The Principal Eigenvalue Problems for Perturbed Fractional Laplace Operators

Guangyu Zhao

Abstract. We study a variety of basic properties of the principal eigenvalue of a perturbed fractional Laplace operator and weakly coupled cooperative systems involving fractional Laplace operators. Our work extends a number of well-known properties regarding the principal eigenvalues of linear second order elliptic operators with Dirichlet boundary condition to perturbed fractional Laplace operators. The establish results are also utilized to investigate the spatio-temporary dynamics of population models.

1 Introduction

In this paper, we are concerned with the principal eigenvalue problem for a perturbed fractional Laplace operator:

$$\begin{cases} (-\Delta)^s u + c(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary, $c \in L^\infty(\Omega)$, and

$$(-\Delta)^s u = c_{n,s}\text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy := c_{n,s} \lim_{\epsilon \to 0} \int_{|x-y|>\epsilon} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy, \quad s \in (0,1),$$

where $c_{n,s} = \frac{4^s s \Gamma(\frac{N}{2}+s)}{\pi^{\frac{n}{2}} \Gamma(1-s)}$ is the normalization constant. As shown in Proposition 4.4 of [20], $\lim_{s \to 1^-} (-\Delta)^s u = -\Delta u$ for any $u \in C_0^\infty(\mathbb{R}^n)$. The paper also addresses the principal eigenvalue problems for weakly coupled cooperative systems involving fractional Laplace operators as follows:

$$\begin{cases} (-\Delta)^{s_i} u_i + \sum_{j=1}^k c_{i,j}(x)u_j = \lambda u_i & \text{in } \Omega \quad (1 \le i \le k) \\ u_i = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.2}$$

where $s_i \in (0,1)$, and $c_{i,j} \in L^\infty(\Omega)$. Its main goal is to examine basic properties of the principal eigenvalues of the perturbed fractional Laplace operators given above, which would be similar to

those for the principal eigenvalues of linear second order elliptic operators with Dirichlet boundary condition.

The principal eigenvalue problems of linear second order elliptic operators are fundamental to the theory and applications of partial differential equations and have been extensively studied over past few decades (see [1],[2],[11],and [17]). It is well known that the principal eigenvalues of linear second order elliptic operators play a crucial role in the study of spatio-temporary dynamics of population models. As shown in [8], just like the principal eigenvalues of other operators describing the dispersal and demographics of populations, the principal eigenvalues of linear second order elliptic operators can provide the intrinsic growth rates for spatially structured populations described by reaction-diffusion models. Meanwhile nonlocal diffusion equations have gained considerable popularity recently, due to their demonstrated applications in numerous diverse and widespread fields of science and engineering (see [3], [4], [6], [7], [15], [10], [21], [23], [24], [26] and references therein). In fact, perturbed fractional Laplace operators like the ones given in (1.1) and (1.2) often arise as a result of the linearization at an equilibrium of a semi-linear equation involving the fractional Laplacian. Therefore, under suitable boundary conditions, it is a natural question to ask if the principal eigenvalues of perturbed fractional Laplace operators still enjoy the essential features of the principal eigenvalues of linear second order elliptic operators.

In Dyda [12], a set of formulas are established for fractional calculus of power functions, and these formulas are employed to estimate upper and lower bounds of the principal eigenvalue of $(-\Delta)^s$ in a ball. Servadei and Valdinoci [23] investigates the eigenvalue problem of a non-local integro-differential operator with homogeneous Dirichlet boundary conditions, which along with [17] form the basis of our study. More recently, Massaccesi and Valdinoci [19] examined the distinction between local and non-local dispersal strategies in an attempt to analyze and to understand the impacts of different diffusive strategies on interacting species. Their work focus on a spatial model that describes the competition between two populations dispersing in different manners. More precisely, they assumed that the motion of one population is governed by a random walk while the other's movement is subject to nonlocal dispersal that follows a power law. A reverse Poincaré-Sobolev condition is given in [19], which serves as an indicator for the abundance of resources. Among other things, the condition was repeatedly used to study the existence and linear stability of an equilibrium of the model revealing the crucial role of the condition in determining the evolutionary stability of different dispersal strategies. It will become clear in this paper that the validity of the reverse Poincaré-Sobolev condition of [19] is equivalent to the negativity of the principal eigenvalue of (1.1).

Inspired by the above works and motivated by a characterization theorem in [17], which manifests the equivalence between the validity of a strong maximum principle and the positivity of the principal eigenvalue of linear second order elliptic operators under suitable conditions, the present work extends the characterization theorem and other basic properties regarding principal

eigenvalue for linear second order elliptic operators to perturbed fractional Laplace operators with homogeneous Dirichlet boundary condition given in (1.1). The same characterization is also established for (1.2). This paper is organized as follows: in section 2, we investigate the equivalence between the validity of a strong maximum principle and the positivity of the principal eigenvalue of (1.1). As immediate consequences of this equivalence, it can be shown that the principal eigenvalue of (1.1) is monotone and continuous with respect to c, and it is also continuous with respect to a regular class of perturbations of Ω around its boundary.

We also study the existence and multiplicity of principal eigenvalues for a linear weighted eigenvalue problem. Section 3 is focused on the principal eigenvalue of (1.2). The aforementioned equivalence is obtained for (1.2) as well under certain conditions on $c_{i,j}$. This equivalence reveals the interplays between the sign of the principal eigenvalue of (1.2), the validity of the strong maximum principle, and the existence of a positive super-solution. As a result of the equivalence, the maximum principle for domains of small volume is also established for weakly coupled cooperative systems. Finally, in section 4, the results established in the previous sections are utilized to study several prototype models that arise in mathematical ecology. More specifically, we study a nonlocal Ross-Macdonald model. It is shown that the principal eigenvalue plays a key role similar to the basic reproduction number when it comes to determining infection dynamics.

Throughout the paper, let Ω be an open and bounded subset of $\mathbb{R}^n$, we always write

$$d(x, \partial\Omega) = \inf\{|x - y| \mid y \in \mathbb{R}^n \setminus \Omega\}$$

for the distance of x to the boundary $\partial\Omega$ of Ω. Given $x \in \mathbb{R}^n$, $B_r(x) := \{y \in \mathbb{R}^n \mid |y - x| < r\}$. Given $u \in L^\infty(\Omega)$, $u \gneq 0$ in Ω means that $u \geq 0$ in Ω, and there exists $x' \in \Omega$ and $r > 0$ for which $B_r(x') \subset \Omega$ and $\inf_{x \in B_r(x')} u > 0$. $u \lneq 0$ in Ω if $-u \gneq 0$ in Ω. $L^\infty_{\text{loc}}(\mathbb{R}^n)$ denotes the space of functions $u \in L^\infty(D)$ for any bounded measurable subset $D \subset \mathbb{R}^n$.

2   The existence of principal eigenvalue and its properties

In the sequel, we let $H^s_0(\Omega)$ be the Hilbert space defined as the closure of $C^\infty_0(\Omega)$ with respect to the norm

$$\|u\|_{H^s_0(\Omega)} := \left(\int_\Omega |u|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx \right)^{\frac{1}{2}}.$$

Notice that $H^s_0(\Omega)$ is a subspace of $H^s(\mathbb{R}^n)$ consisting of functions that vanish outside Ω. As shown in [24], $H^s_0(\Omega)$ is equipped with the following equivalent norm

$$\|u\|_{H^s_0(\Omega)} := \left(\int_\Omega |u|^2 dx \right)^{\frac{1}{2}} + \left(\int_{\mathbb{R}^{2n} \setminus \Omega^c \times \Omega^c} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx \right)^{\frac{1}{2}},$$

where $\mathbb{R}^{2n} := \mathbb{R}^n \times \mathbb{R}^n$ and $\Omega^c = \mathbb{R}^n \setminus \Omega$.

Definition 1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Given $f \in L^2(\Omega)$, a function $w \in H_0^s(\Omega)$ is said to be a weak solution to the boundary value problem

$$\begin{cases} (-\Delta)^s u + c(x)u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{2.1}$$

provided that

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[w(x) - w(y)][\phi(x) - \phi(y)]}{|x-y|^{n+2s}} \, dy \, dx + \int_{\Omega} cw\phi \, dx = \int_{\Omega} f\phi \, dx$$

for any $\phi \in H_0^s(\Omega)$.

