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ARGUMENT ESTIMATES OF STRONGLY STARLIKE FUNCTIONS ASSOCIATED WITH NOOR-INTEGRAL OPERATOR

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Abstract. The purpose of this present paper is to derive some properties of a certain new subclasses of strongly starlike functions defined by the Noor integral operator. Relevant connections of the results, which are presented in the paper, with various known results are also considered.

1. Introduction and definitions

Let $\mathscr{A}$ denote the class of normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. We also denote by $\mathscr{S}$, the subclass of $\mathscr{A}$ consisting of functions which are univalent in Δ. A function $f \in \mathscr{A}$ is said to be strongly starlike of order γ and type (α,β) in Δ, denoted by $\mathscr{S}^*(\alpha,\beta,\gamma)$, if it satisfies,

$$\frac{-\pi\beta}{2} < \arg\left\{\frac{zf'(z)}{f(z)} - \gamma\right\} < \frac{\pi\alpha}{2}, \tag{1.2}$$
$$(z \in \Delta,\ 0 < \alpha \le 1, 0 < \beta \le 1,\ 0 \le \gamma < 1).$$

A function $f \in \mathscr{A}$ is said to be strongly convex of order γ and type (α,β) in Δ, denoted by $C(\alpha,\beta,\gamma)$, if it satisfies,

$$\frac{-\pi\beta}{2} < \arg\left\{1 + \frac{zf''(z)}{f'(z)} - \gamma\right\} < \frac{\pi\alpha}{2}, \tag{1.3}$$
$$(z \in \Delta,\ 0 < \alpha \le 1, 0 < \beta \le 1,\ 0 \le \gamma < 1).$$

A function $f \in \mathscr{A}$ is said to be strongly Mocanu convex of order γ and type (α,β) in Δ, denoted by $Q(\alpha,\beta,\gamma,\lambda)$, if it satisfies,

$$\frac{-\pi\beta}{2} < \arg\left\{(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left[1 + \frac{zf''(z)}{f'(z)}\right] - \gamma\right\} < \frac{\pi\alpha}{2}, \tag{1.4}$$
$$(z \in \Delta,\ 0 < \alpha \le 1, 0 < \beta \le 1,\ 0 \le \gamma < 1, 0 \le \lambda \le 1).$$

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Also, further, we note that

$$Q(\alpha,\beta,0,0)=\mathscr{S}^*(\alpha,\beta)$$

and

$$Q(\alpha,\beta,0,1)=C(\alpha,\beta),$$

the classes introduced by Takahashi and Nunokawa [6].

Let $f\in\mathscr{A}$. Denote by $D^\alpha:\mathscr{A}\to\mathscr{A}$ the operator defined by

$$D^\alpha f(z)=\frac{z}{(1-z)^{\alpha+1}}*f(z),\quad \alpha>-1$$

which implies that,

$$D^n f(z)=\frac{z\left(z^{n-1}f(z)\right)^n}{n!}$$

for $n\in\mathbb{N}_0=\{0,1,2,\ldots\}$ and the operator $*$ stands for the Hadamard Product or Convolution. We note that $D^0f(z)=f(z)$ and $D^1f(z)=zf'(z)$. The operator $D^nf(z)$ is called the Ruscheweyh derivative [5] of the n^{th} order of f. Recently, Noor [2] and Noor and Noor [3] introduced and studied an integral operator $I_n:\mathscr{A}\to\mathscr{A}$ analogous to D^nf as follows. Let $f_n(z)=\frac{z}{(1-z)^{n+1}},\quad n\in\mathbb{N}_0$, and let $f_n{}^{-1}$ be defined such that

$$f_n(z)*f_n^{-1}(z)=\frac{z}{1-z}. \tag{1.5}$$

Then

$$I_nf=f_n^{-1}(z)*f(z)=\left[\frac{z}{(1-z)^{n+1}}\right]^{-1}*f(z). \tag{1.6}$$

