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ON L_1-BIHARMONIC TIMELIKE HYPERSURFACES IN PSEUDO-EUCLIDEAN SPACE $\mathbb{E}_1^4$

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Abstract. A well-known conjecture of Bang-Yen Chen says that the only biharmonic submanifolds in the Euclidean spaces are minimal ones. In this paper, we consider an extended condition (namely, L_1-biharmonicity) on non-degenerate timelike hypersurfaces of the pseudo-Euclidean space $\mathbb{E}_1^4$. A Lorentzian hypersurface $x : M_1^3 \to \mathbb{E}_1^4$ is called L_1-biharmonic if it satisfies the condition $L_1^2 x = 0$, where L_1 is the linearized operator associated with the first variation of 2th mean curvature vector field on M_1^3. According to the multiplicities of principal curvatures, the L_1-extension of Chen's conjecture is proved for Lorentzian hypersurfaces with constant ordinary mean curvature in the pseudo-Euclidean space $\mathbb{E}_1^4$. Additionally, we show that there is no proper L_1-biharmonic L_1-finite type connected orientable Lorentzian hypersurface in $\mathbb{E}_1^4$.

1. Introduction

Biharmonic surfaces in the Euclidean spaces play fundamental roles in the theory of elastics and also in fluid mechanics. For instance, the solutions of plane elastic problems can be stated in terms of biharmonic functions. In general, biharmonic maps appear in the theory of partial differential equations as the solutions of some 4-order strongly elliptic semilinear equations. Also, one can find the role of biharmonic Bezier surfaces in computational geometry. It is clear that the harmonic maps are biharmonic but not vis versa. By definition, a biharmonic map, which is not harmonic, is said to be *proper-biharmonic*. For instance, in the homotopy class of Brower of degree ± 1, one may not find a harmonic map as $\mathbb{T}^2 \to \mathbb{S}^2$. Although, there exists a proper-biharmonic map from $\mathbb{T}^2$ into $\mathbb{S}^2$ (introduced in [10]). From a geometric point of view, the variational problem associated with the bi-energy functional on the set of Riemannian metrics on a domain has given rise to the biharmonic stress-energy tensor. It is useful to obtain new examples of proper-biharmonic maps for the study of submanifolds with certain geometric properties, like pseudo-umbilical and parallel submanifolds.

A differential geometric motivation of the subject of biharmonic maps is a well-known result of Bang-Yen Chen (1987) which states that there is no proper-biharmonic surface in the

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Euclidean 3-spaces $\mathbb{E}^3$. Later on, Dimitrić in his doctoral thesis proved that any biharmonic hypersurface in $\mathbb{E}^m$ with at most two distinct principal curvatures is minimal ([9]). In 1995, Hasanis and Vlachos proved an extension of Chen's result to the hypersurfaces in Euclidean 4-spaces ([11]). In 2013, Akutagawa and Maeta ([1]) have generalized Chen's conjecture on biharmonic submanifolds Euclidean spaces. On the other hand, Chen himself had found a nice relationship between the finite type hypersurfaces and biharmonic ones. The theory of finite type hypersurfaces is a well-known subject interested by Chen (for instance, in [5, 6]) and also followed by L.J. Alias, S.M.B. Kashani and others. In [12], Kashani has studied the notion of L_1-finite type Euclidean hypersurfaces as an extension of finite type ones. One can see main results in Chapter 11 of Chen's book ([5]).

The map L_k, as an extension of the Laplace operator $L_0 = \Delta$, stands for the linearized operator of the first variation of the $(k+1)$th mean curvature of the hypersurface (see, for instance, [2, 13, 17, 18, 20]). This operator is defined by $L_k(f) = tr(P_k \circ \nabla^2 f)$ for any $f \in C^\infty(M)$, where P_k denotes the k-th Newton transformation associated to the second fundamental from of the hypersurface and $\nabla^2 f$ is the hessian of f.

As an extended case, a hypersurface $x : M_p^3 \to \mathbb{E}_1^4$, whose mean curvature vector field is an eigenvector of the Laplace operator Δ, has been studied, for instance, in [7, 8] for the Euclidean case (where, $p = s = 0$), and for the Lorentz case in [3, 4] (for $s = 1$ and $p = 0, 1$). It is interesting to generalize the definition of biharmonic hypersurface by replacing Δ by L_1. Recently, in [15], we have studied the L_1-biharmonic spacelike hypersurfaces in 4-dimentional Minkowski space $\mathbb{E}_1^4$. In this paper, we study the Lorentzian hypersurfaces in Einstein space $\mathbb{E}_1^4$. We pay attention to proper-L_1-biharmonic timelike hypersurfaces as $x : M_1^3 \to \mathbb{E}_1^4$ with three mutually distinct principal curvatures and constant mean curvature. Now, we state here our main results.

Theorem 1.1. *Each L_1-biharmonic connected orientable timelike hypersurface in the Lorentz-Minkowski 4-space with three distinct real principal curvatures and constant mean curvature is 1-minimal.*

Theorem 1.2. *Each L_1-biharmonic connected orientable timelike hypersurface in the Lorentz-Minkowski 4-space with two distinct complex and one constant real principal curvatures and constant mean curvature is 1-minimal.*

Theorem 1.3. *Each L_1-biharmonic connected orientable timelike hypersurface in the Lorentz-Minkowski 4-space with at most two distinct principal curvatures is 1-minimal.*

Theorem 1.4. *There is no L_1-biharmonic L_1-finite type connected orientable Lorentzian hypersurface in the Lorentz-Minkowski 4-space.*

2. Preliminaries

In this section, we recall preliminaries from [2, 13, 14] and [16]-[19]. The 4-dimensional pseudo-Euclidean space, denoted by $\mathbb{E}_1^4$, is the real vector space $\mathbb{R}^4$ endowed with the indefinite inner product defined by $< x, y >:= -x_1 y_1 + \Sigma_{i=2}^4 x_i y_i$, for every $x, y \in \mathbb{R}^4$. In $\mathbb{E}_1^4$, any nondegenerate hypersurface M_p^3 can be endowed with a Riemannian or a Lorentzian induced metric of index $p = 0$ or $p = 1$, according to whether the induced metric is positive definite or indefinite.

Throughout the paper, we study on a Lorentz hypersurface in $\mathbb{E}_1^4$, denoted by an isometric immersion $x : M_1^3 \to \mathbb{E}_1^4$. The symbols $\tilde{\nabla}$ and $\bar{\nabla}$ stand for the Levi-Civita connection on M_1^3 and $\mathbb{E}_1^4$, respectively. For every tangent vector fields X and Y on M, the Gauss formula is given by $\bar{\nabla}_X Y = \tilde{\nabla}_X Y + < SX, Y > \mathbf{n}$, for every $X, Y \in \chi(M)$, where, $\mathbf{n}$ is a (locally) unit normal vector field on M and S is the shape operator of M relative to $\mathbf{n}$. For each non-zero vector $X \in \mathbb{E}_1^4$, the real value $< X, X >$ may be a negative, zero or positive number and then, the vector X is said to be time-like, light-like or space-like, respectively. According to whether the induced metric on a nondegenerate hypersurface M_r^3 of index r in $\mathbb{E}_1^4$ is positive definite or indefinite, M_r^3 is called Riemannian (when $r = 0$) or Lorentzian (when $r = 1$), and therefore every normal vector on M_r^3 is time-like or light-like, respectively.

For a Lorentzian vector space V_1^3, a basis $\mathscr{B} := \{e_1, e_2, e_3\}$ is said to be *orthonormal* if it satisfies $< e_i, e_j >= \epsilon_i \delta_i^j$ for $i, j = 1, 2, 3$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, 3$. As usual, δ_i^j stands for the Kronecker function. $\mathscr{B}$ is called *pseudo-orthonormal* if it satisfies $< e_1, e_1 >=< e_2, e_2 >= 0$, $< e_1, e_2 >= -1$ and $< e_i, e_j >= \delta_i^j$, for $i = 1, \ldots, n$ and $j = 3, \ldots, n$.