Thereafter, for the sake of convenience, we shall write

$$\begin{cases} (L_c u)(x) := (-\Delta)^s u + c(x)u, & x \in \Omega, \\ \mathcal{D}u := u, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.2}$$

Let $\lambda \in \mathbb{R}$, λ is said to be an eigenvalue of the eigenvalue problem (1.1), or equivalently, an eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$ if there exists a function $u \in H_0^s(\Omega)$ such that u is a weak solution of (2.1) with $f = \lambda u$. Depending on our needs, we also consider classical solutions of (2.1), following [21], assume that $f \in L^\infty(\Omega)$, a classical solution to (2.1) is a continuous function $u \in C(\mathbb{R}^n)$ such that $(-\Delta)^s u + c(x)u$ is well defined in Ω and $(-\Delta)^s u + c(x)u = f$ pointwise in Ω.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Suppose that $n > 2s$ and $w \in H_0^s(\Omega)$ is a weak solution to (2.1). Assume that $c, f \in L^p(\Omega)$ and $p > \max\{2, n/2s\}$. Then*

$$\|w\|_{L^\infty(\Omega)} \leq C$$

for some positive constant C depending only on $n, s, p, \Omega, \|c\|_{L^p(\Omega)}$, and $\|f\|_{L^p(\Omega)}$.

Proof. The Moser's iteration presented in [6] is utilized to prove the proposition. Given $m > 0$, let w_m be defined by $w_m = \min\{|w|, m\}$. Clearly, $w_m \in H_0^s(\Omega)$. In view of Lemma A.1 of [6], there holds

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[|w(x)| - |w(y)|][\phi(x) - \phi(y)]}{|x-y|^{n+2s}} \, dy \, dx \leq \int_{\Omega} |c||w|\phi \, dx + \int_{\Omega} |f|\phi \, dx$$

for any positive $\phi \in H_0^s(\Omega)$. Now choose $\phi = (w_m + 1)^\beta - 1$, then following the line of Theorem 3.1 of [6] gives that

$$\left(\int_{\Omega} (w_m+1)^{\frac{(1+\beta)n}{n-2s}} \, dx \right)^{\frac{n-2s}{n}} \leq C(n,s) \left(\frac{\beta+1}{2\sqrt{\beta}} \right)^2 \left\{ K(c,f,\Omega) \left(\int_{\Omega} [(|w|+1)(w_m+\delta)^\beta]^q \, dx \right)^{\frac{1}{q}} \right\},$$

where $q = \frac{p}{p-1}$, $K(c, f, \Omega) = \|c\|_{L^p(\Omega)} + \|f\|_{L^p(\Omega)} + |\Omega|^{\frac{n-2s}{n} - \frac{1}{q}}$, and $C(n, s) > 0$ is constant depending only on n and s. Now let $m \to \infty$, thanks to the monotone convergence theorem, by passing the limits in the above inequality, we find that

$$\left(\int_\Omega (|w| + 1)^{\frac{(1+\beta)n}{n-2s}} dx \right)^{\frac{n-2s}{n}} \leq C(n, s) \left(\frac{\beta+1}{2\sqrt{\beta}} \right)^2 \left\{ K(c, f, \Omega) \left(\int_\Omega [(|w| + 1)^{(1+\beta)q}] dx \right)^{\frac{1}{q}} \right\}.$$

Let $\theta = (1+\beta)q$ and $\chi = \frac{n}{(n-2s)q}$, note that there exists a constant γ_q, depending on q only, such that $\frac{\beta+1}{2\sqrt{\beta}} \leq \sqrt{\gamma_q(\beta + 1)}$ as long as $\beta \geq \frac{2-q}{q}$. If $\beta \geq \frac{2-q}{q}$, write $C(n, s, q) = C(n, s)\gamma_q$, then

$$\left(\int_\Omega (|w| + 1)^{\chi\theta} dx \right)^{\frac{1}{\chi\theta}} \leq \left[K(c, f, \Omega) \right]^{\frac{q}{\theta}} \left(\frac{C(n, s, q)\theta}{q} \right)^{\frac{q}{\theta}} \left(\int_\Omega (|w| + 1)^\theta dx \right)^{\frac{1}{\theta}}.$$

It follows from the assumption that $\chi > 1$. Now choose $\theta_0 = 2$, and let $\theta_{n+1} = \chi\theta_n$ with $n \in \mathbb{N}^+$, after m iterations, we have

$$\|(|w| + 1)\|_{L^{\theta_{m+1}}(\Omega)} \leq \left[K(c, f, \Omega) \right]^{\sum_{i=0}^m \frac{q}{\theta_i}} \prod_{i=0}^m \left(\frac{C(n, s, q)\theta_i}{q} \right)^{\frac{q}{\theta_i}} \|(|w| + 1)\|_{L^2(\Omega)}.$$

By passing the limit as $m \to \infty$, the desired conclusion follows. The proof is completed. $\qquad\square$

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let L_c and $\mathcal{D}$ be given by (2.2). Then $\{L_c, \mathcal{D}, \Omega\}$ has a countable family of eigenvalues $\{\lambda_k\}_{k=1}^\infty$ that can be written as an increasing sequence approaching $+\infty$ as $k \to \infty$. Namely,*

$$-\infty < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots.$$

In addition, there holds

$$\lambda_1 = \inf_{u \in H_0^s(\Omega), \|u\|_{L^2(\Omega)}=1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx + \int_\Omega c(x)u^2 dx, \qquad (2.3)$$

$$\lambda_k = \inf_{u \in X_{k-1}^\perp, \|u\|_{L^2(\Omega)}=1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx + \int_\Omega c(x)u^2 dx, \quad k \geq 2,$$

where $X_k = \bigoplus_{i=1}^k \ker(L_c - \lambda_i I)$. Each λ_k has finite geometric multiplicity. Moreover, the eigenfunctions associated with $\{\lambda_k\}_{k=1}^\infty$ form an orthonormal basis of $H_0^s(\Omega)$. Assume further that Ω satisfies the exterior ball condition, and $\varphi_k \in X_k$. Then $\varphi_k \in C^s(\mathbb{R}^n)$, and

$$L_c\varphi_k = \lambda_k\varphi_k \quad in \ \Omega, \quad \varphi_k = 0 \quad in \ \mathbb{R}^n \setminus \Omega.$$

Proof. The proof of the existence of $\{\lambda_k\}_{k=1}^\infty$ is the same as that in [23], while λ_1 is referred to as the principal eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$, which is a simple eigenvalue. Let φ_i be an eigenvalue associated with λ_i. Owing to Proposition 2.1, $\varphi_i \in L^\infty(\Omega)$. In particular, similar to [23], we can show that $\varphi_1(x) > 0$ (or $\varphi_1(x) < 0$) for all $x \in \Omega$. In case that Ω is a Lipschitz domain satisfying the exterior ball condition, then it follows from [21] that $\varphi_i \in C^s(\mathbb{R}^n)$ satisfies $L_c\varphi_i = \lambda_i\varphi_i$ point-wise in Ω. The proof is completed. $\qquad\square$

Let λ_1 be given by (2.3). In what follows, we denote λ_1 by λ_s, and λ_s is referred to as the principle eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$. Namely, λ_s is the least eigenvalue of the eigenvalue problem (1.1), and it is the only eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$ with positive eigenfunctions.

Definition 2. [17] Let $\Omega \subset \mathbb{R}^n$ be a domain. Ω is said to satisfy the interior sphere property at a single point $x \in \partial\Omega$ if there exist $z_x \in \Omega$ and $r > 0$ such that $|x - z_x| = r$ and $B_r(z_x) \subset \Omega$.

Definition 3. [17] Let $\Omega \subset \mathbb{R}^n$ be a domain. Ω is said to satisfy the uniform interior sphere property in the strong sense if there exists $r > 0$ such that for every $x \in \Omega$ with $d(x, \partial\Omega) \le r$, there exists a point $z_x \in \partial\Omega$ for which $|x - z_x| = d(x, \partial\Omega)$ and $B_r\left(z_x + r\frac{x - z_x}{|x - z_x|}\right) \subset \Omega$.