We note that $I_0f=zf'$ and $I_1f=f$. The operator I_n defined by (1.6) is called the Noor integral operator of nth order of f. For the properties and applications of the Noor integral operator, see Noor [2] and Noor and Noor [3]. From (1.5) and (1.6), and a well-known identity for D^nf, it follows that

$$z(I_{n+1}f)'=(n+1)I_nf-nI_{n+1}f. \tag{1.7}$$

Using the Noor integral, we introduce and study the properties of some new classes of the analytic functions:

$$M_n\left(\alpha,\beta,\gamma,\lambda\right)=\left\{f\in\mathscr{A}:I_nf\in Q(\alpha,\beta,\gamma,0),\ \frac{z\left(I_nf(z)\right)'}{I_nf(z)}\neq\gamma,\quad z\in\Delta\right\}$$

and

$$P_n\left(\alpha,\beta,\gamma,\lambda\right)=\left\{f\in\mathscr{A}:I_nf\in Q(\alpha,\beta,\gamma,1),\ 1+\frac{z\left(I_nf(z)\right)''}{(I_nf(z))'}\neq\gamma,\quad z\in\Delta\right\}.$$

Note that $M_n\left(\alpha,\beta,\gamma,0\right)\equiv S_n^*(\beta,\gamma)$ and $P_n\left(\alpha,\beta,\gamma,\lambda\right)\equiv C_n(\beta,\gamma)$, where the classes $S_n^*(\beta,\gamma)$ and $C_n(\beta,\gamma)$ were introduced and studeid by Liu [1].

2. Main results

In order to prove our results, we need the following result of Nunokawa et al. [4].

Lemma 2.1.([4]) *Let $p(z)$ be analytic in Δ with $p(0) = 1$ and $p(z) \neq 0$. If there exists two points $z_1, z_2 \in \Delta$ such that*

$$-\frac{\beta\pi}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\alpha\pi}{2} \tag{2.1}$$

for $\beta > 0$, $\alpha > 0$, and for $|z| < |z_1| = |z_2|$, then we have

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\frac{\alpha+\beta}{2}m \tag{2.2}$$

and

$$\frac{z_2 p'(z_2)}{p(z_2)} = i\frac{\alpha+\beta}{2}m \tag{2.3}$$

where

$$m \geq \frac{1-|a|}{1+|a|} \quad \text{and} \quad a := i\tan\frac{\pi}{4}\left(\frac{\alpha-\beta}{\alpha+\beta}\right). \tag{2.4}$$

Theorem 2.2.

$$M_n(\alpha,\beta,\gamma,\lambda) \subset M_{n+1}(\alpha,\beta,\gamma,\lambda) \quad \textit{for each} \quad n \in \mathbb{N}_0.$$

Proof. Let $f \in M_n(\alpha,\beta,\gamma,\lambda)$. Then we set

$$\frac{z(I_{n+1}f(z))'}{(I_{n+1}f(z))} = \gamma + \big(1-\gamma\big)\,p(z). \tag{2.5}$$

where $p(z) = 1 + c_1 z + c_2 z^2 + ...$ is analytic in Δ, $p(0) = 1$, and $p(z) \neq 0$ for all $z \in \Delta$. Using (1.7) and (2.5), we have

$$(n+1)\frac{I_n f(z)}{I_{n+1}f(z)} = (n+\gamma) + (1-\gamma)p(z). \tag{2.6}$$

Differentiating both sides of (2.5), it follows that

$$\begin{aligned}\frac{z(I_n f(z))'}{(I_n f(z))} &= \frac{z(I_{n+1}f(z))'}{(I_{n+1}f(z))} + \frac{(1-\gamma)zp'(z)}{(n+\gamma)+(1-\gamma)p(z)}\\ &= \gamma + (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(n+\gamma)+(1-\gamma)p(z)}\end{aligned}$$

or

$$\frac{z(I_n f(z))'}{(I_n f(z))} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(n+\gamma)+(1-\gamma)p(z)}$$

Suppose that there exist two points, $z_1, z_2 \in \Delta$ such that,

$$\frac{-\pi\beta}{2} = \arg p(z_1) < \arg p(z) < \arg p(z_2) = \frac{\pi\alpha}{2} \quad \text{for} \quad |z| < |z_1| = |z_2|.$$