As well-known, the shape operator of the Lorentzian hypersurface M_1^3, as a self-adjoint linear map on the tangent space of M_1^3, can be put into one of four possible canonical matrix forms, usually denoted by I, II, III and IV. Where, in cases I and IV, with respect to an orthonormal basis of the tangent space of M_1^3, the matrix representation of the induced metric on M_1^3 is

$$G_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the shape operator S of M_1^3 can be put into matrix forms

$$B_1 = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \text{and} \qquad B_4 = \begin{pmatrix} \kappa & \lambda & 0 \\ -\lambda & \kappa & 0 \\ 0 & 0 & \eta \end{pmatrix}, \quad (\lambda \neq 0)$$

respectively. For cases II and III, using a pseudo-orthonormal basis of the tangent space of

M_1^3, the induced metric on M_1^3 has matrix form

$$G_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and the shape operator S of M_1^3 can be put into matrix forms

$$B_2 = \begin{pmatrix} \kappa & 0 & 0 \\ 1 & \kappa & 0 \\ 0 & 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad B_3 = \begin{pmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{pmatrix},$$

respectively. In case IV, the matrix B_4 has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.

Remark 2.1. In two cases II and III, one can substitute the pseudo-orthonormal basis $\mathscr{B} := \{e_1, e_2, e_3\}$ by a new orthonormal basis $\tilde{\mathscr{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3\}$ where $\tilde{e_1} := \frac{1}{2}(e_1+e_2)$ and $\tilde{e_2} := \frac{1}{2}(e_1-e_2)$. Therefore, we obtain new matrix representations $\tilde{B_2}$ and $\tilde{B_3}$ (instead of B_2 and B_3, respectively) as

$$\tilde{B_2} = \begin{pmatrix} \kappa+\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \kappa-\frac{1}{2} & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{and} \quad \tilde{B_3} = \begin{pmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\sqrt{2}/2 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{pmatrix}$$

After this changes, to unify the notations we denote the orthonormal basis by $\mathscr{B}$ in all cases.

Notation: According to four possible matrix representations of the shape operator of M_1^3, we define its principal curvatures, denoted by unified notations κ_i for $i = 1,2,3$, as follow.

In case I, we put $\kappa_i := \lambda_i$, for $i = 1,2,3$, where λ_i's are the eigenvalues of B_1.

In cases II, where the matrix representation of S is $\tilde{B_2}$, we take $\kappa_i := \kappa$ for $i = 1,2$, and $\kappa_3 := \lambda$.

In case III, where the shape operator has matrix representation $\tilde{B_3}$, we take $\kappa_i := \kappa$ for $i = 1,2,3$.

Finally, in the case IV, where the shape operator has matrix representation $\tilde{B_4}$, we put $\kappa_1 = \kappa + i\lambda$, $\kappa_2 = \kappa - i\lambda$, and $\kappa_3 := \eta$.

The characteristic polynomial of S on M_1^3 is of the form $Q(t) = \prod_{i=1}^{3}(t-\kappa_i) = \sum_{j=0}^{3}(-1)^j s_j t^{3-j}$, where, $s_0 := 1$, $s_1 = \sum_{j=1}^{3}\kappa_j$, $s_2 := \sum_{1\le i_1<i_2\le 3}\kappa_{i_1}\kappa_{i_2}$ and $s_3 := \kappa_1\kappa_2\kappa_3$.

For $j = 1,2,3$, the *j-th mean curvature* H_j of M is defined by $H_j = \frac{1}{\binom{3}{j}}s_j$. When H_j is identically null, M_1^n is said to be *$(j-1)$-minimal*.

Definition 2.2.

(i) A timelike hypersurface $x : M_1^3 \to \mathbb{E}_1^4$, with diagonalizable shape operator, is said to be *isoparametric* if all of its principal curvatures are constant.

(ii) A timelike hypersurface $x : M_1^3 \to \mathbb{E}_1^4$, with non-diagonalizable shape operator, is said to be *isoparametric* if the minimal polynomial its shape operator has constant coefficients.

Remark 2.3. Here we remind Theorem 4.10 from [14], which assures us that there is no isoparametric timelike hypersurface of $\mathbb{E}_1^4$ with complex principal curvatures.

The well-known Newton transformations on the hypersurface, $P_j : \chi(M) \to \chi(M)$, is defined by

$$P_0 = I, \;\; P_j = s_j I - S \circ P_{j-1}, \quad (j = 1,2,3), \tag{2.1}$$

where, I is the identity map. Using its explicit formula, $P_j = \sum_{i=0}^{j} (-1)^i s_{j-i} S^i$ (where $S^0 = I$), it can be seen that, P_j is self-adjoint and commutative with S (see [2, 17]).

Now, we define a notation as

$$\mu_{j;k} = \sum_{l=0}^{k} (-1)^l \binom{n}{k-l} H_{k-l} \kappa_j^l. \qquad (1 \le j \le 3,\ 1 \le k < 3) \tag{2.2}$$

Corresponding to the four possible forms $\tilde{B}_i$ (for $1 \le i \le 4$) of S, the Newton transformation P_j has different representations. In the case I, where $S_p = \tilde{B}_1$, we have
$P_j(p) = diag[\mu_{1;j}(p), \mu_{2;j}(p), \mu_{3;j}(p)]$, for $j = 1,2$.

When $S = B_2$ (in the case II), we have

$$P_1 = \begin{pmatrix} \kappa + \lambda - \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \kappa + \lambda + \frac{1}{2} & 0 \\ 0 & 0 & 2\kappa \end{pmatrix}, \qquad P_2 = \begin{pmatrix} (\kappa - \frac{1}{2})\lambda & -\frac{1}{2}\lambda & 0 \\ \frac{1}{2}\lambda & (\kappa + \frac{1}{2})\lambda & 0 \\ 0 & 0 & \kappa^2 \end{pmatrix}.$$

In the case III, we have $S_p = B_3$, and

$$P_1 = \begin{pmatrix} 2\kappa & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 2\kappa & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 2\kappa \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \kappa^2 - \frac{1}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2}\kappa \\ \frac{1}{2} & \kappa^2 + \frac{1}{2} & \frac{\sqrt{2}}{2}\kappa \\ \frac{\sqrt{2}}{2}\kappa & \frac{\sqrt{2}}{2}\kappa & \kappa^2 \end{pmatrix}.$$

In the case IV, $S = B_4$,

$$P_1 = \begin{pmatrix} \kappa + \eta & -\lambda & 0 \\ \lambda & \kappa + \eta & 0 \\ 0 & 0 & 2\kappa \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \kappa\eta & -\lambda\eta & 0 \\ \lambda\eta & \kappa\eta & 0 \\ 0 & 0 & \kappa^2 + \lambda^2 \end{pmatrix}.$$

Fortunately, in all cases we have the following important identities for $j=1,2$, similar to those in [2, 17].

$$tr(P_1)=6H_1,\quad tr(P_2)=3H_2,\quad tr(P_1\circ S)=6H_2,\quad tr(P_2\circ S)=3H_3, \tag{2.3}$$

$$trS^2=9H_1^2-6H_2,\quad tr(P_1\circ S^2)=9H_1H_2-3H_3,\quad tr(P_2\circ S^2)=3H_1H_2. \tag{2.4}$$

The *linearized operator* of the $(j+1)$th mean curvature of M, $L_j:\mathscr{C}^\infty(M)\to\mathscr{C}^\infty(M)$ is defined by the formula $L_j(f):=tr(P_j\circ\nabla^2 f)$, where, $<\nabla^2 f(X),Y>=<\nabla_X\nabla f,Y>$ for every $X,Y\in\chi(M)$.

Associated to the orthonormal frame $\{e_1,e_2,e_3\}$ of tangent space on a local coordinate system in the hypersurface $x:M_1^3\to\mathbb{E}_1^4$, $L_1(f)$ has an explicit expression as

$$L_1(f)=\sum_{i=1}^{3}\epsilon_i\mu_{i,1}(e_ie_if-\nabla_{e_i}e_if). \tag{2.5}$$

For a Lorentzian hypersurface $x:M_1^3\to\mathbb{E}_1^4$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a}\in\mathbb{E}_1^4$ we use the decomposition $\mathbf{a}=\mathbf{a}^T+\mathbf{a}^N$ where $\mathbf{a}^T\in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N\perp TM$, and we have the following formulae from [2, 17].

$$\nabla<x,\mathbf{a}>=\mathbf{a}^T,\quad \nabla<\mathbf{n},\mathbf{a}>=-S\mathbf{a}^T. \tag{2.6}$$

$$L_1x=c_1H_2\mathbf{n},\quad L_1\mathbf{n}=-3\nabla(H_2)-3[3H_1H_2-H_3]\mathbf{n}, \tag{2.7}$$

$$L_1^2x=6L_1(H_2\mathbf{n})=-6[9H_2\nabla H_2-2P_2\nabla H_2]-6[9H_1H_2^2+3H_2H_3-L_1H_2]\mathbf{n}. \tag{2.8}$$

Assume that a hypersurface $x:M_1^3\to\mathbb{E}_1^4$ satisfies the condition $L_1^2x=0$, then it is said to be L_1-*biharmonic*. An L_1-biharmonic hypersurface $x:M_1^3\to\mathbb{E}_1^4$ is said to be *proper-L_1-biharmonic*, if it satisfies the condition $L_1x\neq 0$.

