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# CONSONANCY OF DYNAMIC INEQUALITIES CORRELATED ON TIME SCALE CALCULUS

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**Abstract**. In this paper, discrete and continuous versions of some inequalities such as Radon's Inequality, Bergström's Inequality, Nesbitt's Inequality, Rogers-Hölder's Inequality and Schlömilch's Inequality are unified on time scale calculus in extended form.

## 1. Introduction

The following inequality is a refinement of Radon's Inequality as given in [10].

**Theorem 1.1.** *If $n \in \mathbb{N} - \{1\}$, $c_1 \geq 0$, $c_2, c_3, c_4, x_k > 0$, $k \in \{1, 2, \ldots, n\}$, $X_n = \sum_{k=1}^{n} x_k$, $c_3 X_n > c_4 \max_{1\leq k\leq n} x_k$, $\zeta \geq 1$ and $\xi > 0$, then*

$$\frac{(c_1 n + c_2)^{\zeta}}{(c_3 n - c_4)^{\xi}} n^{\xi-\zeta+1} X_n^{\zeta-\xi} \leq \sum_{k=1}^{n} \frac{(c_1 X_n + c_2 x_k)^{\zeta}}{(c_3 X_n - c_4 x_k)^{\xi}}. \tag{1.1}$$

The following result is given in Problem 11634 from the American Mathematical Monthly, March 2012 as given in [9].

**Problem.** If $c_1 \geq 0$, $c_2, c_3, c_4, x_k > 0$, $k \in \{1, 2, \ldots, n\}$, $X_n = \sum_{k=1}^{n} x_k$, $\zeta \geq 1$, $\xi \geq 0$ and $c_3 X_n^{\zeta} > c_4 \max_{1\leq k\leq n} x_k^{\zeta}$, then

$$\frac{(c_1 n + c_2)}{(c_3 n^{\zeta} - c_4)^{\xi}} n^{\zeta\xi} X_n^{1-\zeta\xi} \leq \sum_{k=1}^{n} \frac{c_1 X_n + c_2 x_k}{\left(c_3 X_n^{\zeta} - c_4 x_k^{\zeta}\right)^{\xi}}. \tag{1.2}$$

We shall prove these results on time scales. The calculus of time scales was initiated by Hilger as given in [12]. A *time scale* is an arbitrary nonempty closed subset of the real numbers. In time scale calculus, results are unified and extended. The theory of time scales is applied to unify discrete and continuous analysis and to combine them in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference

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calculus, and quantum calculus, i.e., when $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{N}$ and $\mathbb{T}=q^{\mathbb{N}_0}=\{q^t : t\in\mathbb{N}_0\}$ where $q>1$. The time scale calculus is studied as delta calculus, nabla calculus and diamond-$\alpha$ calculus. This hybrid theory is also widely applied on dynamic inequalities. Basic work on dynamic inequalities is done by Agarwal, Anastassiou, Bohner, Peterson, O'Regan, Saker and many other authors.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a,b\in\mathbb{T}$ with $a<b$ and an interval $[a,b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

## 2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adapted from monographs [6, 7].

For $t\in\mathbb{T}$, the *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ is defined by

$$\sigma(t):=\inf\{s\in\mathbb{T}: s>t\}.$$

The mapping $\mu:\mathbb{T}\to\mathbb{R}_0^+=[0,+\infty)$ such that $\mu(t):=\sigma(t)-t$ is called the *forward graininess function*. The *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ is defined by

$$\rho(t):=\sup\{s\in\mathbb{T}: s<t\}.$$

The mapping $\nu:\mathbb{T}\to\mathbb{R}_0^+=[0,+\infty)$ such that $\nu(t):=t-\rho(t)$ is called the *backward graininess function*. If $\sigma(t)>t$, we say that $t$ is *right-scattered*, while if $\rho(t)<t$, we say that $t$ is *left-scattered*. Also, if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, then $t$ is called *right-dense*, and if $t>\inf\mathbb{T}$ and $\rho(t)=t$, then $t$ is called *left-dense*. If $\mathbb{T}$ has a left-scattered maximum $M$, then $\mathbb{T}^k=\mathbb{T}-\{M\}$, otherwise $\mathbb{T}^k=\mathbb{T}$.