Then from the proof of the Nunokawa lemma [4]

$$\frac{z_1 p'(z_1)}{p(z_1)} = -i\frac{\alpha+\beta}{4}.\frac{1+t_1^2}{t_1}m \quad \text{and} \quad \frac{z_2 p'(z_2)}{p(z_2)} = i\frac{\alpha+\beta}{4}.\frac{1+t_2^2}{t_2}m \tag{2.7}$$

where

$$\begin{aligned} e^{i\left(\frac{\pi}{2}\right)\left(\frac{\alpha-\beta}{\alpha+\beta}\right)} p(z_1)^{2/(\alpha+\beta)} &= -i\,t_1, \\ e^{i\left(\frac{\pi}{2}\right)\left(\frac{\alpha-\beta}{\alpha+\beta}\right)} p(z_2)^{2/(\alpha+\beta)} &= i\,t_2 \end{aligned} \tag{2.8}$$

and $t_1, t_2 > 0$ and $m > \frac{1-|a|}{1+|a|,}$ $a > 0$. Put $z = z_2$. Using (2.7) and (2.8), we have

$$\begin{aligned} \frac{z_2(I_n f(z_2))'}{(I_n f(z_2))} - \gamma &= (1-\gamma)p(z_2) + \frac{(1-\gamma)zp'(z_2)}{(n+\gamma)+(1-\gamma)p(z_2)} \\ &= (1-\gamma)p(z_2)\left[1 + \frac{zp'(z_2)/p(z_2)}{(n+\gamma)p(z_2)+(1-\gamma)}\right] \\ &= (1-\gamma)t_2^{\frac{(\alpha+\beta)}{2}} e^{i\frac{\pi\alpha}{2}}\left[1 + \frac{i\frac{\alpha+\beta}{4}.\frac{1+t_2^2}{t_2}m}{(1-\gamma)t_2^{\frac{(\alpha+\beta)}{2}} e^{i\frac{\pi\alpha}{2}} + (n+\gamma)}\right]. \end{aligned}$$

This implies that

$$\begin{aligned} &\arg\left\{\frac{z_2(I_n f(z_2))'}{(I_n f(z_2))} - \gamma\right\} \\ &= \frac{\pi}{2}\alpha + \arg\left\{1 + \frac{i\frac{\alpha+\beta}{4}.\frac{1+t_2^2}{t_2}m}{(1-\gamma)t_2^{\frac{(\alpha+\beta)}{2}} e^{i\frac{\pi\alpha}{2}} + (n+\gamma)}\right\} \\ &= \frac{\pi}{2}\alpha + \tan^{-1}\left\{\frac{m\frac{\alpha+\beta}{4}(t_2^{-1}+t_2)\left[(n+\gamma)+(1-\gamma)t_2^{\frac{(\alpha+\beta)}{2}}\cos\left(\frac{\pi\alpha}{2}\right)\right]}{\eta(\alpha,\beta,t_2)}\right\} \\ &\geq \frac{\pi\alpha}{2} \end{aligned}$$

where

$$\begin{aligned} \eta(\alpha,\beta,t_2) = (n+\gamma)^2 &+ 2(n+\gamma)(1-\gamma)t_2^{\frac{(\alpha+\beta)}{2}}\cos\left(\frac{\pi\alpha}{2}\right) \\ &+ (1-\gamma)^2 t_2^{\alpha+\beta} + m\frac{\alpha+\beta}{4}(t_2^{-1}+t_2)(1-\gamma)t_2^{\frac{(\alpha+\beta)}{2}}\sin\left(\frac{\pi\alpha}{2}\right) \end{aligned}$$

and $m > \frac{1-|a|}{1+|a|}$, which contradicts the hypothesis that $f \in M_n(\alpha,\beta,\gamma,\lambda)$.
Similarly, if $\arg p(z_1) = \frac{-\pi\beta}{2}$, then we obtain that

$$\arg\left\{\frac{z_2(I_n f(z_2))'}{(I_n f(z_2))} - \gamma\right\} \leq \frac{\pi\beta}{2}$$

This completes the proof of the Theorem 2.2.