Definition 4. Let $u \in L^\infty_{\text{loc}}(\mathbb{R}^n) \cap C(\overline{\Omega})$, u is said to be a (classical) super-solution of $\{L_c, \mathcal{D}, \Omega\}$ if $L_c u \in L^\infty(\Omega)$, and

$$(-\Delta)^s u + c(x)u \ge 0 \ \text{ in } \ \Omega, \quad \text{and} \ \ u \ge 0 \ \text{ in } \ \mathbb{R}^n \setminus \Omega.$$

In addition, u is said to be a strict super-solution of $\{L_c, \mathcal{D}, \Omega\}$ provided that u is a (classical) super-solution of $\{L_c, \mathcal{D}, \Omega\}$ while u is not classical solution of $L_c u = 0$ in Ω and $u = 0$ in $\mathbb{R}^n \setminus \Omega$.

Proposition 2.3. *[15] Let $u \in L^\infty_{loc}(\mathbb{R}^n) \cap C(\overline{\Omega})$ be a super-solution of $\{L_c, \mathcal{D}, \Omega\}$. Assume that $c \in L^\infty(\Omega)$ and $c \ge 0$. Then either $u = 0$ in $\mathbb{R}^n$ or $u > 0$ in Ω.*

Proof. The proof can be found in [15](see the proof for Theorem 2.1 of [15]). $\qquad\square$

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain (either a bounded or an unbounded domain). Suppose that Ω satisfies the interior sphere property at a point $x_0 \in \partial\Omega$. Let $z_{x_0} \in \Omega$ such that $|x_0 - z_{x_0}| = r$ and $B_r(z_{x_0}) \subset \Omega$. Assume that $u \in C(\overline{B}_r(z_{x_0})) \cap L^\infty_{loc}(\mathbb{R}^n)$ is a super-solution of $\{L_c, \mathcal{D}, B_r(z_{x_0})\}$ and $u \gneqq 0$ in $\mathbb{R}^n$. Then there exist positive constants ρ and r_1 with $0 < r_1 < r$ such that $u(x) \ge \theta\rho d(x, \partial B_r(z_{x_0}))^s$ for all $x \in B_r(z_{x_0})$, where $\theta = \inf_{x \in \overline{B}_{r_1}(z_{x_0})} u$, and r_1, ρ depend on r, s, n and $|c_+|_{L^\infty(\Omega)}$ only.*

Proof. Set

$$w_1(x) = (r^2 - |x - z_{x_0}|^2)_+^s, \quad w_2(x) = (r^2 - |x - z_{x_0}|^2)_+^{1+s}, \quad x \in \mathbb{R}^n.$$

It then follows from Table 3 of [12] that

$$(-\Delta)^s w_1(x) = 2^{2s}\Gamma(1+s)\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(\frac{n}{2}\right)^{-1} = k_1, \quad x \in B_r(z_{x_0}),$$

$$
\begin{aligned}
(-\Delta)^s w_2(x) &= 2^{2s}\Gamma(2+s)\Gamma\left(\frac{n}{2}+s\right)\Gamma\left(\frac{n}{2}\right)^{-1}\left(r^2 - \left(1+\frac{2s}{n}\right)|x - z_{x_0}|^2\right), \quad x \in B_r(z_{x_0}) \\
&= k_2\left(r^2 - \left(1+\frac{2s}{n}\right)|x - z_{x_0}|^2\right).
\end{aligned}
$$

Since

$$\lim_{|x-z_{x_0}|\to r}\left(r^2 - \left(1+\frac{2s}{n}\right)|x - z_{x_0}|^2\right) = -\frac{2s}{n}r^2, \quad \lim_{|x-z_{x_0}|\to r} w_1 = \lim_{|x-z_{x_0}|\to r} w_2 = 0,$$

there exists $r_1 \in (0, r)$ such that $k_2(r^2 - (1+\frac{2s}{n})|x - z_{x_0}|^2) + |c_+|_{L_\infty}(w_1 + w_2) \leq -\frac{sk_2r^2}{n}$ for all $x \in B_r(z_{x_0}) \setminus \overline{B}_{r_1}(z_{x_0})$. Apparently, r_1 is determined by r, s, n, and $|c_+|_{L_\infty}$ only, and is independent of z_{x_0}. Note that $\frac{sk_2r^2}{nk_1} = \frac{s\Gamma(1+s)r^2}{n}$. Let $k_3 = \min\{1, \frac{s\Gamma(1+s)r^2}{n}\}$, it is easy to see that

$$(-\Delta)^s[k_3 w_1 + w_2] + c_+(x)[k_3 w_1 + w_2] \leq 0 \quad \text{in } B_r(z_{x_0}) \setminus \overline{B}_{r_1}(z_{x_0}).$$

Now let $\theta = \inf_{x\in\overline{B}_{r_1}(z_{x_0})} u$. Since $u \gneq 0$ in $\mathbb{R}^n$, and $(-\Delta)^s u + c_+ u \geq 0$ for all $x \in B_r(z_{x_0})$, it follows from Proposition 2.3 that $u > 0$ in $B_r(z_{x_0})$. Moreover, as $\overline{B}_{r_1}(z_{x_0})$ is compact, we have $\theta > 0$. Write $v = \frac{\theta}{r^{2s}+r^{2+2s}}(k_3 w_1 + w_2)$. Clearly, $\sup_{x\in\Omega} v \leq \theta$, it then follows that $u \geq v$ for all $x \in \overline{B}_{r_1}(z_{x_0})$. In addition, the assumption shows that $u \geq v = 0$ for all $x \in \mathbb{R}^n \setminus B_r(z_{x_0})$. Meanwhile, there holds that $(-\Delta)^s(u - v) + c_+(u - v) \geq 0$ in $B_r(z_{x_0}) \setminus \overline{B}_{r_1}(z_{x_0})$. Thus Proposition 2.3 implies that $u > v$ for all $x \in B_r(z_{x_0}) \setminus \overline{B}_{r_1}(z_{x_0})$. Notice that

$$v \geq \frac{\theta}{r^{2s}+r^{2+2s}}(k_3 w_1) \geq \frac{r^s k_3 \theta}{r^{2s}+r^{2+2s}}(r - |x - z_{x_0}|)_+^s.$$

Let $\rho = \frac{r^s k_3}{r^{2s}+r^{2+2s}}$, we readily conclude that $u(x) \geq \theta\rho d(x, \partial B_r(z_{x_0}))^s$ for all $x \in B_r(z_{x_0}) \setminus \overline{B}_{r_1}(z_{x_0})$. In particular, as a result of the fact that $u \geq v$ for all $x \in \overline{B}_{r_1}(z_{x_0})$, there hold that $u \geq v$ for all $x \in B_r(z_{x_0})$. In other words, $u \geq \theta\rho d(x, \partial B_r(z_{x_0}))^s$ for all $x \in B_r(z_{x_0})$. The proof is completed. $\qquad\square$

We now give a uniform decay property of E. Hopf for super-solutions of $\{L_c, \mathcal{D}, \Omega\}$, which will be used in serval places of this paper.

Lemma 2.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying the uniform interior sphere property in the strong sense. Let $u \in L_{loc}^{\infty}(\mathbb{R}^n) \cap C(\overline{\Omega})$ be a nonnegative super-solution of $\{L_c, \mathcal{D}, \Omega\}$, and $u \gneq 0$ in $\mathbb{R}^n$. Then there exists $\beta > 0$ such that $u(x) \geq \beta d(x, \partial\Omega)^s$ for all $x \in \Omega$. Assume further that $\partial\Omega$ is of class $\mathcal{C}^1$, and $u = 0$ in $\mathbb{R}^n \setminus \Omega$. Then $\lim_{d(x,\partial\Omega)\to 0} \frac{u(x)}{d(x,\partial\Omega)^s} \geq \beta$.*