By equalities (2.7) and (2.8), from the condition $L_1(H_2\mathbf{n})=0$ (which is equivalent to L_1-biharmonicity) we obtain simpler conditions on M_1^3 to be a L_1-biharmonic hypersurface in $\mathbb{E}_1^4$, as:

$$\text{(i) } L_1H_2=3(3H_1H_2^2-H_2H_3)=H_2tr(S^2\circ P_1),\qquad \text{(ii) } P_2\nabla H_2=\frac{9}{2}H_2\nabla H_2. \tag{2.9}$$

The structure equations of $\mathbb{E}_1^4$ are given by

$$d\omega_i=\sum_{j=1}^{4}\omega_{ij}\wedge\omega_j,\quad \omega_{ij}+\omega_{ji}=0, \tag{2.10}$$

$$d\omega_{ij}=\sum_{l=1}^{4}\omega_{il}\wedge\omega_{lj}. \tag{2.11}$$

With restriction to M, we have $\omega_4=0$ and then,

$$0=d\omega_4=\sum_{i=1}^{3}\omega_{4,i}\wedge\omega_i. \tag{2.12}$$

By Cartan's lemma, there exist functions h_{ij} such that

$$\omega_{4,i} = \sum_{j=1}^{3} h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.13}$$

This gives the second fundamental form of M, as $B = \sum_{i,j} h_{ij}\omega_i\omega_j e_4$. The mean curvature H is given by $H = \frac{1}{3}\sum_{i=1}^{3} h_{ii}$. From (2.10)–(2.13) we obtain the structure equations of M as follow.

$$d\omega_i = \sum_{j=1}^{3} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.14}$$

$$d\omega_{ij} = \sum_{k=1}^{3} \omega_{ik} \wedge \omega_{kj} - \frac{1}{2}\sum_{k,l=1}^{3} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.15}$$

for $i, j = 1, 2, 3$, and the Gauss equations

$$R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{2.16}$$

where R_{ijkl} denotes the components of the Riemannian curvature tensor of M.

Let h_{ijk} denote the covariant derivative of h_{ij}. We have

$$dh_{ij} = \sum_{k=1}^{3} h_{ijk}\omega_k + \sum_{k=1}^{3} h_{kj}\omega_{ik} + \sum_{k=1}^{3} h_{ik}\omega_{jk}. \tag{2.17}$$

Thus, by exterior differentiation of (2.13), we obtain the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{2.18}$$

Now we recall the definition of an L_1-finite type hypersurface from [12], which is the basic notion of the paper.

Definition 2.4. An isometrically immersed hypersurface $x : M_1^3 \to \mathbb{E}_1^4$ is said to be of L_1-finite type if x has a finite decomposition $x = \sum_{i=0}^{m} x_i$, for some positive integer m, satisfying the condition $L_1 x_i = \tau_i x_i$, where , $\tau_i \in \mathbb{R}$ and $x_i : M^3 \to \mathbb{E}_1^4$ is smooth maps, for $i = 1, 2, \ldots, m$, and x_0 is constant. If all τ_i's are mutually different, M^n is said to be of L_1-m-type. An L_1-m-type hypersurface is said to be null if for at least one i $(1 \le i \le m)$ we have $\tau_i = 0$.

3. L_1-biharmonic timelike hypersurfeces of $\mathbb{E}_1^4$

The next lemma can be proved by a similar proof as in [20].

Lemma 3.1. *Let M_1^3 be a timelike hypersurface in $\mathbb{E}_1^4$ of type I with principal curvatures of constant multiplicities. Then the distribution of the space of principal directions corresponding to the principal curvatures is completely integrable. In addition, if a principal curvature is of multiplicity greater than one, then it will be constant on each integral submanifold of the corresponding distribution.*

Theorem 3.2. *There is no L_1-biharmonic L_1-finite type Lorentzian hypersurface in $\mathbb{E}_1^4$.*

Proof. Let $x : M_1^3 \to \mathbb{E}_1^4$ be an L_1-biharmonic L_1-finite type Lorentzian hypersurface in $\mathbb{E}_1^4$. By assumption, the map x has a finite decomposition as

$$x = x_0 + x_{t_1} + \cdots + x_{t_k}, \tag{3.1}$$

with $L_1 x_0 = 0$ and $L_1 x_{t_i} = \lambda_{t_i} x_{t_i}$ for nonzero distinct eigenvalues $\lambda_{t_1}, \ldots, \lambda_{t_k}$ of L_1. By the action of L_1^s on both sides of equality (3.1), we obtain

$$0 = L_1^s x = \lambda_{t_1}^s x_{t_1} + \cdots + \lambda_{t_k}^s x_{t_k}, \tag{3.2}$$

for $s = 1, 2, 3, \ldots$. Since $\lambda_{t_1}, \ldots, \lambda_{t_k}$ are distinct eigenvalues of L_1, equation (3.2) can not hold. □

Corollary 3.3. *There is no L_1-biharmonic L_1-finite type connected orientable Riemannian or Lorentzian hypersurface in the Lorentz-Minkowski 4-space.*

For instance, we see two examples of non L_1-biharmonic timelike hypersurfaces in $\mathbb{E}_1^4$.

Example 3.4. Let $M_1^3(r)$ be the product $\mathbb{S}_1^2(r) \times \mathbb{E}^1 \subset \mathbb{E}_1^4$ where $r > 0$. It has another representation as

$$M_1^3(r) = \{(y_1, \ldots, y_4) \in \mathbb{E}_1^4 | -y_1^2 + y_2^2 + y_3^2 = r^2\},$$

having the spacelike vector field $\mathbf{n}(y) = -\frac{1}{r}(y_1, y_2, y_3, 0)$ as the Gauss map. Clearly, it has two distinct principal curvatures $\kappa_1 = \kappa_2 = \frac{1}{r}$, $\kappa_3 = 0$, and the constant higher order mean curvatures $H_1 = \frac{2}{3} r^{-1}$, $H_2 = \frac{1}{3} r^{-2}$ and $H_3 = 0$. One can see that $L_1^2 x \neq 0$.

Example 3.5. Let $\bar{M}_1^3(r)$ be the product $\mathbb{E}_1^2 \times \mathbb{S}^1(r) \subset \mathbb{E}_1^4$ where $r > 0$. It can be represented as

$$\bar{M}_1^3(r) = \{(y_1, \ldots, y_4) \in \mathbb{R}_1^4 | y_3^2 + y_4^2 = r^2\},$$

with the Gauss map $\mathbf{n}(y) = -\frac{1}{r}(0, 0, y_3, y_4)$. it has two distinct principal curvatures $\kappa_1 = \kappa_2 = 0$, $\kappa_3 = \frac{1}{r}$, and the constant higher order mean curvatures $H_1 = \frac{1}{3r}$, and $H_k = 0$ for $k = 2, 3$. So, Also, one can see that $L_1^2 x \neq 0$ we have $L_k^2 x = 0$ for $k = 2, 3$.

3.1. Timelike hypersurfeces of type I

Proposition 3.6. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be an L_1-biharmonic connected orientable timelike hypersurface with three distinct real principal curvatures and constant ordinary mean curvature. Then, the 2nd mean curvature of M has to be constant.*

Proof. Suppose that, H_2 is non-constant. Considering the open subset $\mathscr{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathscr{U} = \emptyset$. By the assumption M_1^3 has three distinct principal curvature, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form B_1, such that $Se_i = \lambda_i e_i$ and then, $P_2 e_i = \mu_{i,2} e_i$ for $i = 1,2,3$. Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (2.9(ii)) we get

$$e_i(H_2)(\mu_{i,2} - \frac{9}{2} H_2) = 0, \tag{3.3}$$

for $i = 1,2,3$. Each point of $\mathscr{U}$ has an open neighborhood on which we have $e_i(H_2) \neq 0$ for at least one i. So, without loss of generality, we can assume that $e_1(H_2) \neq 0$ and then we have $\mu_{1,2} = \frac{9}{2} H_2$, (locally) on $\mathscr{U}$, which gives $\lambda_2 \lambda_3 = \frac{9}{2} H_2$. Now, we prove three simple claims.