Theorem 2.3.
$$P_n(\alpha,\beta,\gamma,\lambda) \subset P_{n+1}(\alpha,\beta,\gamma,\lambda) \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Proof.

$$\begin{aligned} f \in P_n(\alpha,\beta,\gamma,\lambda) &\Longleftrightarrow I_n f(z) \in Q(\alpha,\beta,\gamma,1) \\ &\Longleftrightarrow z(I_n f(z))' \in Q(\alpha,\beta,\gamma,0) \\ &\Longleftrightarrow I_n(zf'(z)) \in Q(\alpha,\beta,\gamma,0) \\ &\Longleftrightarrow zf'(z) \in M_n(\alpha,\beta,\gamma,\lambda) \\ &\Longleftrightarrow zf'(z) \in M_{n+1}(\alpha,\beta,\gamma,\lambda) \\ &\Longleftrightarrow I_{n+1}(zf'(z)) \in Q(\alpha,\beta,\gamma,0) \\ &\Longleftrightarrow z(I_{n+1} f(z))' \in Q(\alpha,\beta,\gamma,0) \\ &\Longleftrightarrow I_{n+1}(f(z)) \in Q(\alpha,\beta,\gamma,1) \\ &\Longleftrightarrow f \in P_{n+1}(\alpha,\beta,\gamma,\lambda). \end{aligned}$$

It is an easy observation that Alexander's type relationship holds between the classes $P_n(\alpha,\beta,\gamma,\lambda)$ and $M_n(\alpha,\beta,\gamma,\lambda)$ which we state formally as

Theorem 2.4.
$$f \in P_n(\alpha,\beta,\gamma,\lambda) \Longleftrightarrow zf' \in M_n(\alpha,\beta,\gamma,\lambda).$$

3. Integral operators

Theorem 3.1. *Let $v > -\gamma$ and $0 \le \gamma < 1$. If $f \in M_n(\alpha,\beta,\gamma,\lambda)$ with*

$$\frac{z\big(I_n(J_v f(z))\big)'}{I_n(J_v f(z))} \neq \gamma, \quad \textit{for all} \quad z \in \Delta,$$

then $J_v f \in M_n(\alpha,\beta,\gamma,\lambda)$, where $J_v f(z)$ is given by

$$J_v f(z) = \frac{v+1}{z^v} \int_0^z t^{v-1} f(t)\,dt, \quad (v > -1; f \in \mathscr{A}). \tag{3.1}$$

Proof. Let

$$\frac{z(I_n J_v f(z))'}{(I_n J_v f(z))} = \gamma + \big(1-\gamma\big)\, p(z). \tag{3.2}$$

where $p(z)$ is analytic in Δ, $p(0) = 1$, and $p(z) \neq 0\,(z \in \Delta)$. Using (3.1), we have

$$z(I_n J_v f)' = (v+1) I_n J_v f - v I_n J_v f. \tag{3.3}$$

By (3.2) and (3.3), we get

$$\frac{I_n f(z)}{I_n J_v f(z)} = \frac{1}{v+1}\left[(v+\gamma)+(1-\gamma)p(z)\right]. \tag{3.4}$$

Differentiating (3.4) logarithmically, it follows that

$$\frac{z(I_n f(z))'}{(I_n J_v f(z))} - \gamma = (1-\gamma)p(z) + \frac{(1-\gamma)zp'(z)}{(v+\gamma)+(1-\gamma)p(z)} \tag{3.5}$$

The remaining part of the proof is similar to that of Theorem 2.2 and so is omitted.

Theorem 3.2. *Let* $v > -\gamma$ *and* $0 \le \gamma < 1$. *If* $f \in P_n(\alpha,\beta,\gamma,\lambda)$ *with*

$$1 + \frac{z\big(I_n(J_v f(z))\big)''}{\big(I_n(J_v f(z))\big)'} \neq \gamma, \quad \textit{for all} \quad z \in \Delta, J_v f \in P_n(\alpha,\beta,\gamma,\lambda)$$

Proof.

$$\begin{aligned} f \in P_n(\alpha,\beta,\gamma,\lambda) &\iff zf'(z) \in M_n(\alpha,\beta,\gamma,\lambda) \\ &\iff J_v(zf'(z)) \in M_n(\alpha,\beta,\gamma,\lambda) \\ &\iff zJ_v(f'(z)) \in M_n(\alpha,\beta,\gamma,\lambda) \\ &\iff J_v(f(z)) \in P_n(\alpha,\beta,\gamma,\lambda). \end{aligned}$$

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