Proof. Let $r > 0$ be the constant specified in Definition 3. Given $x \in \Omega$ with $d(x, \partial\Omega) \leq r$, let $z_x \in \partial\Omega$ such that $d(x, \partial\Omega) = |x - z_x|$, we write $x' = z_x + r\frac{x-z_x}{|x-z_x|}$. Due to the assumption, $B_r(x') \subset \Omega$. Clearly, $z_x \in \overline{B}_r(x') \cap \partial\Omega$. Let $r_1 \in (0, r)$ be the constant found in the proof of Lemma 2.1 for $B_r(x')$. Again, r_1 depends on r, n, s and $|c_+|_{L^{\infty}(\Omega)}$ only. Let $\Theta := \{x \in \Omega \mid d(x, \partial\Omega) \geq \frac{r-r_1}{2}\}$. Notice that $d(x', \partial\Omega) = |x' - z_x| = r$. It hence follows that $\overline{B}_{r_1}(x') \subset \Theta$. In fact, if there were a $y \in \overline{B}_{r_1}(x')$ and y were not in Θ, that is, $d(y, \partial\Omega) < \frac{r-r_1}{2}$. Let $z_y \in \partial\Omega$ such that $d(y, \partial\Omega) = |y - z_y|$. Then we would have $|x' - z_y| \leq |x' - y| + |y - z_y| \leq r_1 + \frac{r-r_1}{2} = \frac{r+r_1}{2} < r$, which is obviously a contradiction as $d(x', \partial\Omega) = r$. The contradiction confirms that $\overline{B}_{r_1}(x') \subset \Theta$. Slightly abusing the notation, we let $\theta = \inf_{x\in\Theta} u$. Then it follows from Lemma 2.1 that $u(y) \geq \theta\rho d(y, \partial B_r(x'))^s$ for $y \in B_r(x')$. Namely, $u(x) \geq \theta\rho d(x, \partial\Omega)^s$. Note that $x \in \Omega$ with $d(x, \partial\Omega) \leq r$ is arbitrary, and ρ depends on r, s, n and $|c_+|_{L^{\infty}(\Omega)}$ only. It is easy to see that $u(x) \geq \theta\rho d(x, \partial\Omega)^s$ for all $x \in \Omega$ with $d(x, \partial\Omega) \leq r$. Now let $\Theta' = \{x \in \Omega \mid d(x, \partial\Omega) \geq r\}$. Clearly, Θ' is a compact subset of Ω. Let $\vartheta = \inf_{x\in\Theta'} u$, then $\vartheta > 0$ as $u > 0$ in Ω. Since $d(x, \partial\Omega)$ is bounded, there exists $\gamma > 0$ such that $u \geq \gamma d(x, \partial\Omega)^s$ for all $x \in \Theta'$. Set $\beta = \min\{\theta\rho, \gamma\}$, we readily conclude that $u(x) \geq d(x, \partial\Omega)^s$ for all $x \in \Omega$. Assume further that $\partial\Omega$ is of class $\mathcal{C}^1$, and $u = 0$ in $\mathbb{R}^n \setminus \Omega$, Then it follows from [21] that $\frac{u}{d(x,\partial\Omega)^s} \in C^{\alpha}(\overline{\Omega})$ for some $\alpha \in (0, s)$. Hence $\lim_{d(x,\partial\Omega)\to 0} \frac{u(x)}{d(x,\partial\Omega)^s} \geq \beta$. The proof is completed. $\square$

Remark 1. Under the assumptions that are slightly weaker than those of Lemma 2.1, by deriving a contradiction, it is established in [15] that $\liminf_{d(x,\partial\Omega)\to 0} \frac{u(x)}{d(x,\partial\Omega)^s} > 0$. While Lemma 2.1 presents a bit more explicit lower bound for $\frac{u(x)}{d(x,\partial\Omega)^s}$. In case that $\partial\Omega$ is sufficiently smooth, then the Hopf's lemma given in [15] can be restated as $\lim_{\epsilon\to 0} \frac{u(x-\epsilon\nu)}{\epsilon^s} \geq \beta$.

Definition 5. Let L_c and $\mathcal{D}$ be defined in (2.2). Then $\{L_c, \mathcal{D}, \Omega\}$ is said to satisfy the strong maximum principle if $u > 0$ in Ω whenever $u \in C^s(\overline{\Omega}) \cap L_{\text{loc}}^{\infty}(\mathbb{R}^n)$ is a nontrivial super-solution of $\{L_c, \mathcal{D}, \Omega\}$.

Theorem 2.1. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in \mathcal{C}^1$. Let λ_s be the principal eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$. Then the following statements are equivalent*

(i) $\lambda_s > 0$,

(ii) $\{L_c, \mathcal{D}, \Omega\}$ has a positive strict super-solution $u \in C(\overline{\Omega}) \cap L_{loc}^{\infty}(\mathbb{R}^n)$,

(iii) $\{L_c, \mathcal{D}, \Omega\}$ satisfies the strong maximum principle;

(iv) the linear boundary value problem

$$\begin{cases} (-\Delta)^s u + c(x)u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

has an unique solution $u \in H_0^s(\Omega)$ for each $f \in L^2(\Omega)$. In particular, $u \geq 0$ provided that $f \geq 0$.

Proof. (i) implies (ii). This is obvious as a positive eigenfunction associated with λ_s is a strict super-solution of $\{L_c, \mathcal{D}, \Omega\}$. (ii) implies (iii). Let $u \in C(\overline{\Omega}) \cap L_{\text{loc}}^\infty(\mathbb{R}^n)$ be the positive strict super-solution of $\{L_c, \mathcal{D}, \Omega\}$. Let $v \in C^s(\overline{\Omega}) \cap L_{\text{loc}}^\infty(\mathbb{R}^n)$ be a super-solution of $\{L_c, \mathcal{D}, \Omega\}$. We shall follow the idea of [1] to show that $v > 0$ in Ω. Let $v(x_0) = \inf_\Omega v(x)$ for some $x_0 \in \overline{\Omega}$. It is sufficient to show that $v(x_0) \geq 0$. Assume to the contrary that this is not true, that is $v(x_0) < 0$, then as $v \geq 0$ in Ω^c, we have $x_0 \in \Omega$. Given $x \in \Omega$. Let $\overline{x} \in \partial\Omega$ be the point such that $|x - \overline{x}| = d(x, \partial\Omega)$. Since $v \in C^s(\overline{\Omega})$, then there is $\kappa > 0$ for which $v(x) \geq v(\overline{x}) - \kappa d(x, \partial\Omega)^s \geq -\kappa d(x, \partial\Omega)^s$. Consequently, $v(x) \geq -\kappa d(x, \partial\Omega)^s$ for all $x \in \overline{\Omega}$. Meanwhile, Lemma 2.2 shows that $u(x) \geq \beta d(x, \partial\Omega)^s$ for all $x \in \overline{\Omega}$ and some $\beta > 0$. Hence there is $t' > 0$ such that $t'u(x) + v(x) \geq 0$ for all $x \in \overline{\Omega}$. Now let $\overline{t} = \inf\{t \geq 0 \mid tu(x) + v(x) \geq 0 \text{ in } \overline{\Omega}\}$. Clearly, $\overline{t} > 0$ and $\overline{t}u(x) + v(x) \geq 0$ in $\overline{\Omega}$. Since $\overline{t}u(x) + v(x)$ is a strict super-solution of $\{L_c, \mathcal{D}, \Omega\}$, we have $\overline{t}u(x) + v(x) > 0$ in Ω. Thus, Lemma 2.2 shows that $\overline{t}u(x) + v(x) \geq \theta d(x, \partial\Omega)^s$ for all $x \in \overline{\Omega}$ and some $\theta > 0$. Let $\vartheta = \frac{\theta}{2\kappa}$. Then $\overline{t}u(x) + v(x) \geq -\vartheta v(x)$ for all $x \in \overline{\Omega}$. Hence $\frac{\overline{t}}{1+\vartheta}u(x) + v(x) \geq 0$ for all $x \in \overline{\Omega}$, which contradicts the definition of $\overline{t}$. Therefore, $v(x_0) \geq 0$. Particularly, Proposition 2.3 implies that $v(x) > 0$ in Ω. It is apparent that (iii) yields that (ii)

Next we show that (ii) implies (i). Assume to the contrary that $\lambda_s \leq 0$. Let w be the strict super-solution and φ be a positive eigenfunction. It follows from Proposition 1.1 of [21] that $|\varphi(x)| \leq Cd(x, \partial\Omega)^s$. As $w \geq 0$ and $\varphi = 0$ in $\mathbb{R}^n \setminus \Omega$, in particular, Lemma 2.2 shows that there exists $\gamma > 0$ for which $w \geq \gamma d(x, \partial\Omega)^s$ for all $x \in \Omega$. Thus, there exists $t > 0$ such that $tw - \varphi \geq 0$ for all $x \in \mathbb{R}^n$. Again let $\overline{t} = \{t \in \mathbb{R} \mid tw - \varphi \geq 0\}$. Apparently, $\overline{t} > 0$. Note that $tw - \varphi$ is a super-solution of $L_c - \lambda_s I$. In addition, one of the followings must hold: $L_c(tw - \varphi) - \lambda_s(tw - \varphi) \gneq 0$ in Ω or $tw - \varphi \gneq 0$ in Ω^c. With the same reasoning as above, we can reach a contradiction. Hence $\lambda_s > 0$.

