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SIMPLE RINGS OF CHARACTERISTIC NOT 2 WITH ASSOCIATORS IN THE LEFT NUCLEUS ARE ASSOCIATIVE

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Abstract. We prove that if R is a simple ring of characteristic not 2 with associators in the left nucleus then R is associative. This extends our previous result [2].

1. Introduction

Let R be a nonassociative ring. We shall denote the associator by $(x,y,z)=(xy)z-x(yz)$ for all x,y,z in R. In any ring R one has the following nuclei:

$$N=\{n\in R|(n,R,R)=0\} - \text{left nucleus},$$
$$M=\{n\in R|(R,n,R)=0\} - \text{middle nucleus},$$
$$L=\{n\in R|(R,R,n)=0\} - \text{right nucleus}.$$

A ring R is called simple if $R^2\neq 0$ and the only nonzero ideal of R is itself. Since R^2 is a nonzero ideal of R, we have $R^2=R$. A ring R is called semiprime if the only ideal of R which squares to zero is the zero ideal. Note that each associator is linear in each argument. Thus N, M and L are additive subgroups of $(R,+)$. We shall use the Teichmüller identity.

(1) $(wx,y,z)-(w,xy,z)+(w,x,yz)=w(x,y,z)+(w,x,y)z$ for all w,x,y,z in R, which is valid in every ring.
As a consequence of (1), we have that N, M and L are associative subrings of R. Suppose that $n\in N$. Then with $w=n$ in (1) we obtain
(2) $(nx,y,z)=n(x,y,z)$ for all x,y,z in R and n in N.

Definition. Let A be the associator ideal of a ring R.

By (1) A can be characterized as all finite sums of associators and right (or left) multiples of associators. Hence, we otbain

(3) $A=(R,R,R)+(R,R,R)R$.

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In [1], E. Kleinfeld proved that if R is a semiprime ring such that $(R,R,R) \subseteq N \cap M \cap L$ and $(R,+)$ has no elements of order 2 then R is associative. In [2], the author showed the result for the simple ring case under the weaker hypothesis $(R,R,R) \subseteq N \cap M$. In the note, we extends this result. In [3], we generalized E. Kleinfeld's result under the weaker hypothesis $(R,R,R) \subseteq$ two of the three nuclei.

2. Result

Theorem. *Let R be a simple ring of characteristic not 2 and satisfy $(*)$ $(R,R,R) \subseteq N$. Then R is associative.*

Proof. Assume that R is not associative. Then by (3) and $(*)$, we have

(4) $R = R^2 = AR = \{(R,R,R)+(R,R,R)R\}R = (R,R,R)R+(R,R,R)R^2 = (R,R,R)R.$

Using (1) and $(*)$, we get

(5) $w(x,y,z)+(w,x,y)z \in N$ for all w,x,y,z in R.

Then with $x \in (R,R,R)$ in (5), and applying $(*)$ we obtain $(R,(R,R,R),R)R \subseteq N$. Using this, $(*)$ and (2), we have $0 = ((R,(R,R,R),R)R,R,R) = (R,(R,R,R),R)(R,R,R)$ and so $(R,(R,R,R),R)\cdot(R,R,R)R = (R,(R,R,R),R)(R,R,R)\cdot R = 0$. Combined this with (4) results in

(6) $(R,(R,R,R),R)R = 0.$

Assume that $x \in (R,(R,R,R),R)$ and $w,y,z,t \in R$. Then by (6), (1) and $(*)$ we get $(wx,y,z)+(w,x,yz) = (wx,y,z)-(w,xy,z)+(w,x,yz) = w(x,y,z)+(w,x,y)z = 0$ and so $(wx,y,z)t = -(w,x,yz)t = 0$. The last identity implies

(7) $(R(R,(R,R,R),R),R,R)R = 0.$

Then with $y \in (R,R,R)$ in (5) and applying (7), we obtain

(8) $((R,R,(R,R,R))R,R,R)R = 0.$

Using (4), $(*)$, (2) and (8), we have

(9) $(R,R,(R,R,R))R = (R,R,(R,R,R))\cdot(R,R,R)R = (R,R,(R,R,R))(R,R,R)\cdot R = ((R,R,(R,R,R))R,R,R)R = 0.$

Then with $y \in (R,R,R)$ in (5) and applying (9), we get

(10) $R(R,(R,R,R),R) \subset N.$

For all $x \in (R,(R,R,R),R)$ and $w,y,z \in R$, using the previous computation and by (10) we obtain $(w,x,yz) = -(wx,y,z) = 0$.
Since $R^2 = R$, this implies

(11) $(R,(R,R,R),R) \subseteq M$.

Let $T = (R,(R,R,R),R)$. We define V_n inductively by $V_0 = T$, $V_1 = RT$ and $V_{n+1} = RV_n$, $n = 1,2,3,\cdots$. Assume that

(12) $B = \sum_{n=0}^{\infty} V_n$.

We want to prove by induction that

(13) $B \cdot R = 0$

By (6), (11) and (9), we have $V_0R = TR = 0$, $V_1R = RT \cdot R = R \cdot TR = 0$ and $V_2R = R(RT) \cdot R \subseteq ((R,R,T) + R^2T)R = (R,R,T)R + R^2T \cdot R = 0$. Suppose that $V_iR = 0$, $i = 0,1,2,\cdots,m$ and $V_{m+1}R = 0$. Then using these and (9), we get $V_{m+2}R = R(RV_m) \cdot R \subseteq ((R,R,V_m) + R^2V_m)R = (R,R,V_m)R + V_{m+1}R = (R,R,V_m)R = (R,R,R(RV_{m-2}))R \subseteq (R,R,(R,R,V_{m-2}) + R^2V_{m-2})R = (R,R,(R,R,V_{m-2}))R + (R,R,V_{m-1})R = (R,R,V_{m-1})R$. Continuing in this manner, we eventually have $V_{m+2}R \subseteq (R,R,V_m)R \subseteq (R,R,V_{m-1})R \subseteq \cdots \subseteq (R,R,V_2)R \subseteq (R,R,V_1)R = (R,R,RT)R$. By (1) and (9), we get $RT = R(R,(R,R,R),R) \subseteq (R,R,R) + (R,R,(R,R,R))R = (R,R,R)$.

Thus, applying this and (9) we have $V_{m+2}R \subseteq (R,R,RT)R \subseteq (R,R,(R,R,R))R = 0$. Hence, by induction (13) holds. By (13), B is just the ideal of R generated by $(R,(R,R,R),R)$. By the simplicity of R and (13), we get $B = 0$. Thus, $(R,(R,R,R),R) = 0$ and so $(R,R,R) \subseteq N \cap M$. Hence, by Theorem 2 of [2], R is associative. This contradiction proves the theorem.

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