For a function $f:\mathbb{T}\to\mathbb{R}$, the delta derivative $f^{\Delta}$ is defined as follows:

Let $t\in\mathbb{T}^k$, if there exists $f^{\Delta}(t)\in\mathbb{R}$ such that for all $\epsilon>0$, there exists a neighborhood $U$ of $t$, such that

$$|f(\sigma(t))-f(s)-f^{\Delta}(t)(\sigma(t)-s)|\leq\epsilon|\sigma(t)-s|,$$

for all $s\in U$, then $f$ is said to be *delta differentiable* at $t$, and $f^{\Delta}(t)$ is called the *delta derivative* of $f$ at $t$.

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be *right-dense continuous* (*rd-continuous*), if it is continuous at each right-dense point and there exists a finite left limit at every left-dense point. The set of all rd-continuous functions is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.

The next definition is given in [6, 7].

**Definition 2.1.** A function $F:\mathbb{T}\to\mathbb{R}$ is called a delta antiderivative of $f:\mathbb{T}\to\mathbb{R}$, provided that $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^k$. Then the delta integral of $f$ is defined by

$$\int_a^b f(t)\Delta t = F(b)-F(a).$$

The following results of nabla calculus are taken from [2, 6, 7].

If $\mathbb{T}$ has a right-scattered minimum $m$, then $\mathbb{T}_k=\mathbb{T}-\{m\}$, otherwise $\mathbb{T}_k=\mathbb{T}$. A function $f:\mathbb{T}_k\to\mathbb{R}$ is called *nabla differentiable* at $t\in\mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t)\in\mathbb{R}$ such that for any $\epsilon>0$, there exists a neighborhood $V$ of $t$, such that

$$|f(\rho(t))-f(s)-f^{\nabla}(t)(\rho(t)-s)|\le\epsilon|\rho(t)-s|,$$

for all $s\in V$.

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be *left-dense continuous* (*ld-continuous*), provided it is continuous at all left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$. The set of all ld-continuous functions is denoted by $C_{ld}(\mathbb{T},\mathbb{R})$.

The next definition is given in [2, 6, 7].

**Definition 2.2.** A function $G:\mathbb{T}\to\mathbb{R}$ is called a nabla antiderivative of $g:\mathbb{T}\to\mathbb{R}$, provided that $G^{\nabla}(t)=g(t)$ holds for all $t\in\mathbb{T}_k$. Then the nabla integral of $g$ is defined by

$$\int_a^b g(t)\nabla t = G(b)-G(a).$$

Now we present short introduction of diamond-$\alpha$ derivative as given in [1, 19].

**Definition 2.3.** Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the $\Delta$ and $\nabla$ senses. For $t\in\mathbb{T}_k^k$, where $\mathbb{T}_k^k=\mathbb{T}^k\cap\mathbb{T}_k$, the diamond-$\alpha$ dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t)=\alpha f^{\Delta}(t)+(1-\alpha)f^{\nabla}(t),\qquad 0\le\alpha\le 1.$$

Thus $f$ is diamond-$\alpha$ differentiable if and only if $f$ is $\Delta$ and $\nabla$ differentiable.

The diamond-$\alpha$ derivative reduces to the standard $\Delta$-derivative for $\alpha=1$, or the standard $\nabla$-derivative for $\alpha=0$. It represents a weighted dynamic derivative for $\alpha\in(0,1)$.