Claim 1: $e_2(H_2) = e_3(H_2) = 0$.
If $e_2(H_2) \neq 0$ or $e_3(H_2) \neq 0$, then by (3.3) we get $\mu_{1,2} = \mu_{2,2} = \frac{9}{2} H_2$ or $\mu_{1,2} = \mu_{3,2} = \frac{9}{2} H_2$, which give $\lambda_3(\lambda_2 - \lambda_1) = 0$ or $\lambda_2(\lambda_1 - \lambda_3) = 0$. But, since λ_i's are assumed to be mutually distinct, we get $\lambda_3 = 0$ or $\lambda_2 = 0$, which gives $H_2 = 0$ on $\mathscr{U}$. The result is in contradiction with the definition of $\mathscr{U}$.

Claim 2: $e_2(\lambda_1) = e_3(\lambda_1) = 0$.
Since H is assumed to be constant on M, we have $e_2(\lambda_1) = e_2(3H - \lambda_1 - \lambda_2) = -e_2(\lambda_1) - e_2(\lambda_2)$. On the other hand, from two recent results $e_2(H_2) = 0$ and $\lambda_2 \lambda_3 = \frac{9}{2} H_2$ we get

$$e_2(\lambda_1 \lambda_3) + e_2(\lambda_1 \lambda_2) = e_2(3H_2 - \frac{9}{2} H_2) = 0,$$

which gives $\lambda_1 e_2(\lambda_2 + \lambda_3) + (\lambda_2 + \lambda_3) e_2 \lambda_1 = 0$, and then we have

$$\lambda_1 e_2(3H - \lambda_1) + (\lambda_2 + \lambda_3) e_2 \lambda_1 = \lambda_1 e_2(-\lambda_1) + (\lambda_2 + \lambda_3) e_2 \lambda_1 = (-\lambda_1 + \lambda_2 + \lambda_3) e_2 \lambda_1 = 0.$$

Therefore, assuming $e_2(\lambda_1) \neq 0$, we get $\lambda_1 = \lambda_2 + \lambda_3$ which gives contradiction

$$e_2(\lambda_1) = e_2(\lambda_2 + \lambda_3) = e_2(3H - \lambda_1) = -e_2(\lambda_1).$$

Consequently, $e_2(\lambda_1) = 0$.

Similarly, one can show $e_3(\lambda_1) = 0$. So, Claim 2 is proved.

Claim 3: $e_2(\lambda_3) = e_3(\lambda_2) = 0$.
Using the notations

$$\nabla_{e_i} e_j = \sum_{k=1}^{3} \omega_{ij}^k e_k, \ \ (i,j=1,2,3), \tag{3.4}$$

and the compatibility condition $\nabla_{e_k} < e_i, e_j >= 0$, we have

$$\omega_{ki}^i = 0, \quad \omega_{ki}^j + \omega_{kj}^i = 0, \quad (i,j,k=1,2,3) \tag{3.5}$$

and applying the Codazzi equation (see [16], page 115, Corollary 34(2))

$$(\nabla_V S)W = \nabla_W S)V, \quad (\forall V, W \in \chi(M)) \tag{3.6}$$

on the basis $\{e_1, e_2, e_3\}$, we get for distinct $i, j, k = 1,2,3$

$$\text{(a)} \ \ e_i(\lambda_j) = (\lambda_i - \lambda_j)\omega_{ji}^j, \qquad \text{(b)} \ \ (\lambda_i - \lambda_j)\omega_{ki}^j = (\lambda_k - \lambda_j)\omega_{ik}^j. \tag{3.7}$$

Also, by a straightforward computation of components of the identity $\nabla_{e_i} S)e_j - \nabla_{e_j} S)e_i \equiv 0$ for distinct $i, j = 1,2,3$, we get $[e_2,\ e_3](H_2) = 0$, $\omega_{12}^1 = \omega_{13}^1 = \omega_{13}^2 = \omega_{21}^3 = \omega_{32}^1 = 0$ and

$$\omega_{21}^2 = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}, \quad \omega_{31}^3 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}, \omega_{23}^2 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}, \quad \omega_{32}^3 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}. \tag{3.8}$$

Therefore, the covariant derivatives $\nabla_{e_i} e_j$ simplify to $\nabla_{e_1} e_k = 0$ for $k = 1,2,3$, and

$$\begin{aligned}
&\nabla_{e_2} e_1 = \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2} e_2, \qquad \nabla_{e_3} e_1 = \frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} e_3, \qquad \nabla_{e_2} e_2 = \frac{e_1(\lambda_2)}{\lambda_2 - \lambda_1} e_1, \\
&\nabla_{e_3} e_2 = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3} e_3, \qquad \nabla_{e_2} e_3 = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2} e_2, \qquad \nabla_{e_3} e_3 = \frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1} e_1 + \frac{e_2(\lambda_3)}{\lambda_3 - \lambda_2} e_2.
\end{aligned} \tag{3.9}$$

Now, the Gauss equation for $< R(e_2, e_3)e_1,\ e_2 >$ and $< R(e_2, e_3)e_1,\ e_3 >$ show that

$$e_3\left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right) = \frac{e_3(\lambda_2)}{\lambda_3 - \lambda_2}\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right), \tag{3.10}$$

$$e_2\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}\right) = \frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3} - \frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right). \tag{3.11}$$

We also have the Gauss equation for $< R(e_1, e_2)e_1,\ e_2 >$ and $< R(e_3, e_1)e_1,\ e_3 >$, which give the following relations

$$e_1\left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right) + \left(\frac{e_1(\lambda_2)}{\lambda_1 - \lambda_2}\right)^2 = \lambda_1\lambda_2, \quad e_1\left(\frac{e_1(\lambda_3)}{\lambda_1 - \lambda_3}\right) + \left(\frac{e_1(\lambda_3)}{\lambda_3 - \lambda_1}\right)^2 = \lambda_1\lambda_3. \tag{3.12}$$

Finally, we obtain from the Gauss equation for $< R(e_3, e_1)e_2, e_3 >$ that

$$e_1\left(\frac{e_2(\lambda_3)}{\lambda_2 - \lambda_3}\right) = \frac{e_1(\lambda_3)e_2(\lambda_3)}{(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_3)}. \tag{3.13}$$

On the other hand, we consider the L_1-biharmonic condition (2.9). It follows from Claim I that

$$-\mu_{1,1}e_1e_1(H_2)+\left(\mu_{2,1}\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\mu_{3,1}\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)e_1(H_2)-9H_2^2(H-\frac{3}{2}\lambda_1)=0. \tag{3.14}$$

By differentiating (3.14) along on e_2 respectively e_3, and using (3.10), (3.11) we obtain

$$e_2\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}\right)=\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}-\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right), \tag{3.15}$$

$$e_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}\right)=\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}-\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right). \tag{3.16}$$

Using (3.9), we find that

$$[e_1,\ e_2]=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}e_2. \tag{3.17}$$

Applying both sides of the equality (3.17) on $\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}$, using (3.15), (3.12), and (3.13), we deduce that

$$\frac{e_2(\lambda_3)}{\lambda_2-\lambda_3}\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0. \tag{3.18}$$

(3.18) shows that $e_2(\lambda_3)=0$ or

$$\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}=\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}. \tag{3.19}$$

From equation (3.19), by differentiating on its both sides along e_1 and applying (3.12), we get $\lambda_2=\lambda_3$, which is a contradiction, so we have to confirm the result $e_2(\lambda_3)=0$.

Analogously, using (3.9), we find that $[e_1,\ e_3]=\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}e_3$. By a similar manner, we deduce that

$$\frac{e_3(\lambda_2)}{\lambda_3-\lambda_2}\left(\frac{e_1(\lambda_2)}{\lambda_2-\lambda_1}+\frac{e_1(\lambda_3)}{\lambda_1-\lambda_3}\right)=0, \tag{3.20}$$

and one can show that $e_3(\lambda_2)$ necessarily has to be vanished.