We now show that (i) implies (iv). Define

$$I(u) = \frac{c_{n,s}}{4} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx + \frac{1}{2} \int_\Omega cu^2 dx - \int_\Omega fu dx.$$

As

$$0 < \lambda_s = \inf_{u \in H_0^s(\mathbb{R}^n), \|u\|_{L^2} = 1} \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx + \int_\Omega cu^2 dx,$$

it is easy to see that I is bounded from below and coercive. In addition, let $w, v \in X_0$ with $w \neq v$. A straightforward calculation yields that

$$
\begin{aligned}
\langle I'(w) - I'(v), w - v \rangle &= \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{|(w-v)(x) - (w-v)(y)|^2}{|x-y|^{n+2s}} dy dx + \int_\Omega c(x)(w-v)^2 dx \\
&\geq \lambda_s \|w - v\|^2_{L^2(\Omega)} > 0.
\end{aligned}
$$

Namely, I is strictly convex, and it follows from [13] that I has a unique minimizer still denoted by u. That is, the boundary value problem has a unique solution. Given that $f \in L^2(\Omega)$ with $f \geq 0$. Let $u_- = \min\{u, 0\}$, Clearly, $u_- \in H_0^s(\Omega)$. We show that $u_- \equiv 0$,

$$
\begin{aligned}
&\langle I'(u), u_- \rangle \\
&= \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(u_-(x) - u_-(y))}{|x-y|^{n+2s}} dy dx - \int_\Omega c(x)(u_-)^2 dx - \int_\Omega f u_- dx \\
&= -\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{|u_-(x) - u_-(y)|^2}{|x-y|^{n+2s}} dy dx \\
&\quad + \frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{(u_+(x) - u_+(y))(u_-(x) - u_-(y))}{|x-y|^{n+2s}} dy dx - \int_\Omega c(x)(u_-)^2 dx - \int_\Omega f u_- dx \\
&= -\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{|u_-(x) - u_-(y)|^2}{|x-y|^{n+2s}} dy dx - c_{n,s} \int_{\mathbb{R}^{2n}} \frac{u_+(x) u_-(y)}{|x-y|^{n+2s}} dy dx - \int_\Omega c(x)(u_-)^2 dx \\
&\quad - \int_\Omega f u_- dx = 0.
\end{aligned}
$$

Here $\mathbb{R}^{2n} := \mathbb{R}^n \times \mathbb{R}^n$. Hence,

$$
\lambda_s \|u_-\|^2_{L^2(\Omega)} \leq -c_{n,s} \int_{\mathbb{R}^{2n}} \frac{u_+(x) u_-(y)}{|x-y|^{n+2s}} dy dx - \int_\Omega f u_- dx \leq 0.
$$

This implies that $u_- = 0$. Finally, we show (iv) implies (i). Choose any $f \in C(\overline{\Omega})$ with $f \gneqq 0$. Then the boundary value problem has a unique solution $u_f \in H_0^s(\Omega)$ and $u_f \gneqq 0$. Again, Proposition 2.1 and Proposition 1.1 of [21] imply that $u_f \in C^s(\mathbb{R}^n)$. In particular, we have $u_f > 0$ in Ω. Thus u_f is a strict super-solution of $\{L_c, \mathcal{D}, \Omega\}$. Namely, (ii) holds, and hence (i) follows. $\qquad\square$

Remark 2. Assume that $\lambda_s > 0$. Let u_f be the unique weak solution of (2.1). Then straightforward calculation shows that $\|u_f\|_{L^2(\Omega)} \leq \lambda_s^{-1} \|f\|_{L^2(\Omega)}$.

Proposition 2.4. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary. Let λ_s be the principal eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$. Then*

$$
\lambda_s \geq \frac{1}{r^{2s}} \frac{4^s \Gamma(s+1) \Gamma(s + \frac{n}{2})(2s+2)(2s+n)(6-2s)}{\Gamma(\frac{n}{2})[12n + (16-2n)s]} + \inf_{x \in \Omega} c,
$$

where $r > 0$ is the positive number such that $|\Omega| = |B_r(0)|$.

Proof. The idea of the proof is similar to that of the proof for Proposition 8.6 of [17]. We shall present a sketch. Let ψ be an eigenfunction corresponding to λ_s with $\|\psi\|_{L^2(\Omega)} = 1$, then the Faber-Krahn inequality of [5] along with Proposition 2.2 imply that

$$\lambda_s \geq \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\psi(x) - \psi(y)|^2}{|x-y|^{n+2s}} dy dx + \inf_{x \in \Omega} c \geq \left(\frac{|B_r(0)|}{|\Omega|} \right)^{\frac{2s}{n}} \lambda_1(B_r) + \inf_{x \in \Omega} c,$$

where $\lambda_1(B_r)$ is the principal eigenvalue of $\{(-\Delta)^s, \mathcal{D}, B_r(0)\}$. Note that $\lambda_1(B_r) = r^{-2s}\lambda_1(B_1)$. Now choose $r > 0$ such that $|B_r(0)| = |\Omega|$, the desired conclusion follows from Lemma 9 of [12]. The proof is completed. □

Proposition 2.5. *Let Ω_i $(i = 1,2) \subset \mathbb{R}^n$ be two bounded open domains with $\partial\Omega_i \in \mathcal{C}^1$ such that $\Omega_2 \subseteq \Omega_1$. Let λ_{Ω_i} be the principal eigenvalues of $\{L_c, \mathcal{D}, \Omega_i\}$, respectively. Then $\lambda_{\Omega_1} \leq \lambda_{\Omega_2}$. In case that $\Omega_2 \subsetneq \Omega_1$, then $\lambda_{\Omega_1} < \lambda_{\Omega_2}$. Let λ_{c_i} be the principal eigenvalue of $\{L_{c_i}, \mathcal{D}, \Omega\}$ with $c_i \in L^\infty(\Omega); (i = 1,2)$. Assume that $c_1 \leq c_2$. Then $\lambda_{c_1} \leq \lambda_{c_2}$. In case that $c_i \lneq c_2$, then $\lambda_{c_1} < \lambda_{c_2}$. Furthermore, λ_c is Lipschitz continuous with respect to c.*

Proof. Clearly, $H_0^s(\Omega_2) \subseteq H_0^s(\Omega_1)$. Thus, it follows from Proposition 2.2 that $\lambda_{\Omega_2} \geq \lambda_{\Omega_1}$. In case that $\Omega_2 \subsetneq \Omega_1$, let φ_1 be a positive eigenfunction associated with λ_{Ω_1}, then we have $L_c\varphi_1 - \lambda_{\Omega_1}\varphi_1 = 0$ in Ω_2 and $\varphi_1 \gneq 0$ in $\mathbb{R}^n \setminus \Omega_2$. Namely, φ_1 is a strict super-solution of $\{L_c - \lambda_{\Omega_1}, \mathcal{D}, \Omega_2\}$. As $\lambda_{\Omega_2} - \lambda_{\Omega_1}$ is the principal eigenvalue of $\{L_c - \lambda_{\Omega_1}, \mathcal{D}, \Omega_2\}$, Theorem 2.1 yields that $\lambda_{\Omega_2} > \lambda_{\Omega_1}$. The monotonicity of λ_c with respect to c is also an immediate consequence of Theorem 2.1. We now proceed to show the Lipschitz continuity of λ_c with respect to c. Notice that

$$\int_{\mathbb{R}^{2n}} \frac{[\varphi_i(x) - \varphi_i(y)][\varphi_j(x) - \varphi_j(y)]}{|x-y|^{n+2s}} dy dx + \int_\Omega c_i\varphi_i\varphi_j dx = \int_\Omega \lambda_{c_i}\varphi_i\varphi_j dx.$$

Here either $i = 1$ and $j = 2$, or $i = 2$ and $j = 1$, again λ_{c_i} is the principal eigenvalue of L_{c_i} in Ω. By subtracting the two equations, we have

$$(\lambda_{c_1} - \lambda_{c_2}) = \frac{\int_\Omega (c_1 - c_2)\varphi_1\varphi_2 dx}{\int_\Omega \varphi_1\varphi_2 dx}.$$