**Theorem 2.4** ([19])**.** *Let $f,g:\mathbb{T}\to\mathbb{R}$ be diamond-$\alpha$ differentiable at $t\in\mathbb{T}$ and we write $f^{\sigma}(t)=f(\sigma(t))$, $g^{\sigma}(t)=g(\sigma(t))$, $f^{\rho}(t)=f(\rho(t))$ and $g^{\rho}(t)=g(\rho(t))$. Then*

(i) *$f\pm g:\mathbb{T}\to\mathbb{R}$ is diamond-$\alpha$ differentiable at $t\in\mathbb{T}$, with*

$$(f\pm g)^{\diamond_\alpha}(t)=f^{\diamond_\alpha}(t)\pm g^{\diamond_\alpha}(t).$$

(ii) *$fg : \mathbb{T} \to \mathbb{R}$ is diamond-$\alpha$ differentiable at $t \in \mathbb{T}$, with*

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^{\sigma}(t)g^{\Delta}(t) + (1-\alpha)f^{\rho}(t)g^{\nabla}(t).$$

(iii) *For $g(t)g^{\sigma}(t)g^{\rho}(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond-$\alpha$ differentiable at $t \in \mathbb{T}$, with*

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^{\sigma}(t)g^{\rho}(t) - \alpha f^{\sigma}(t)g^{\rho}(t)g^{\Delta}(t) - (1-\alpha)f^{\rho}(t)g^{\sigma}(t)g^{\nabla}(t)}{g(t)g^{\sigma}(t)g^{\rho}(t)}.$$

**Definition 2.5** ([19])**.** Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond-$\alpha$ integral from $a$ to $t$ of $h$ is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s, \quad 0 \le \alpha \le 1,$$

provided that there exist delta and nabla integrals of $h$ on $\mathbb{T}$.

**Theorem 2.6** ([19])**.** *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$-integrable functions on $[a, b]_{\mathbb{T}}$. Then*

(i) $\displaystyle\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s;$

(ii) $\displaystyle\int_a^t cf(s) \diamond_\alpha s = c\int_a^t f(s) \diamond_\alpha s;$

(iii) $\displaystyle\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s;$

(iv) $\displaystyle\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s;$

(v) $\displaystyle\int_a^a f(s) \diamond_\alpha s = 0.$

We need the following result.

**Theorem 2.7** ([1])**.** *Let $a, b \in \mathbb{T}$ and $c, d \in \mathbb{R}$. Suppose that $g \in C([a, b]_{\mathbb{T}}, (c, d))$ and $h \in C([a, b]_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |h(s)| \diamond_\alpha s > 0$. If $\Phi \in C((c, d), \mathbb{R})$ is convex, then generalized Jensen's Inequality is*

$$\Phi\left(\frac{\int_a^b |h(s)|g(s) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}\right) \le \frac{\int_a^b |h(s)|\Phi\big(g(s)\big) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}. \tag{2.1}$$

*If $\Phi$ is strictly convex, then the inequality sign " $\le$ " in the above inequality can be replaced by " $<$ ".*

## 3. Main results

In order to present our main results, first we prove an extension of Radon's Inequality via time scales.

**Theorem 3.1.** *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$-integrable functions, $c_1 \geq 0$, $c_2, c_3, c_4 > 0$ and $c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x > c_4 \sup_{x \in [a,b]_{\mathbb{T}}} |f(x)|$. If $\zeta \geq 1$ and $\xi > 0$, then*

$$\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\xi-\zeta+1} \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\zeta-\xi} \frac{\left(c_1 \int_a^b |w(x)| \diamond_\alpha x + c_2\right)^{\zeta}}{\left(c_3 \int_a^b |w(x)| \diamond_\alpha x - c_4\right)^{\xi}}$$
$$\leq \int_a^b |w(x)| \frac{\left(c_1 \int_a^b |w(x)||f(x)| \diamond_\alpha x + c_2 |f(x)|\right)^{\zeta}}{\left(c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)|\right)^{\xi}} \diamond_\alpha x. \tag{3.1}$$

**Proof.** First we note that

$$\begin{aligned} & c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x > c_4 \sup_{x \in [a,b]_{\mathbb{T}}} |f(x)| \\ & \Rightarrow c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x > c_4 |f(x)|, \qquad \forall x \in [a,b]_{\mathbb{T}} \\ & \Rightarrow c_3 \int_a^b |w(x)| \diamond_\alpha x > c_4. \end{aligned}$$

If $|f(x)| = k(x) \int_a^b |w(x)||f(x)| \diamond_\alpha x$, then $\int_a^b |w(x)| k(x) \diamond_\alpha x = 1$.