Hence, we have obtained $e_2(\lambda_3)=e_3(\lambda_2)=0$ which, by applying the Gauss equation for $<R(e_2,e_3)e_1,e_3>$, gives the following equality

$$\frac{e_1(\lambda_3)e_1(\lambda_2)}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}=\lambda_2\lambda_3. \tag{3.21}$$

Finally, using (3.12), differentiating (3.21) along e_1 gives

$$\lambda_2\lambda_3\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)=0, \tag{3.22}$$

which implies $\lambda_2\lambda_3=0$ (since we have seen above that $\left(\frac{e_1(\lambda_3)}{\lambda_3-\lambda_1}+\frac{e_1(\lambda_2)}{\lambda_1-\lambda_2}\right)\neq 0$). Therefore, we obtain $H_2=0$ on $\mathcal{U}$, which is a contradiction. Hence H_2 is constant on M^3. □

Theorem 3.7. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be an L_1-biharmonic timlike hypersurface with shape operator of type I in $\mathbb{E}_1^4$. If M_1^3 has constant mean curvature and three distinct principal curvatures, then it is 1-minimal.*

Proof. By assumption H_1 is assumed to be constant and then, by Theorem 3.6 it is proved that H_2 has to be constant. We claim that H_2 is null. If $H_2 \neq 0$, by using [(2.9)(i)] we obtain that H_3 is constant. Therefore, all of mean curvatures H_i (for $i = 1,2,3$) are constant, which means that M^3 is isoparametric. By Corollary 2.7 in [14], an isoparametric Lorentzian hypersurface of Case I in the Einstein space has at most one nonzero principal curvature distinct principal, which contradicts with the assumption that, three principal curvatures of M are assumed to be mutually distinct. So $H_2 \equiv 0$. □

Theorem 3.8. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be a L_1-biharmonic Lorentzian hypersurfaces of $\mathbb{E}_1^4$ with diagonalizable shape operator (i.e of type I). If M_1^3 has exactly two distinct principal curvatures, then it is 1-minimal.*

Proof. By assumption, M_1^3 has two distinct principal curvatures λ_1 and λ_2 of multiplicities 1 and 2, respectively. Defining the open subset $\mathscr{U}$ of M as $\mathscr{U} := \{p \in M_1^3 : \nabla H_2^2(p) \neq 0\}$, we prove that $\mathscr{U}$ is empty. Assuming $\mathscr{U} \neq \varnothing$, we consider $\{e_1, e_2, e_3\}$ as a local orthonormal frame of principal directions of S on $\mathscr{U}$ such that $Se_i = \lambda_i e_i$ for $i = 1,2,3$. By assumption, we have

$$\lambda_1 = \lambda_2 = \lambda, \qquad \lambda_3 = \mu.$$

Therefore, we obtain

$$\mu_{1,2} = \mu_{2,2} = \lambda\mu, \quad \mu_{3,2} = \lambda^2, \quad 3H_2 = \lambda^2 + 2\lambda\mu. \tag{3.23}$$

By condition [(2.9)(ii)], we have

$$P_2(\nabla H_2) = \frac{9}{2} H_2 \nabla H_2. \tag{3.24}$$

Then, using the polar decomposition

$$\nabla H_2 = \sum_{i=1}^{3} \epsilon_i < \nabla H_2, e_i > e_i, \tag{3.25}$$

we see that (3.24) is equivalent to

$$\epsilon_i < \nabla H_2, e_i > (\mu_{i,2} - \frac{9}{2} H_2) = 0$$

on $\mathscr{U}$ for $i = 1,2,3$. Hence, for every i such that $< \nabla H_2, e_i > \neq 0$ on $\mathscr{U}$ we get

$$\mu_{i,2} = \frac{9}{2} H_2. \tag{3.26}$$

By definition, we have $\nabla H_2 \neq 0$ on $\mathscr{U}$, which gives one or both of the following states.

State 1. $< \nabla H_2, e_i > \neq 0$, for $i = 1$ or $i = 2$. By equalities (3.23) and (3.26), we obtain

$$\lambda\mu = \frac{9}{2}(\frac{2}{3}\lambda\mu + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(2\mu + \frac{3}{2}\lambda) = 0. \tag{3.27}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we get $\mu = -\frac{3}{4}\lambda$, $H_2 = -\frac{1}{6}\lambda^2$.

State 2. $< \nabla H_2, e_3 > \neq 0$. By equalities (3.23) and (3.26), we obtain

$$\lambda^2 = \frac{9}{2}(\frac{2}{3}\lambda\mu + \frac{1}{3}\lambda^2),$$

which gives

$$\lambda(3\mu + \frac{1}{2}\lambda) = 0. \tag{3.28}$$

If $\lambda = 0$ then $H_2 = 0$. Otherwise, we have $\mu = -\frac{1}{6}\lambda$, $H_2 = \frac{2}{9}\lambda^2$.

Both states requires the same calculation, so we consider for instance State 2.
By Lemma 3.1, let us denote the maximal integral submanifold through $x \in \mathscr{U}$, corresponding to λ by $\mathscr{U}_1^{n-1}(x)$. We write

$$d\lambda = \sum_{i=1}^{3} \lambda_{,i}\omega_i \qquad d\mu = \sum_{j=1}^{3} \mu_{,j}\omega_j. \tag{3.29}$$

Then, Lemma 3.1 implies that $\lambda_{,1} = \lambda_{,2} = 0$. We can assume that $\lambda > 0$ on $\mathscr{U}$, then we have (in State 2)

$$\mu = \frac{-1}{6}\lambda < 0. \tag{3.30}$$

By means of (2.17), we obtain

$$\sum_{k=1}^{3} h_{ijk}\omega_k = \delta_{ij}d\lambda_j + (\lambda_i - \lambda_j)\omega_{ij}, \tag{3.31}$$

for $i, j, k = 1, 2, 3$. Here, we adopt the notational convention that $a, b, c = 1, 2$.
From (3.29) and (3.31), we have

$$\begin{aligned} &h_{12k} = h_{21k} = 0, \\ &h_{aab} = 0, \quad h_{aa3} = \lambda_{,3}, \\ &h_{33a} = 0, \quad h_{333} = \mu_{,3}. \end{aligned} \tag{3.32}$$

Combining this with (2.18) and the formula

$$\sum_{i=1}^{3} h_{a3i}\omega_i = dh_{a3} + \sum_{i=1}^{3} h_{i3}\omega_{ia} + \sum_{i=1}^{3} h_{ai}\omega_{i3} = (\lambda-\mu)\omega_{a3},$$

we obtain from (3.30)

$$\omega_{a3} = \frac{\lambda_{,3}}{\lambda-\mu}\omega_a = \frac{6\lambda_{,3}}{7\lambda}\omega_a. \tag{3.33}$$

Therefore we have

$$d\omega_3 = \sum_{a=1}^{2} \omega_{3a}\wedge\omega_a = 0.$$

Notice that we may consider λ to be locally a function of the parameter s, where s is the arc length of an orthogonal trajectory of the family of the integral submanifolds corresponding to λ. We may put $\omega_3 = ds$.
Thus, for $\lambda = \lambda(s)$, we have

$$d\lambda = \lambda_{,3}ds, \quad \lambda_{,3} = \lambda'(s),$$

so from (3.33), we get

$$\omega_{a3} = \frac{\lambda_{,3}}{\lambda-\mu}\omega_a = \frac{6\lambda'(s)}{7\lambda}\omega_a. \tag{3.34}$$

According to the structure equations of $\mathbb{E}_1^4$ and (3.34), we may compute

$$\begin{aligned}
&\text{(i):}\ d\omega_{a3} = \sum_{b=1}^{2}\omega_{ab}\wedge\omega_{b3} + \omega_{a4}\wedge\omega_{43} = \left(\frac{6\lambda'}{7\lambda}\right)\sum_{b=1}^{2}\omega_{ab}\wedge\omega_b - \lambda\mu\omega_a\wedge ds,\\
&\text{(ii):}\ d\omega_{a3} = d\left\{\frac{6\lambda'}{7\lambda}\omega_a\right\} = \left(\frac{6\lambda'}{7\lambda}\right)' ds\wedge\omega_a + \left(\frac{6\lambda'}{7\lambda}\right)d\omega_a\\
&\qquad = \left\{-\left(\frac{6\lambda'}{7\lambda}\right)' + \left(\frac{6\lambda'}{7\lambda}\right)^2\right\}\omega_a\wedge ds + \left(\frac{6\lambda'}{7\lambda}\right)\sum_{b=1}^{2}\omega_{ab}\wedge\omega_b.
\end{aligned} \tag{3.35}$$