Therefore,

$$|\lambda_{c_1} - \lambda_{c_2}| \leq \frac{\int_\Omega |c_1 - c_2|\varphi_1\varphi_2 dx}{\int_\Omega \varphi_1\varphi_2 dx} \leq \|c_1 - c_2\|_{L^\infty(\Omega)}.$$

Namely, λ_c is Lipschitz continuous with respect to c. The proof is completed. □

Proposition 2.6. *Assume that $\Omega \subset \mathbb{R}^n$ is a connected and bounded domain with $\partial\Omega \in \mathcal{C}^1$. Let $\Omega_i \subset\subset \Omega$ be open subsets of Ω with smooth boundary satisfying $\Omega_i \subset \Omega_{i+1}$ and $\bigcup_{j=1}^\infty \Omega_j = \Omega$. Let $\phi \in C_0^\infty(\Omega)$. Then there exists a sequence $\{\phi_j\}_{j=1}^\infty$ with $\phi_j \in C_0^\infty(\Omega_j)$ such that $\lim_{j\to\infty} \|\phi_j - \phi\|_{H_0^s(\Omega)} = 0$.*

Proof. Denote that $A_\delta := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) < \frac{1}{\delta}\}$, where $\delta > 0$. Assume without loss of generality that $\Omega_j \subseteq \Omega \setminus A_{4j}$ and that $\Omega \setminus A_{2j} \subseteq \Omega_j$. Following [17], we choose $w_j \in C_0^\infty(\Omega)$ such that $0 \le w_j \le 1$, $w_j = 1$ in $\overline{\Omega} \setminus A_j$, and $w_j = 0$ in A_{2j}. Moreover, $\|\nabla w_j\|_{C(\overline{\Omega})} \le Cj$ for some positive constant C independent of j. Now write $\phi_j = w_j \phi$. Clearly, $\phi_j \in C_0^\infty(\Omega_j)$. It is easy to see that $\phi_j \to \phi$ in $L^2(\Omega)$ as $j \to \infty$. Notice that $(\phi_j - \phi)(x) - (\phi_j - \phi)(y) = 0$ whenever $(x, y) \in (\Omega \setminus A_j) \times (\Omega \setminus A_j), (\Omega \setminus A_j) \times \Omega^c$, and $\Omega^c \times \Omega^c$. Hence, we have

$$\int_{\mathbb{R}^{2n}} \frac{|(\phi_j - \phi)(x) - (\phi_j - \phi)(y)|^2}{|x-y|^{n+2s}} dydx$$

$$= 2 \int_{A_j \times (\Omega \setminus A_j)} + \int_{A_j \times A_j} \frac{|(\phi_j - \phi)(x) - (\phi_j - \phi)(y)|^2}{|x-y|^{n+2s}} dydx$$

$$+ 2 \int_{A_j \times \Omega^c} \frac{|(\phi_j - \phi)(x) - (\phi_j - \phi)(y)|^2}{|x-y|^{n+2s}} dydx.$$

Let $\varrho = \sup\{|x-y| \mid x, y \in \overline{\Omega}\}$. By Fubini's theorem and mean value theorem, we find that

$$\int_{A_j \times \Omega} \frac{|(\phi_j - \phi)(x) - (\phi_j - \phi)(y)|^2}{|x-y|^{n+2s}} dydx$$

$$\le \int_{A_j \times \Omega} 2 \frac{|\phi_j(x) - \phi_j(y)|^2 + |\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} dydx$$

$$\le 4 \int_{A_j} dx \int_\Omega \frac{|\phi(x)|^2 |w_j(x) - w_j(y)|^2 + |w_j(y)|^2 |\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} dy$$

$$+ 2 \int_{A_j \times \Omega} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} dydx$$

$$\le 4 \int_{A_j} dx \int_{B_\varrho(x)} \frac{|\phi(x)|^2 |w_j(x) - w_j(y)|^2}{|x-y|^{n+2s}} dy + 6 \int_{A_j} dx \int_{B_\varrho(x)} \frac{|\phi(x) - \phi(y)|^2}{|x-y|^{n+2s}} dy$$

$$\le 4\omega_n \int_{A_j} j^2 |\phi(x)|^2 dx \int_0^\varrho \frac{r^{n+1}}{r^{(n+2s)}} dr + 6\omega_n \int_{A_j} \|\nabla\phi\|_{L^\infty(\Omega)}^2 dx \int_0^\varrho \frac{r^{n+1}}{r^{(n+2s)}} dr$$

$$\le 4\omega_n \varrho^{2(1-s)} \|\nabla\phi\|_{L^2(A_j)}^2 + 6\omega_n \varrho^{2(1-s)} \|\nabla\phi\|_{L^\infty(\Omega)}^2 |A_j|.$$

Here $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(n/2)}$, and we used the fact that $\|\phi\|_{L^2(A_j)} \le \frac{1}{j} \|\nabla\phi\|_{L^2(A_j)}$ (see Remark 4.1 of [17]). Next let $\Omega_1^c = \{x \in \Omega^c \mid \text{dist}(x, \partial\Omega) \le 1\}$, and $\iota = \sup\{|x-y| \mid x, y \in \overline{A_j \cup \Omega_1^c}\}$. Straightforward calculation shows that

$$\int_{A_j \times \Omega^c} \frac{|(\phi_j - \phi)(x) - (\phi_j - \phi)(y)|^2}{|x-y|^{n+2s}} dydx$$

$$= \int_{A_j \times \Omega_1^c} + \int_{A_j \times (\Omega^c \setminus \Omega_1^c)} \frac{|(\phi_j - \phi)(x) - (\phi_j - \phi)(y)|^2}{|x-y|^{n+2s}} dydx$$

$$\le \omega_n \iota^{2(1-s)} \left[4\|\nabla\phi\|_{L^2(A_j)}^2 + 6\|\nabla\phi\|_{L^\infty(\Omega)}^2 |A_j| \right]$$

$$+ \int_{A_j \times (\Omega^c \setminus \Omega_1^c)} \frac{|(\phi_j - \phi)(x) - (\phi_j - \phi)(y)|^2}{|x - y|^{n+2s}} dy dx$$

$$\leq \omega_n \iota^{2(1-s)} \left[4 \|\nabla \phi\|_{L^2(A_j)}^2 + 6 \|\nabla \phi\|_{L^\infty(\Omega)}^2 |A_j| \right] + 4\omega_n \int_{A_j} \|\phi\|_{L^\infty(\Omega)}^2 dx \int_1^\infty \frac{r^{n-1}}{r^{n+2s}} dr$$

$$\leq \omega_n \iota^{2(1-s)} \left[4 \|\nabla \phi\|_{L^2(A_j)}^2 + 6 \|\nabla \phi\|_{L^\infty(\Omega)}^2 |A_j| \right] + 4\omega_n \|\phi\|_{L^\infty(\Omega)}^2 |A_j|.$$

Since $\|\nabla \phi\|_{L^2(A_j)} \to 0$, and $|A_j| \to 0$ as $j \to \infty$, we readily conclude that $\lim_{j \to \infty} \|\phi_j - \phi\|_{H_0^s(\Omega)} = 0$. The proof is completed. $\qquad\qquad\square$

Theorem 2.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a connected and bounded domain with $\partial \Omega \in \mathcal{C}^1$. Let λ_{Ω_i} be the principal eigenvalue of $\{L_c, \mathcal{D}, \Omega_i\}$, where $\Omega_i \subset \mathbb{R}^n$ are a sequence of bounded domains with smooth boundary. (i) Assume that $\Omega_i \subset\subset \Omega$ satisfying $\Omega_i \subset \Omega_{i+1}$ and $\bigcup_{j=1}^\infty \Omega_j = \Omega$. Then $\lim_{j \to \infty} \lambda_{\Omega_j} = \lambda_\Omega$. (ii) Assume that $\Omega_i \subset\subset \Omega$ satisfying $\Omega_{i+1} \subset \Omega_i$ and $\bigcap_{j=1}^\infty \Omega_j = \Omega_0 \subset \Omega$. Then $\lim_{j \to \infty} \lambda_{\Omega_j} = \lambda_{\Omega_0}$.*