The right-hand side of (3.1) can be written as

$$\begin{aligned} & \int_a^b |w(x)| \frac{\left(c_1 \int_a^b |w(x)||f(x)| \diamond_\alpha x + c_2 |f(x)|\right)^{\zeta}}{\left(c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)|\right)^{\xi}} \diamond_\alpha x \\ & = \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\zeta-\xi} \int_a^b |w(x)| \frac{(c_1 + c_2 k(x))^{\zeta}}{(c_3 - c_4 k(x))^{\xi}} \diamond_\alpha x. \end{aligned} \tag{3.2}$$

Let $\Phi(k(x)) = \frac{(c_1 + c_2 k(x))^{\zeta}}{(c_3 - c_4 k(x))^{\xi}} = (c_1 + c_2 k(x))^{\zeta} (c_3 - c_4 k(x))^{-\xi}$.

Clearly $\Phi(k(x))$ is a convex function on $\left(0, \frac{c_3}{c_4}\right)$ because it is the product of two convex functions.

Then we can apply Jensen's Inequality given in (2.1) and obtain

$$\frac{\left(c_1 + c_2 \frac{\int_a^b |w(x)| k(x) \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x}\right)^{\zeta}}{\left(c_3 - c_4 \frac{\int_a^b |w(x)| k(x) \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x}\right)^{\xi}} \leq \frac{\int_a^b |w(x)| \left(\frac{(c_1 + c_2 k(x))^{\zeta}}{(c_3 - c_4 k(x))^{\xi}}\right) \diamond_\alpha x}{\int_a^b |w(x)| \diamond_\alpha x}. \tag{3.3}$$

Inequality (3.1) is clear from (3.3).

Thus, the proof of Theorem 3.1 is completed. ∎

**Remark 3.2.** If we take $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n+1\}$, $n \in \mathbb{N} - \{1\}$, $X_n = \sum_{k=1}^{n} x_k$ and $c_3 X_n > c_4 \max_{1 \le k \le n} x_k$, then (3.1) reduces to (1.1).

**Corollary 3.3.** *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\diamond_\alpha$-integrable functions, $c_1 \ge 0$, $c_2, c_3, c_4 > 0$ and $c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x > c_4 \sup_{x \in [a,b]_{\mathbb{T}}} |f(x)|$. If $\xi > 0$, then*

$$\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right) \frac{\left(c_1 \int_a^b |w(x)| \diamond_\alpha x + c_2\right)^{\xi+1}}{\left(c_3 \int_a^b |w(x)| \diamond_\alpha x - c_4\right)^{\xi}}$$
$$\le \int_a^b |w(x)| \frac{\left(c_1 \int_a^b |w(x)||f(x)| \diamond_\alpha x + c_2 |f(x)|\right)^{\xi+1}}{\left(c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)|\right)^{\xi}} \diamond_\alpha x. \tag{3.4}$$

**Proof.** Applying Theorem 3.1 for $\zeta = \xi + 1$, we obtain Corollary 3.3. □

**Remark 3.4.** Let $c_1 = 0$, $c_2 = 1$ and $g(x) = c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)|$.

The inequality (3.4) takes the form

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\xi+1}}{\left(\int_a^b |w(x)| g(x) \diamond_\alpha x\right)^{\xi}} \le \int_a^b |w(x)| \frac{|f(x)|^{\xi+1}}{g^{\xi}(x)} \diamond_\alpha x. \tag{3.5}$$

If we take $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in (0, +\infty)$ and $g(k) = y_k \in (0, +\infty)$ for $k \in \{1, 2, \dots, n+1\}$, $n \in \mathbb{N}$, then discrete version of (3.5) reduces to the following Radon's Inequality

$$\frac{\left(\sum_{k=1}^{n} x_k\right)^{\xi+1}}{\left(\sum_{k=1}^{n} y_k\right)^{\xi}} \le \sum_{k=1}^{n} \frac{x_k^{\xi+1}}{y_k^{\xi}}, \tag{3.6}$$

as given in [16].