Comparing equalities [(3.35)(i)] and [(3.35)(ii)], we get $\left(\frac{6\lambda'}{7\lambda}\right)' - \left(\frac{6\lambda'}{7\lambda}\right)^2 - \lambda\mu = 0$, which, by combining with (3.30), gives

$$\left(\frac{6\lambda'}{7\lambda}\right)' - \left(\frac{6\lambda'}{7\lambda}\right)^2 - \left(\frac{-1}{6}\right)\lambda^2 = 0. \tag{3.36}$$

Defining function $\beta(s) := \left(\frac{1}{\lambda(s)}\right)^{\frac{6}{7}}$ for $s\in(-\infty,+\infty)$, from (3.36) we get $\beta'' = \left(\frac{1}{6}\right)\beta^{\frac{-8}{6}}$, which by integrating, gives $(\beta')^2 = -\beta^{\frac{-2}{6}} + c$, where c is the constant of integration. The last equation is equivalent to

$$(\lambda')^2 = -\left(\frac{7}{6}\right)^2\lambda^4 + c\left(\frac{7}{6}\right)^2\lambda^{\frac{26}{7}}. \tag{3.37}$$

Now, in order to compare two sides of condition [(2.9)(i)], we need to compute $\nabla_{e_i}\nabla H_2$ and $P_1(e_i)$ for $i=1,2,3$. From (3.27) we have $\nabla H_2 = \frac{4}{9}\lambda\lambda' e_3$, which by using (3.34), gives

$$\begin{aligned}\nabla_{e_a}\nabla H_2 &= \frac{4}{9}\lambda\lambda'\nabla_{e_a}e_3 = \frac{4}{9}\lambda^r\lambda'\sum_b \omega_{3b}(e_a)e_b = -\frac{8}{21}\lambda'^2 e_a,\\ \nabla_{e_3}\nabla H_2 &= \frac{4}{9}\nabla_{e_3}(\lambda\lambda' e_3) = \frac{4}{9}\lambda'^2 e_3 + \frac{4}{9}\lambda\lambda'' e_3.\end{aligned} \tag{3.38}$$

By using (3.23) and (3.30), we compute $P_1(e_a)$ and $P_1(e_3)$.

$$P_1(e_1) = \frac{5}{6}\lambda e_1, \quad P_1(e_2) = \frac{5}{6}\lambda e_2 \quad P_1(e_3) = 2\lambda e_3. \tag{3.39}$$

From (3.38) and (3.39), we get

$$L_1 H_2 = 6H_2\left(\frac{-10(\lambda')^2}{21\lambda} + \frac{2(\lambda')^2}{3\lambda} + \frac{2}{3}\lambda''\right). \tag{3.40}$$

From [(2.9)(i)], we have $L_1 H_2 = H_2\, tr(S^2\circ P_1) = 2H_2\frac{11}{6}\lambda^3$, which Combining with (3.40), gives

$$\lambda\lambda'' + \left(1+\frac{-5}{7}\right)\lambda'^2 - 2\frac{33}{12}\lambda^4 = 0. \tag{3.41}$$

On the other hand, the equality (3.36) is equivalent to

$$\lambda\lambda'' = \frac{13}{7}\lambda'^2 + \frac{-7}{36}\lambda^4. \tag{3.42}$$

Now, substituting (3.42) and (3.41), we obtain

$$\frac{15}{7}\lambda'^2 + \frac{191}{36}\lambda^4 = 0. \tag{3.43}$$

From equations (3.37), (3.43) and (3.27), we get that H_2 is locally constant on $\mathscr{U}$, which is a contradiction with the definition of $\mathscr{U}$. Hence H_2 is constant on M.

By a similar discussion, one can get the same result in State 1. □

3.2. Timelike hypersurfeces of types II

Proposition 3.9. *Let $x: M_1^3 \to \mathbb{E}_1^4$ be an L_1-biharmonic connected orientable timelike hypersurface with shape operator of type II in $\mathbb{E}_1^4$. If M_1^3 has constant ordinary mean curvature, then its 2nd mean curvature has to be constant.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathscr{U} = \{p\in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathscr{U} = \emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_2$, such that $Se_1 = (\kappa+\frac{1}{2})e_1 - \frac{1}{2}e_2$, $Se_2 = \frac{1}{2}e_1 + (\kappa-\frac{1}{2})e_2$, $Se_3 = \lambda e_3$ and then, we have $P_2 e_1 = (\kappa-\frac{1}{2})\lambda e_1 + \frac{1}{2}\lambda e_2$, $P_2 e_2 = -\frac{1}{2}\lambda e_1 + (\kappa+\frac{1}{2})\lambda e_2$ and $P_2 e_3 = \kappa^2 e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (2.9(ii)) we get

$$\begin{aligned}
&\text{(i)} \quad \epsilon_1 e_1(H_2)[(\kappa - \frac{1}{2})\lambda - \frac{9}{2}H_2] = \epsilon_2 e_2(H_2)\frac{\lambda}{2} \\
&\text{(ii)} \quad \epsilon_2 e_2(H_2)[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2] = -\epsilon_1 e_1(H_2)\frac{\lambda}{2}, \\
&\text{(iii)} \quad \epsilon_3 e_3(H_2)(\kappa^2 - \frac{9}{2}H_2) = 0.
\end{aligned} \tag{3.44}$$

Now, we prove some simple claims.

Claim 1: $e_1(H_2) = e_2(H_2) = e_3(H_2) = 0$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities [(3.44)(i , ii)] by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned}
&\text{(i)} \quad \Big(\kappa - \frac{1}{2}\Big)\lambda - \frac{9}{2}H_2 = \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\frac{\lambda}{2}, \\
&\text{(ii)} \quad \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\Big[(\kappa + \frac{1}{2})\lambda - \frac{9}{2}H_2\Big] = -\frac{\lambda}{2},
\end{aligned} \tag{3.45}$$

which, by substituting (i) in (ii), gives $\frac{\lambda}{2}(1+u)^2 = 0$, where $u := \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}$. Then $\lambda = 0$ or $u = -1$. If $\lambda = 0$, then we get $H_2 = 0$ from [(3.45)(i)]. Also, by assumption $\lambda \neq 0$ we get $u = -1$ which gives $\kappa\lambda = \frac{9}{2}H_2$, then $\kappa(3\kappa + 4\lambda) = 0$ and finally $\kappa = -\frac{4}{3}\lambda$ (since $\kappa = 0$ gives $H_2 = 0$ again). Hence, we have $H_2 = \frac{2}{9}\kappa\lambda = -\frac{8}{27}\lambda^2$ and $H_1 = -\frac{5}{9}\lambda$, and since H_1 is assumed to be constant, H_2 has to be constant and we have $e_1(H_2) = 0$, which is a contradiction. Therefore, the first claim is proved. The second claim (i.e. $e_2(H_2) = 0$) can be proven by a similar manner.

Now, if $e_3(H_2) \neq 0$, then by [(3.44)(iii)] we get $\kappa^2 = \frac{9}{2}H_2$, then $\kappa(\kappa + 6\lambda) = 0$, which gives $\kappa = 0$ or $\kappa = -6\lambda$. If $\kappa = 0$, then $H_2 = 0$, and if $\kappa = -6\lambda$ then since $H_1 = -\frac{11}{3}\lambda$ is assumed to be constant, we get that H_2 is constant and then $e_3(H_2) = 0$. Which is a contradiction, so we have $e_3(H_2) = 0$. □

Theorem 3.10. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be an L_1-biharmonic timlike hypersurface with shape operator of type II in $\mathbb{E}_1^4$. If M_1^3 has constant mean curvature, then it is 1-minimal.*

Proof. By assumption H_1 is assumed to be constant and then, by Proposition 3.9 it is proved that H_2 has to be constant. We claim that H_2 is null. Since the shape operator is of type II, there exist two possible cases as:

Case 1: M_1^3 has a principal curvature κ of multiplicity 3;

Case 2: M_1^3 has two principal curvatures κ and λ of multiplicities 2 and 1, respectively.