Proof. To establish (i), we let φ_i be a positive eigenfunction corresponding to λ_{Ω_i}. Namely,

$$(-\Delta)^s \varphi_i + c(x)\varphi_i = \lambda_{\Omega_i} \varphi_i$$

in Ω_i and $\varphi_i = 0$ in Ω^c. As a result of Proposition 2.5, there holds that $\lambda_i \geq \lambda_{i+1} > \lambda_\Omega$. Hence λ_{Ω_i} are bounded and $\lim_{i \to \infty} \lambda_{\Omega_i}$ exists. Assume without loss of generality that $\|\varphi_i\|_{L^2(\Omega)} = 1$. Then, in view of Proposition 2.1, we have $\|\varphi_i\|_{C^s(\mathbb{R}^n)} \leq C$ for some positive constant C. Since $\varphi_i = 0$ in Ω_i^c, it follows from Ascoli-Arzelà theorem that there exists $\varphi^* \in C(\mathbb{R}^n)$ for which $\lim_{i \to \infty} \|\varphi_i - \varphi^*\|_{C(\mathbb{R}^n)} = 0$. Clearly, $\varphi^* = 0$ in Ω^c and $\varphi^* \gneq 0$ in Ω. We next show that $\lim_{i \to \infty} \|\varphi_i - \varphi^*\|_{H_0^s(\Omega)} = 0$. To this end, we show that $\{\varphi_i\}$ is a Cauchy sequence of $H_0^s(\Omega)$. Note that

$$\int_{\mathbb{R}^{2n}} \frac{|(\varphi_j - \varphi_i)(x) - (\varphi_j - \varphi_i)(y)|^2}{|x - y|^{n+2s}} dy dx = \int_{\mathbb{R}^{2n}} \frac{|\varphi_j(x) - \varphi_j(y)|^2}{|x - y|^{n+2s}} dy dx$$
$$+ \int_{\mathbb{R}^{2n}} \frac{|\varphi_i(x) - \varphi_i(y)|^2}{|x - y|^{n+2s}} dy dx - 2 \int_{\mathbb{R}^{2n}} \frac{[\varphi_j(x) - \varphi_j(y)][\varphi_i(x) - \varphi_i(y)]}{|x - y|^{n+2s}} dy dx.$$

Assume that $j > i$, then $\varphi_i \in H_0^s(\Omega_j)$, and

$$\frac{c_{n,s}}{2} \int_{\mathbb{R}^{2n}} \frac{|(\varphi_j - \varphi_i)(x) - (\varphi_j - \varphi_i)(y)|^2}{|x - y|^{n+2s}} dy dx$$
$$= \int_{\Omega_j} (\lambda_{\Omega_j} - c)\varphi_j^2 dx + \int_{\Omega_i} (\lambda_{\Omega_i} - c)\varphi_i^2 dx - 2 \int_{\Omega_j} (\lambda_{\Omega_j} - c)\varphi_j \varphi_i dx$$
$$\leq \|\varphi_j - \varphi_i\|_{L^2(\Omega)} \|(\lambda_{\Omega_j} - c)\varphi_j\|_{L^2(\Omega)} + \|(\lambda_{\Omega_j} - c)\varphi_j - (\lambda_{\Omega_i} - c)\varphi_i\|_{L^2(\Omega)} \|\varphi_i\|_{L^2(\Omega)}.$$

Thus, $\{\varphi_i\}$ is a Cauchy sequence in $H_0^s(\Omega)$ as φ_i converges to φ^* in $L^2(\Omega)$ and $\lim_{i \to \infty} \lambda_{\Omega_i}$ exists. In particular, in view of Poincaré- Sobolev inequality, we have $\lim_{i \to \infty} \|\varphi_i - \varphi^*\|_{H_0^s(\Omega)} = 0$.

Given $\phi \in C_0^\infty(\Omega)$, in view of Proposition 2.6, there exists $\{\phi_i\}$ with $\phi_i \in C_0^\infty(\Omega_i)$ such that ϕ_i converges to ϕ in $H_0^s(\Omega)$ as $i \to \infty$. Then

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^{2n}} \frac{[\varphi_i(x) - \varphi_i(y)][\phi_i(x) - \phi_i(y)]}{|x-y|^{n+2s}}dydx + \int_\Omega c(x)\varphi_i\phi_i dx = \lambda_{\Omega_i}\int_\Omega \varphi_i\phi_i dx.$$

By passing the limits in the above equation, we find that

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^{2n}} \frac{[\varphi^*(x) - \varphi^*(y)][\phi(x) - \phi(y)]}{|x-y|^{n+2s}}dydx + \int_\Omega c(x)\varphi^*\phi_i dx = \lambda^*\int_\Omega \varphi^*\phi_i dx.$$

Here $\lambda^* = \lim \lambda_{\Omega_i}$. This yields that λ^* is an eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$ and $L_c\varphi^* = \lambda^*\varphi^*$ in Ω. Again, Proposition 2.3 implies that $\varphi^* > 0$ in Ω. Thus $\lambda^* = \lambda_\Omega$. We now proceed to show (ii). With slight abuse of the notations, we still let φ_i be the positive eigenfunction corresponding to the eigenvalue λ_i with $\|\varphi_i\|_{L^2(\Omega)} = 1$. It is easy to see that $\lambda_i \leq \lambda_{i+1} < \lambda_\Omega$. In particular, it follows from the same reasoning that $\lim_{i\to\infty}\|\varphi_i - \varphi_*\|_{H^s(\mathbb{R}^n)} = 0$ and $\lim_{i\to\infty}\|\varphi_i - \varphi_*\|_{C(\mathbb{R}^n)} = 0$ for some $\varphi_* \in H_0^s(\Omega_0) \cap C(\mathbb{R}^n)$. Note that $H_0^s(\Omega_0) \subset H_0^s(\Omega_i)$. Let $\phi \in H_0^s(\Omega_0)$, then

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^{2n}} \frac{[\varphi_i(x) - \varphi_i(y)][\phi(x) - \phi(y)]}{|x-y|^{n+2s}}dydx + \int_{\Omega_0} c(x)\varphi_i\phi_i dx = \lambda_{\Omega_i}\int_{\Omega_0} \varphi_i\phi dx.$$

Let $\lambda_* = \lim_{i\to\infty}\lambda_{\Omega_i}$. By passing the limits in the above equation, we find that

$$\frac{c_{n,s}}{2}\int_{\mathbb{R}^{2n}} \frac{[\varphi_*(x) - \varphi_*(y)][\phi(x) - \phi(y)]}{|x-y|^{n+2s}}dydx + \int_{\Omega_0} c(x)\varphi_*\phi dx = \lambda_*\int_{\Omega_0} \varphi_*\phi dx.$$

Again, the same argument as that for (i) yields that $\lambda_* = \lambda_{\Omega_0}$. The proof is completed. $\qquad\square$

Motivated by [2] and [17], a point-wise min-max characterization of λ_s can be given as follows

Proposition 2.7. *Suppose that $\Omega \subset \mathbb{R}^n$ is a connected and bounded domain with $\partial\Omega \in \mathcal{C}^1$. Set $P := \{u \in C(\overline{\Omega}) \cap L_{loc}^\infty(\mathbb{R}^n) \mid \phi \gneq 0 \text{ in } \mathbb{R}^n, L_c\phi \in L^\infty(\Omega)\}$. Let $\lambda_s' := \sup\{\lambda \mid \exists\phi \in P, L_c\phi \geq \lambda\phi \text{ in } \Omega\}$. Then $\lambda_s' = \lambda_s$, where λ_s is the principle eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$. In addition, $\lambda_s = \sup_{\phi\in P}\inf_\Omega\left(\frac{L_c\phi}{\phi}\right)$. Furthermore, let $\widetilde{P} := \{\phi \in P \mid \phi > 0 \text{ in } \overline{\Omega}\}$, and $\lambda_s'' := \sup\{\lambda \mid \exists\phi \in \widetilde{P}, L_c\phi \geq \lambda\phi \text{ in } \Omega\}$. Then $\lambda_s'' = \lambda_s$. Additionally, $\lambda_s = \sup_{\phi\in\widetilde{P}}\inf_\Omega\left(\frac{L_c\phi}{\phi}\right)$.*