Inequality (3.6) is widely studied by many authors because it is used in many applications.

**Remark 3.5.** Let $c_1 = 0$, $c_2 = 1$ and $g(x) = c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)|$.

If $\xi = 1$, then inequality (3.5) takes the form

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{2}}{\int_a^b |w(x)| g(x) \diamond_\alpha x} \le \int_a^b |w(x)| \frac{|f(x)|^{2}}{g(x)} \diamond_\alpha x. \tag{3.7}$$

If we take $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in \mathbb{R} - \{0\}$ and $g(k) = y_k \in (0, +\infty)$ for $k \in \{1, 2, \ldots, n+1\}$, $n \in \mathbb{N}$, then discrete version of (3.7) reduces to

$$\frac{\left(\sum\limits_{k=1}^{n} x_k\right)^2}{\sum\limits_{k=1}^{n} y_k} \leq \sum_{k=1}^{n} \frac{x_k^2}{y_k}, \tag{3.8}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{x_n}{y_n}$.

The inequality from (3.8) is called Bergström's Inequality or Titu Andreescu's Inequality or also Engel's Inequality in literature as given in [3, 4, 5, 13].

**Corollary 3.6.** *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\diamond_\alpha$-integrable functions, $c_1 \geq 0$, $c_2, c_3, c_4 > 0$ and $c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x > c_4 \sup\limits_{x \in [a,b]_{\mathbb{T}}} |f(x)|$. Then*

$$\begin{aligned} &\left(\int_a^b |w(x)| \diamond_\alpha x\right)\left(\frac{c_1 \int_a^b |w(x)| \diamond_\alpha x + c_2}{c_3 \int_a^b |w(x)| \diamond_\alpha x - c_4}\right) \\ &\quad \leq \int_a^b |w(x)| \left(\frac{c_1 \int_a^b |w(x)||f(x)| \diamond_\alpha x + c_2 |f(x)|}{c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)|}\right) \diamond_\alpha x. \end{aligned} \tag{3.9}$$

**Proof.** Applying Theorem 3.1 for $\zeta = \xi = 1$, we obtain Corollary 3.6. □

**Remark 3.7.** If we take $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in (0, +\infty)$ for $k \in \{1, 2, \ldots, n+1\}$, $n \in \mathbb{N} - \{1\}$, $X_n = \sum\limits_{k=1}^{n} x_k$ and $c_3 X_n > c_4 \max\limits_{1 \leq k \leq n} x_k$, then discrete version of (3.9) becomes

$$\frac{n(c_1 n + c_2)}{c_3 n - c_4} \leq \sum_{k=1}^{n} \frac{c_1 X_n + c_2 x_k}{c_3 X_n - c_4 x_k}, \tag{3.10}$$

as given in [10].

The upcoming result is generalized Nesbitt's Inequality on time scales.

**Corollary 3.8.** *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\diamond_\alpha$-integrable functions, $c_3, c_4 > 0$ and $c_3 \displaystyle\int_a^b |w(x)||f(x)| \diamond_\alpha x > c_4 \sup\limits_{x \in [a,b]_{\mathbb{T}}} |f(x)|$. Then*

$$\begin{aligned} &\int_a^b |w(x)| \left(\frac{1}{c_3 \int_a^b |w(x)| \diamond_\alpha x - c_4}\right) \diamond_\alpha x \\ &\quad \leq \int_a^b |w(x)| \left(\frac{|f(x)|}{c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)|}\right) \diamond_\alpha x. \end{aligned} \tag{3.11}$$