In Case 1, we have $H_1 = \kappa$, $H_2 = \kappa^2$ and $H_3 = \kappa^3$. By [(2.9)(i)], we have $3H_1H_2^2 = H_2H_3$, which gives $\kappa^5 = 0$, and then $H_2 = 0$.

In Case 2, we have $H_1=\frac{1}{3}(2\kappa+\lambda)$, $H_2=\frac{1}{3}(\kappa^2+2\kappa\lambda)$ and $H_3=\kappa^2\lambda$. We assume that $H_2\neq 0$ and continue in two subcases as follow. Since $H_2\neq 0$, then $\kappa\neq 0$ and by using [(2.9)(i)] we obtain that H_3 is constant. Therefore, all of mean curvatures H_i (for $i=1,2,3$) are constant, which means that M_1^3 is isoparametric. By Corollary 2.7 in [14], an isoparametric Lorentzian hypersurface of Case II in the Einstein space has at most one nonzero principal curvature, so we get $\lambda=0$. Then $H_1=\frac{2}{3}\kappa$, $H_2=\frac{1}{3}\kappa^2$ and $H_3=0$, hence, by [(2.9))(i)], we get $\kappa=0$, which contradicts with the assumption of this case. Therefore $H_2=0$. □

3.3. Timelike hypersurfeces of types III

Proposition 3.11. *Let $x: M_1^3\to\mathbb{E}_1^4$ be an L_1-biharmonic connected orientable timelike hypersurface with shape operator of type III in $\mathbb{E}_1^4$. Then M_1^3 has constant 2nd mean curvature.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathscr{U}=\{p\in M:\nabla H_2^2(p)\neq 0\}$, we try to show $\mathscr{U}=\emptyset$. By the assumption, with respect to a suitable (local) orthonormal tangent frame $\{e_1,e_2,e_3\}$ on M, the shape operator S has the matrix form $\tilde{B}_3$, such that $Se_1=\kappa e_1+\frac{\sqrt{2}}{2}e_3$, $Se_2=\kappa e_2-\frac{\sqrt{2}}{2}e_3$, $Se_3=-\frac{\sqrt{2}}{2}e_1-\frac{\sqrt{2}}{2}e_2+\kappa e_3$ and then, we have $P_2e_1=(\kappa^2-\frac{1}{2})e_1-\frac{1}{2}e_2-\frac{\sqrt{2}}{2}\kappa e_3$, $P_2e_2=\frac{1}{2}e_1+(\kappa^2+\frac{1}{2})e_2+\frac{\sqrt{2}}{2}\kappa e_3$ and $P_2e_3=\frac{\sqrt{2}}{2}\kappa e_1+\frac{\sqrt{2}}{2}\kappa e_2+\kappa^2e_3$.

Using the polar decomposition $\nabla H_2=\sum_{i=1}^{3}\epsilon_i e_i(H_2)e_i$, from condition (2.9(ii)) we get

$$\begin{aligned}
&\text{(i)}\quad \epsilon_1e_1(H_2)[(\kappa^2-\frac{1}{2})-\frac{9}{2}H_2]+\frac{1}{2}\epsilon_2e_2(H_2)+\frac{\sqrt{2}}{2}\epsilon_3e_3(H_2)\kappa=0\\
&\text{(ii)}\quad \frac{-1}{2}\epsilon_1e_1(H_2)+\epsilon_2e_2(H_2)[(\kappa^2+\frac{1}{2})-\frac{9}{2}H_2]+\frac{\sqrt{2}}{2}\epsilon_3e_3(H_2)\kappa=0\\
&\text{(iii)}\quad \epsilon_1e_1(H_2)\frac{-\sqrt{2}}{2}\kappa+\epsilon_2e_2(H_2)\frac{\sqrt{2}}{2}\kappa+\epsilon_3e_3(H_2)(\kappa^2-\frac{9}{2}H_2)=0.
\end{aligned}\tag{3.46}$$

Now, we prove some simple claims.

Claim: $e_1(H_2)=e_2(H_2)=e_3(H_2)=0$.
If $e_1(H_2)\neq 0$, then by dividing both sides of equalities [(3.44)(i, ii, iii)] by $\epsilon_1e_1(H_2)$, and using the identity $H_2=\kappa^2$ in Case III, we get

$$\begin{aligned}
&\text{(i)}\quad -\frac{1}{2}-\frac{7}{2}\kappa^2+\frac{1}{2}u_1+\frac{\sqrt{2}}{2}u_2\kappa=0\\
&\text{(ii)}\quad \frac{-1}{2}+u_1(\frac{1}{2}-\frac{7}{2}\kappa^2)+\frac{\sqrt{2}}{2}u_2\kappa=0\\
&\text{(i)}\quad \frac{-\sqrt{2}}{2}\kappa+\frac{\sqrt{2}}{2}u_1\kappa-\frac{7}{2}\kappa^2)u_2=0,
\end{aligned}\tag{3.47}$$

where $u_1:=\frac{\epsilon_2e_2(H_2)}{\epsilon_1e_1(H_2)}$ and $u_2:=\frac{\epsilon_3e_3(H_2)}{\epsilon_1e_1(H_2)}$, which, by comparing (i) and (ii), gives $\kappa^2(u_1-1)=0$. If $\kappa=0$, then $H_2=0$. Assuming $\kappa\neq 0$, we get $u_1=1$, which, using [(3.47)(iii)], gives $u_2=0$.

Substituting $u_1 = 1$ and $u_2 = 0$ in [(3.47)(i)], we obtain again $\kappa = 0$, which is a contradiction. Hence $e_1(H_2) \equiv 0$.

Therefore, using the result $e_1(H_2) \equiv 0$, the system of equations (3.46) gives

$$\begin{aligned}
&\text{(i)} \quad \frac{1}{2}\epsilon_2 e_2(H_2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0\\
&\text{(ii)} \quad \epsilon_2 e_2(H_2)(\frac{1}{2} - \frac{7}{2}\kappa^2) + \frac{\sqrt{2}}{2}\epsilon_3 e_3(H_2)\kappa = 0\\
&\text{(iii)} \quad \epsilon_2 e_2(H_2)\frac{\sqrt{2}}{2}\kappa - \epsilon_3 e_3(H_2)\frac{7}{2}\kappa^2 = 0.
\end{aligned} \tag{3.48}$$

Comparing (i) and (ii), we get $\kappa e_2(H_2) = 0$, which using iii gives $\kappa e_3(H_2) = 0$, and then, using (i), gives $e_2(H_2) = 0$. Then, the second claim (i.e. $e_2(H_2) = 0$) is proved.

Now, using the results $e_1(H_2) = e_2(H_2) = 0$, we get $\kappa e_3(H_2) = 0$, which, using $H_2 = \kappa^2$, implies $\kappa e_3(\kappa^2) = 0$ and then $e_3(\kappa^3) = 0$, and finally $e_3(H_2) = 0$. □

Theorem 3.12. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be a L_1-biharmonic timlike hypersurface with shape operator of type III in $\mathbb{E}_1^4$. Then, it is 1-minimal. Furthermore, all of mean curvatures of M_1^3 are null.*

Proof. By Proposition 3.11, the 2th mean curvature of M_1^3 is constant, which, by [(2.9)(i)], gives $L_1 H_2 = 9H_1 H_2^2 - 3H_2 H_3 = 0$, and then $3H_1 H_2^2 = H_2 H_3$, which, using $H_1 = \kappa$, $H_2 = \kappa^2$ and $H_3 = \kappa^3$, gives $\kappa^5 = 0$, and then $H_1 = H_2 = H_3 = 0$. □

3.4. Timelike hypersurfeces of types IV

Proposition 3.13. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be an L_1-biharmonic connected orientable timelike hypersurface with shape operator of type IV in $\mathbb{E}_1^4$. If M_1^3 has constant mean curvature and a constant real principal curvature, then its 2nd and 3rd mean curvatures are constant.*

Proof. Suppose that, H_2 be non-constant. Considering the open subset $\mathcal{U} = \{p \in M : \nabla H_2^2(p) \neq 0\}$, we try to show $\mathcal{U} = \emptyset$. By the assumption M_1^3 has three distinct principal curvature, then, with respect to a suitable (local) orthonormal tangent frame $\{e_1, e_2, e_3\}$ on M, the shape operator S has the matrix form B_4, such that $Se_1 = \kappa e_1 - \lambda e_2$, $Se_2 = \lambda e_1 + \kappa e_2$, $Se_3 = \eta e_3$ and then, we have $P_2 e_1 = \kappa\eta e_1 + \lambda\eta e_2$, $P_2 e_2 = -\lambda\eta e_1 + \kappa\eta e_2$ and $P_2 e_3 = (\kappa^2 + \lambda^2)e_3$.

