\mu\pi}{\rho}+\frac{\pi}{s(n+1)}}^{\frac{(2\mu+1)\pi}{\rho}}\left(\frac{\left|\psi_x(t)\right|}{\widetilde{\omega}(t)\left(t-\frac{2\mu\pi}{\rho}\right)^{\gamma}}\left|\sin\frac{\rho t}{2}\right|^{\beta}\right)^p dt\right]^{\frac{1}{p}}\right.\\ &\left[\int_{\frac{2\mu\pi}{\rho}+\frac{\pi}{s(n+1)}}^{\frac{(2\mu+1)\pi}{\rho}}\left(\frac{\widetilde{\omega}(t)\left(t-\frac{2\mu\pi}{\rho}\right)^{\gamma}}{t\left(\frac{\rho t}{\pi}-2\mu\right)\left|\sin\frac{\rho t}{2}\right|^{\beta}}\right)^q dt\right]^{\frac{1}{q}}\end{aligned}$$

$$+ A_{n,\rho} \sum_{\mu=\left[\frac{\rho m}{r}\right]}^{\left[\frac{\rho(m+1)}{r}\right]-1} \left[\int_{\frac{(2\mu+1)\pi}{\rho}}^{\frac{2(\mu+1)\pi}{\rho}-\frac{\pi}{s(n+1)}} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}(t)\left(\frac{2(\mu+1)\pi}{r}-t\right)^{\gamma}} \left|\sin\frac{\rho t}{2}\right|^{\beta} \right)^p dt \right]^{\frac{1}{p}}$$

$$\left[\int_{\frac{(2\mu+1)\pi}{\rho}}^{\frac{2(\mu+1)\pi}{\rho}-\frac{\pi}{s(n+1)}} \left(\frac{\widetilde{\omega}(t)\left(\frac{2(\mu+1)\pi}{r}-t\right)^{\gamma}}{t\left(2(\mu+1)-\frac{\rho t}{\pi}\right)\left|\sin\frac{\rho t}{2}\right|^{\beta}} \right)^q dt \right]^{\frac{1}{q}} \Bigg\}$$

$$+ \sum_{m=0}^{[r/2]} \left\{ \sum_{\mu=\left[\frac{\rho m}{r}\right]+1}^{\left[\frac{\rho(m+1)}{r}\right]-1} \left[\int_{\frac{2\mu\pi}{\rho}}^{\frac{2\mu\pi}{\rho}+\frac{\pi}{s(n+1)}} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}\left(t-\frac{2\mu\pi}{\rho}\right)} \right)^p \sin^{\beta p}\frac{\rho t}{2} dt \right]^{\frac{1}{p}} \right.$$

$$\left[\int_{\frac{2\mu\pi}{\rho}}^{\frac{2\mu\pi}{\rho}+\frac{\pi}{s(n+1)}} \left(\frac{\widetilde{\omega}\left(t-\frac{2\mu\pi}{\rho}\right)}{\frac{\pi}{\rho}\sin^{\beta}\frac{\rho t}{2}} \right)^q dt \right]^{\frac{1}{q}}$$

$$+ \sum_{\mu=\left[\frac{\rho m}{r}\right]}^{\left[\frac{\rho(m+1)}{r}\right]-2} \left[\int_{\frac{2(\mu+1)\pi}{\rho}-\frac{\pi}{s(n+1)}}^{\frac{2(\mu+1)\pi}{\rho}} \left(\frac{|\psi_x(t)|}{\widetilde{\omega}\left(\frac{2(\mu+1)\pi}{\rho}-t\right)} \right)^p \sin^{\beta p}\frac{\rho t}{2} dt \right]^{\frac{1}{p}}$$

$$\left. \left[\int_{\frac{2(\mu+1)\pi}{\rho}-\frac{\pi}{s(n+1)}}^{\frac{2(\mu+1)\pi}{\rho}} \left(\frac{\widetilde{\omega}\left(\frac{2(\mu+1)\pi}{\rho}-t\right)}{\frac{\pi}{\rho}\sin^{\beta}\frac{\rho t}{2}} \right)^q dt \right]^{\frac{1}{q}} \right\}.$$

Further, by Lemmas 5 and 6 with (13) and Lemmas 7 and 8 with (14), we get

$$|I_3(x)| + \left|I_3'(x)\right| + |I_4(x)|$$

$$= O_x(1) \sum_{m=0}^{[r/2]} \left\{ A_{n,\rho} \sum_{\mu=\left[\frac{\rho m}{r}\right]}^{\left[\frac{\rho(m+1)}{r}\right]-1} 2\frac{\pi}{\rho} \left[\int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}} \left(\frac{\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right) t^{\gamma}}{t\frac{\pi}{s(n+1)}\left|\sin\frac{\rho t}{2}\right|^{\beta}} \right)^q dt \right]^{\frac{1}{q}} \right\}$$

$$+ O_x(1) \sum_{m=0}^{[r/2]} \left\{ \sum_{\mu=\left[\frac{\rho m}{r}\right]+1}^{\left[\frac{\rho(m+1)}{r}\right]-1} 2\frac{\rho}{\pi} \left[\int_0^{\frac{\pi}{s(n+1)}} \left(\frac{\widetilde{\omega}(t)}{\frac{\pi}{\rho}\sin^{\beta}\frac{\rho t}{2}} \right)^q dt \right]^{\frac{1}{q}} \right\}$$

$$= O_x(1)\,\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right) \left\{ (n+1)^{1+\gamma} A_{n,\rho} \left[\int_{\frac{\pi}{s(n+1)}}^{\frac{\pi}{\rho}} \left(t^{\gamma-1-\beta}\right)^q dt \right]^{\frac{1}{q}} + \left[\int_0^{\frac{\pi}{s(n+1)}} \left(t^{-\beta}\right)^q dt \right]^{\frac{1}{q}} \right\}$$

$$= O_x(1)\,\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right) \left\{ (n+1)^{1+\gamma} A_{n,\rho} (n+1)^{-\gamma+\beta+\frac{1}{p}} + (n+1)^{\beta-\frac{1}{q}} \right\},$$

for $0 \le \beta < 1 - \frac{1}{p}$. Finally, by (15)

$$|I_3(x)| + \left|I_3'(x)\right| + |I_4(x)| = O_x(1)\,\widetilde{\omega}\left(\frac{\pi}{s(n+1)}\right)(n+1)^{1+\beta+\frac{1}{p}} A_{n,\rho}$$

and thus our statement follows. □

4.2. Proof of Theorem 2

The proof is the same as above, but for the estimate of $|I_0(x)|$ when $\rho = 1$ we use the inequality $\left|\widetilde{D}_{k,1}(t)\right| \leq \frac{1}{2}k(k+1)|t|$ from Lemma 2, and the conditions (9) and (16) instead of (11) and (13). □

4.3. Proof of Theorem 4

We can note that for the estimate of $\left\|\widetilde{T}_{n,A}f(\cdot) - \widetilde{f}_r\left(\cdot, \frac{\pi}{s(n+1)}\right)\right\|_{L^p_{2\pi/\rho}}$ we need the conditions on $\widetilde{\omega}$ from the assumptions of Theorem 3. These conditions always hold with $\left\|\psi_{\cdot}(t)\right\|_{L^p_{2\pi/\rho}}$ instead of $\left|\psi_x(t)\right|$ and thus, the desired result follows. □

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