**Proof.** Applying Corollary 3.6 for $c_1 = 0$ and $c_2 = 1$, we obtain Corollary 3.8. □

**Remark 3.9.** If we take $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in (0, +\infty)$ for $k \in \{1, 2, \ldots, n+1\}$, $n \in \mathbb{N} - \{1\}$, $X_n = \sum_{k=1}^{n} x_k$ and $c_3 X_n > c_4 \max_{1 \le k \le n} x_k$, then discrete version of (3.11) takes the form

$$\frac{n}{c_3 n - c_4} \le \sum_{k=1}^{n} \frac{x_k}{c_3 X_n - c_4 x_k}, \tag{3.12}$$

as given in [8].

Furthermore, if we take $n = 3$ and $c_3 = c_4$, then (3.12) takes the form

$$\frac{3}{2} \le \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_1} + \frac{x_3}{x_1 + x_2}, \tag{3.13}$$

for any $x_1, x_2, x_3 > 0$. The inequality from (3.13) is called Nesbitt's Inequality as given in [14].

The upcoming result is Rogers-Hölder's Inequality, which is given in [1, 15].

**Corollary 3.10.** *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\diamond_\alpha$-integrable functions. If $\frac{1}{\zeta} + \frac{1}{\gamma} = 1$ with $\zeta > 1$, then*

$$\int_a^b |w(x)||f(x)g(x)| \diamond_\alpha x \le \left( \int_a^b |w(x)||f(x)|^\zeta \diamond_\alpha x \right)^{\frac{1}{\zeta}} \left( \int_a^b |w(x)||g(x)|^\gamma \diamond_\alpha x \right)^{\frac{1}{\gamma}}. \tag{3.14}$$

**Proof.** Let $\zeta = \xi + 1$, $\xi > 0$, $c_1 = 0$, $c_2 = 1$ and $g(x) = c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)| > 0$. Then (3.1) takes the form

$$\frac{\left( \int_a^b |w(x)||f(x)| \diamond_\alpha x \right)^\zeta}{\left( \int_a^b |w(x)||g(x)| \diamond_\alpha x \right)^{\zeta - 1}} \le \int_a^b \frac{|w(x)||f(x)|^\zeta}{|g(x)|^{\zeta - 1}} \diamond_\alpha x. \tag{3.15}$$

Replacing $|w(x)|$ by $|w(x)||g(x)|^{\zeta-1}$ in (3.15), we get

$$\frac{\left( \int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x \right)^\zeta}{\left( \int_a^b |w(x)||g(x)|^\zeta \diamond_\alpha x \right)^{\zeta - 1}} \le \int_a^b |w(x)||f(x)|^\zeta \diamond_\alpha x. \tag{3.16}$$

Taking power $\frac{1}{\zeta} > 0$ on both sides, we get

$$\int_a^b |w(x)||f(x)||g(x)|^{\zeta-1} \diamond_\alpha x \le \left( \int_a^b |w(x)||f(x)|^\zeta \diamond_\alpha x \right)^{\frac{1}{\zeta}} \left( \int_a^b |w(x)||g(x)|^\zeta \diamond_\alpha x \right)^{1 - \frac{1}{\zeta}}. \tag{3.17}$$

Replacing $|g(x)|$ by $|g(x)|^{\frac{\gamma}{\zeta}}$ and using the fact that $\frac{1}{\zeta} + \frac{1}{\gamma} = 1$, we obtain our required result. $\square$

Now we present Schlömilch's Inequality on time scales. Symmetric versions of Schlömilch's Inequality are also given in [11, 17].