Using the polar decomposition $\nabla H_2 = \sum_{i=1}^{3} \epsilon_i e_i(H_2) e_i$, from condition (2.9(ii)) we get

$$\begin{aligned}
&\text{(i)} \quad \epsilon_1 e_1(H_2)(\kappa\eta - \frac{9}{2}H_2) = \epsilon_2 e_2(H_2)\lambda\eta,\\
&\text{(ii)} \quad \epsilon_2 e_2(H_2)(\kappa\eta - \frac{9}{2}H_2) = -\epsilon_1 e_1(H_2)\lambda\eta,\\
&\text{(iii)} \quad \epsilon_3 e_3(H_2)(\kappa^2 + \lambda^2 - \frac{9}{2}H_2) = 0.
\end{aligned} \tag{3.49}$$

Now, we prove three simple claims.

Claim 1: $e_1(H_2) = e_2(H_2) = 0$.
If $e_1(H_2) \neq 0$, then by dividing both sides of equalities [(3.49)(i , ii)] by $\epsilon_1 e_1(H_2)$ we get

$$\begin{aligned} &\text{(i)} \quad \kappa\eta - \frac{9}{2}H_2 = \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\lambda\eta, \\ &\text{(ii)} \quad \frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}(\kappa\eta - \frac{9}{2}H_2) = -\lambda\eta, \end{aligned} \tag{3.50}$$

which, by substituting (i) in (ii), gives $\lambda\eta\left(1+\left(\frac{\epsilon_2 e_2(H_2)}{\epsilon_1 e_1(H_2)}\right)^2\right) = 0$, then $\lambda\eta = 0$. Since by assumption $\lambda \neq 0$, we get $\eta = 0$. So, by [(3.50)(i)], we have $H_2 = 0$.

Similarly, if $e_2(H_2) \neq 0$, then by dividing both sides of equalities [(3.49)(i , ii)] by $\epsilon_2 e_2(H_2)$ we get

$$\begin{aligned} &\text{(i)} \quad \frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}(\kappa\eta - \frac{9}{2}H_2) = \lambda\eta, \\ &\text{(ii)} \quad \kappa\eta - \frac{9}{2}H_2 = -\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}\lambda\eta, \end{aligned} \tag{3.51}$$

which, by substituting (i) in (ii), gives $\lambda\eta\left(1+\left(\frac{\epsilon_1 e_1(H_2)}{\epsilon_2 e_2(H_2)}\right)^2\right) = 0$, then $\lambda\eta = 0$. Since by assumption $\lambda \neq 0$, we get $\eta = 0$. So, by [(3.51)(ii)], we have $H_2 = 0$.

Claim 2: $e_3(H_2) = 0$.
If $e_3(H_2) \neq 0$, then from equality [(3.49)(iii)] we have $\kappa^2 + \lambda^2 = \frac{9}{2}H_2$, which gives $\kappa^2 + \lambda^2 = -6\kappa\eta$, where $\eta = 3H_1 - 2\kappa$ and η and H_1 are assumed to be constant on $\mathscr{U}$. So, κ is also constant on $\mathscr{U}$, and then, we obtain $H_2 = \frac{-4}{3}\kappa\eta = \frac{8}{3}\kappa^2 - 4H_1\kappa$ and $H_3 = -6\kappa\eta^2 = -6\kappa(3H_1 - 2\kappa)^2$. are constant on $\mathscr{U}$. □

Theorem 3.14. *Let $x : M_1^3 \to \mathbb{E}_1^4$ be an L_1-biharmonic connected orientable timelike hypersurface with shape operator of type IV in $\mathbb{E}_1^4$. If M_1^3 has constant mean curvature and a constant real principal curvature, then it is* 1*-minimal.*

Proof. By Proposition 3.13, the 2th mean curvature of M_1^3 is constant, which gives $L_1 H_2 = 0$. Then, by [(2.9)(i)], we have $9H_1H_2^2 - 3H_2H_3 = 0$, which gives $(7\eta - 4\kappa)\kappa^2\eta^2 = 0$.

Now, if $7\eta = 4\kappa$, then from $\kappa^2 + \lambda^2 = -6\kappa\eta$ we get $\frac{31}{7}\kappa^2 + \lambda^2 = 0$, and then $\kappa = \lambda = 0$, which gives $H_2 = H_3 = 0$. Also, if $\kappa^2\eta^2 = 0$, then we have $H_2 = H_3 = 0$. □

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References

[1] K. Akutagawa and S. Maeta, *Biharmonic properly immersed submanifolds in Euclidean spaces*, Geom. Dedicata, **164** (2013), 351–355.

[2] L. J. Alias, Gürbüz, N., *An extension of Takahashi theorem for the linearized operators of the higher order mean curvatures*, Geom. Dedicata, **121** (2006), 113–127.

[3] A. Arvanitoyeorgos, F. Defever, G. Kaimakamis and B. J. Papantoniou, *Biharmonic Lorentz hypersurfaces in* E_1^4, Pacific J. Math., **229** (2007), 293–306.

[4] A. Arvanitoyeorgos, F. Defever and G. Kaimakamis, *Hypersurfaces in* E_s^4 *with proper mean curvature vector*, J. Math. Soc. Japan, **59** (2007), 797–809.

[5] B. Y. Chen, *Total Mean Curvature and Submanifolds of Finite Type*, Series in Pure Mathematics, 2. World Scientific Publishing Co, Singapore, 2014.

[6] B. Y. Chen, *Some open problems and conjetures on submanifolds of finite type*, Soochow J. Math., **17** (1991), 169–188.

[7] F. Defever, *Hypersurfaces of* E^4 *satisfying* $\Delta\vec{H} = \lambda\vec{H}$, Michigan. Math. J., **44** (1997), 355–363.

[8] F. Defever, *Hypersurfaces of* E^4 *with harmonic mean curvature vector*, Math. Nachr., **196** (1998), 61–69.

[9] I. Dimitrić, *Submanifolds of* E^n *with harmonic mean curvature vector*, Bull. Inst. Math. Acad. Sin., **20** (1992), 53–65.

[10] J. Eells and J. C. Wood, *Restrictions on harmonic maps of surfaces*, Topology, **15** (1976), 263–266.

[11] T. Hasanis and T. Vlachos, *Hypersurfaces in* E^4 *with harmonic mean curvature vector field*, Math. Nachr., **172** (1995), 145–169.

[12] S. M. B. Kashani, *On some* L_1*-finite type (hyper)surfaces in* $\mathbb{R}^{n+1}$, Bull. Korean Math. Soc., **46** (2009), 35–43.

[13] P. Lucas and H. F. Ramirez-Ospina, *Hypersurfaces in the Lorentz-Minkowski space satisfying* $L_k\psi = A\psi + b$, Geom. Dedicata, **153** (2011), 151–175.

[14] M. A. Magid, *Lorentzian isoparametric hypersurfaces*, Pacific J. of Math., **118** (1985), 165–197.

[15] F. Pashaie and A. Mohammadpouri, L_k*-biharmonic spacelike hypersurfaces in Minkowski* 4*-space* $\mathbb{E}_1^4$, Sahand Comm. Math. Anal., **5** (2017), 21–30.

[16] B. O'Neill, Semi-Riemannian Geometry with Applicatins to Relativity, Acad. Press Inc., 1983.

[17] F. Pashaie and S. M. B. Kashani, *Spacelike hypersurfaces in Riemannian or Lorentzian space forms satisfying* $L_k x = Ax + b$, Bull. Iran. Math. Soc., **39** (2013), 195–213.

[18] F. Pashaie and S. M. B. Kashani, *Timelike hypersurfaces in the Lorentzian standard space forms satisfying* $L_k x = Ax + b$, Mediterr. J. Math., **11** (2014), 755–773.

[19] A. Z. Petrov, Einstein Spaces, Pergamon Press, Hungary, Oxford and New York, 1969.

[20] R. C. Reilly, *Variational properties of functions of the mean curvatures for hypersurfaces in space forms*, J. Differential Geom., **8** (1973), 465–477.

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