**Corollary 3.11.** *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$-integrable functions with $\int_a^b |w(x)| \diamond_\alpha x = 1$. If $0 < \eta_1 < \eta_2$, then*

$$\left(\int_a^b |w(x)||f(x)|^{\eta_1} \diamond_\alpha x\right)^{\frac{1}{\eta_1}} \le \left(\int_a^b |w(x)||f(x)|^{\eta_2} \diamond_\alpha x\right)^{\frac{1}{\eta_2}}. \tag{3.18}$$

**Proof.** Let $\zeta = \xi + 1$, $\xi > 0$, $c_1 = 0$, $c_2 = 1$ and $c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)| \equiv 1$. Then (3.1) becomes

$$\frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\zeta}}{\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\zeta-1}} \le \int_a^b |w(x)||f(x)|^{\zeta} \diamond_\alpha x. \tag{3.19}$$

Set $\zeta = \frac{\eta_2}{\eta_1} > 1$. Then (3.19) becomes

$$\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\frac{\eta_2}{\eta_1}} \le \int_a^b |w(x)||f(x)|^{\frac{\eta_2}{\eta_1}} \diamond_\alpha x. \tag{3.20}$$

Replacing $|f(x)|$ by $|f(x)|^{\eta_1}$ and taking power $\frac{1}{\eta_2} > 0$, we obtain (3.18). □

**Remark 3.12.** Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$-integrable functions, $c_1 = 0$, $c_2 = 1$ and $c_3 \displaystyle\int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)| > 0$ on $[a,b]_{\mathbb{T}}$. If $\zeta \ge \xi > 0$ and letting $\zeta$ be replaced by $\zeta - \xi + 1$, then (3.1) takes the form

$$\begin{aligned}
&\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{2\xi-\zeta} \frac{\left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\zeta-2\xi+1}}{\left(c_3 \int_a^b |w(x)| \diamond_\alpha x - c_4\right)^{\xi}} \\
&\quad \le \int_a^b \frac{|w(x)||f(x)|^{\zeta-\xi+1}}{\left(c_3 \int_a^b |w(x)||f(x)| \diamond_\alpha x - c_4 |f(x)|\right)^{\xi}} \diamond_\alpha x,
\end{aligned} \tag{3.21}$$

as given in [18].

**Theorem 3.13.** *Let $w, f \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$-integrable functions. If $\zeta \ge 1$, $\xi \ge 0$, $c_1 \ge 0$, $c_2, c_3, c_4 > 0$ and $c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\zeta} > c_4 \sup_{x \in [a,b]_{\mathbb{T}}} |f(x)|^{\zeta}$, then*

$$\begin{aligned}
&\left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\zeta\xi} \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{1-\zeta\xi} \left[\frac{c_1 \int_a^b |w(x)| \diamond_\alpha x + c_2}{\left\{c_3 \left(\int_a^b |w(x)| \diamond_\alpha x\right)^{\zeta} - c_4\right\}^{\xi}}\right] \\
&\quad \le \int_a^b |w(x)| \frac{c_1 \int_a^b |w(x)||f(x)| \diamond_\alpha x + c_2 |f(x)|}{\left\{c_3 \left(\int_a^b |w(x)||f(x)| \diamond_\alpha x\right)^{\zeta} - c_4 |f(x)|^{\zeta}\right\}^{\xi}} \diamond_\alpha x.
\end{aligned} \tag{3.22}$$

**Proof.** Let $\Lambda = \int_a^b |w(x)||f(x)|\diamond_\alpha x$ and $\Omega = \int_a^b |w(x)|\diamond_\alpha x$. We consider right hand side of (3.22) and by applying Radon's Inequality and Jensen's Inequality, we have that

$$\begin{aligned}
&\int_a^b |w(x)| \frac{c_1\Lambda + c_2|f(x)|}{\left(c_3\Lambda^\zeta - c_4|f(x)|^\zeta\right)^\xi} \diamond_\alpha x \\
&= \int_a^b |w(x)| \frac{\left(c_1\Lambda + c_2|f(x)|\right)^{\xi+1}}{\left(c_1\Lambda + c_2|f(x)|\right)^\xi \left(c_3\Lambda^\zeta - c_4|f(x)|^\zeta\right)^\xi} \diamond_\alpha x \\
&= \int_a^b |w(x)| \frac{\left(c_1\Lambda + c_2|f(x)|\right)^{\xi+1}}{\left(c_1c_3\Lambda^{\zeta+1} - c_1c_4|f(x)|^\zeta\Lambda + c_2c_3|f(x)|\Lambda^\zeta - c_2c_4|f(x)|^{\zeta+1}\right)^\xi} \diamond_\alpha x \\
&\geq \frac{\left\{\int_a^b |w(x)|\left(c_1\Lambda + c_2|f(x)|\right)\diamond_\alpha x\right\}^{\xi+1}}{\left\{\int_a^b |w(x)|\left(c_1c_3\Lambda^{\zeta+1} - c_1c_4|f(x)|^\zeta\Lambda + c_2c_3|f(x)|\Lambda^\zeta - c_2c_4|f(x)|^{\zeta+1}\right)\diamond_\alpha x\right\}^\xi} \\
&\geq \frac{\Lambda^{\xi+1}\left(c_1\Omega + c_2\right)^{\xi+1}}{\left(c_1c_3\Omega\Lambda^{\zeta+1} - c_1c_4\Lambda\frac{\Lambda^\zeta}{\Omega^{\zeta-1}} + c_2c_3\Lambda^{\zeta+1} - c_2c_4\frac{\Lambda^{\zeta+1}}{\Omega^\zeta}\right)^\xi} \\
&= \frac{\Omega^{\zeta\xi}\Lambda^{\xi+1}\left(c_1\Omega + c_2\right)^{\xi+1}}{\left(c_1c_3\Omega^{\zeta+1}\Lambda^{\zeta+1} - c_1c_4\Omega\Lambda^{\zeta+1} + c_2c_3\Omega^\zeta\Lambda^{\zeta+1} - c_2c_4\Lambda^{\zeta+1}\right)^\xi} \\
&= \frac{\Omega^{\zeta\xi}\Lambda^{1-\zeta\xi}\left(c_1\Omega + c_2\right)^{\xi+1}}{\left\{\left(c_1\Omega + c_2\right)\left(c_3\Omega^\zeta - c_4\right)\right\}^\xi} \\
&= \frac{\Omega^{\zeta\xi}\Lambda^{1-\zeta\xi}\left(c_1\Omega + c_2\right)}{\left(c_3\Omega^\zeta - c_4\right)^\xi}.
\end{aligned}$$

The proof of Theorem 3.13 is completed. □

**Remark 3.14.** If $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in (0, +\infty)$ for $k \in \{1, 2, \ldots, n+1\}$, $n \in \mathbb{N} - \{1\}$, $X_n = \sum_{k=1}^{n} x_k$ and $c_3 X_n^\zeta > c_4 \max_{1\leq k\leq n} x_k^\zeta$, then discrete version of (3.22) takes the form of (1.2).

**Remark 3.15.** If we set $\alpha = 1$, then we get delta versions and if we set $\alpha = 0$, then we get nabla versions of diamond-$\alpha$ integral operator inequalities presented in this article.

Also, if we set $\mathbb{T} = \mathbb{Z}$, then we get discrete versions and if we set $\mathbb{T} = \mathbb{R}$, then we get continuous versions of diamond-$\alpha$ integral operator inequalities presented in this article.

## 4. Conclusion and future work

In this research article, we have presented some dynamic inequalities for the diamond-$\alpha$ integral. Our work shows that many classical inequalities such as Radon's Inequality, Bergström's Inequality, Nesbitt's Inequality, Rogers-Hölder's Inequality and Schlömilch's Inequality can be refined and generalized on time scales.

In the future research, we will continue to explore other inequalities on time scales. We can consider dynamic inequalities by using the functional generalization, n-tuple diamond-$\alpha$ integral, fractional Riemann-Liouville integral, quantum calculus and $\alpha, \beta$-symmetric quantum calculus.

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