Proof. It is obvious that $\lambda_s' \geq \lambda_s$. Assume to the contrary that $\lambda_s' > \lambda_s$. Let $\eta = \frac{\lambda_s' - \lambda_s}{2}$. Then, there exists $\psi \in C(\mathbb{R}^n, \mathbb{R}^+)$ with $\psi > 0$ in Ω such that $L_c\psi - \lambda_s\psi \geq \eta\psi > 0$ in Ω. Hence, it follows from Theorem 2.1 that the principal eigenvalue of $\{L_c - \lambda_s I, \mathcal{D}, \Omega\}$ is strictly positive, which is a contradiction. Thus, $\lambda_s' = \lambda_s$. To show that $\lambda_s = \sup_{\phi\in P}\inf_\Omega\left(\frac{L_c\phi}{\phi}\right)$, we fix $\lambda < \lambda_s$. Then Theorem 2.1 implies that the boundary value problem: $(L_c - \lambda)u = 1$ in Ω and $u = 0$ in Ω^c has a unique positive solution $\psi_\lambda \in C^s(\mathbb{R}^n)$. Clearly, $\inf_\Omega(L_c\psi_\lambda/\psi_\lambda) > \lambda$. Hence $\lambda <$

$\sup_{\phi \in P} \inf_\Omega \left(\frac{L_c \phi}{\phi} \right)$. Since this is true for any $\lambda < \lambda_s$, it follows that $\lambda_s \leq \sup_{\phi \in P} \inf_\Omega \left(\frac{L_c \phi}{\phi} \right)$. On the other hand, by means of the same argument given above, we can show that $\lambda'_s = \lambda_s$. We now prove the last part. Given any $\lambda < \lambda_s$, Theorem 2.1 and Proposition 2.6 imply that there exists $\Omega_\lambda \subset \mathbb{R}^n$ with $\Omega \subset\subset \Omega_\lambda$ such that $(L_c - \lambda)u = 1$ in Ω_λ and $u = 0$ in Ω_λ^c has a unique positive solution $\psi_\lambda \in C^s(\mathbb{R}^n)$. Clearly, $\psi_\lambda \in \widetilde{P}$. Thus, it follows from the same reasoning that $\lambda''_s = \lambda_s = \sup_{\phi \in \widetilde{P}} \inf_\Omega \left(\frac{L_c \phi}{\phi} \right)$. The proof is completed. $\square$

We next consider the existence and multiplicity of principal eigenvalues for the linear weighted boundary value problem

$$\begin{cases} L_c u = \lambda a(x)u, & x \in \Omega, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega, \end{cases} \tag{2.4}$$

where $a \in L^\infty(\Omega)$. Let $\mu(\lambda)$ be the principal eigenvalue of $\{L_c - \lambda a, \mathcal{D}, \Omega\}$. Following [17], λ_s is referred to as a principal eigenvalue of (2.4) if $\mu(\lambda_s) = 0$. In view of Proposition 2.1, it is easy to see that $\mu(t\lambda_1 + (1-t)\lambda_2) \geq t\mu(\lambda_1) + (1-t)\mu(\lambda_2)$ for any $t \in [0,1]$ and $\lambda_1, \lambda_2 \in \mathbb{R}$, that is, $\mu(\lambda)$ is concave with respect to λ. In addition, by invoking the same argument as that given in [17] (see Theorem 9.1), we infer that $\mu(\lambda)$ is analytic in $\lambda \in \mathbb{R}$. The following results, Theorem 2.3 and Theorem 2.4, are given without proofs because the proofs are the same as those for theorem 9.3 and Theorem 9.4 in [17] in the presence of Proposition 2.2 and Proposition 2.5.

Theorem 2.3. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary. Let $\mu(\lambda)$ be the principal eigenvalue of $\{L_c - \lambda a, \mathcal{D}, \Omega\}$. (i) Suppose that there exists $\Omega_+ \subset \Omega$ such that $a \gneq 0$ in Ω_+. Then $\lim_{\lambda \to \infty} \mu(\lambda) = -\infty$. In addition, Assume that $a \gneq 0$ in Ω. Then (2.4) has a principal eigenvalue if and only if $\lim_{\lambda \to -\infty} \mu(\lambda) > 0$. Moreover, the principal eigenvalue is unique if it exists. (ii) Suppose that there exists $\Omega_- \subset \Omega$ such that $a \lneq 0$ in Ω_-. Then $\lim_{\lambda \to \infty} \mu(\lambda) = \infty$. In addition, Assume that $a \lneq 0$ in Ω. Then (2.4) has a principal eigenvalue if and only if $\lim_{\lambda \to -\infty} \mu(\lambda) < 0$. Moreover, the principal eigenvalue is unique if it exists.*

Theorem 2.4. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary. Suppose $a \in L^\infty(\Omega)$ changes sign in the sense that there exist $\Omega_\pm \subset \Omega$ such that $a \gneq 0$ in Ω_+, and $a \lneq 0$ in Ω_-. Let $\mu(\lambda)$ be the principal eigenvalue of $\{L_c - \lambda a, \mathcal{D}, \Omega\}$. Let $\mu(\lambda^*) = \max_{\lambda \in \mathbb{R}} \mu(\lambda)$ for some $\lambda^* \in \mathbb{R}$. Then (i) (2.4) has no principal eigenvalues if $\mu(\lambda^*) < 0$. (ii) In case that $\mu(\lambda^*) = 0$, (2.4) has one and only one principal eigenvalue, which is λ^*. (iii) (2.4) possesses exactly two principal eigenvalues $\lambda_s^-(a)$ and $\lambda_s^+(a)$ with $\lambda_s^-(a) < \lambda_s^+(a)$ provided that $\mu(\lambda^*) > 0$. In particular, under the assumption that $\mu(\lambda^*) > 0$, there hold (a) $\lambda_s^-(a) < 0 < \lambda_s^+(a)$ if $\mu(0) > 0$. (b) $0 = \lambda_s^-(a) < \lambda_s^+(a)$ if $\mu(0) = 0$ and $\mu'(0) > 0$. (c) $\lambda_s^-(a) < \lambda_s^+(a) = 0$ if $\mu(0) = 0$ and $\mu'(0) < 0$. (d) $0 < \lambda_s^-(a) < \lambda_s^+(a)$ if $\mu(0) < 0$ and $\mu'(0) > 0$. (e) $\lambda_s^-(a) < \lambda_s^+(a) < 0$ if $\mu(0) < 0$ and $\mu'(0) < 0$.*

Corollary 2.5. *Let all of the assumptions of Theorem 2.4 are satisfied. Suppose that $\mu(0) > 0$. Let $\lambda_s^-(a) < 0 < \lambda_s^+(a)$ be the principal eigenvalues of (2.4) given in Theorem 2.4. Then, there holds that*

$$\lambda_s^+(a) = \inf_{\{u \in H_0^s(\Omega) | I_+(u) = 1\}} J(u), \quad \lambda_s^-(a) = - \inf_{\{u \in H_0^s(\Omega) | I_-(u) = 1\}} J(u),$$

where

$$J(u) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx + \int_\Omega c(x) u^2 dx, \quad I_\pm(u) = \int_\Omega \pm a(x) u^2 dx.$$

Proof. Let $\theta \in \mathbb{R}$. It is easy to see that J is bounded from below, weakly lower semi-continuous, and coercive on the manifolds $I_\pm(u) = \theta$. Write $\lambda_\pm = \inf_{\{u \in H_0^s(\Omega) | I_\pm(u) = 1\}} J(u)$. In view of Theorem 1.1.1 of [9], $\lambda_\pm$ are attained at some $\psi_\pm \in H_0^s(\Omega)$, respectively. Obviously, $\lambda_\pm > 0$, and both $|\psi_\pm|$ are also a minimizer. The same argument as that of proposition 9 of [23] implies that $\psi_\pm$ do not change sign in Ω. Thus, λ_+ and $-\lambda_-$ are two eigenvalues of (2.4) with positive eigenfunctions. That is, $\mu(\lambda_+) = \mu(-\lambda_-) = 0$. As $\mu(\lambda)$ has only two λ-intercepts at $\lambda_s^-(a)$ and $\lambda_s^+(a)$, we readily infer that $\lambda_s^-(a) = -\lambda_-$ and $\lambda_s^+(a) = \lambda_+$. The proof is completed. $\qquad\square$

Note that the same argument can be easily adopted to give a similar variational characterization for the principal eigenvalues of (2.4) provided that $\mu(\lambda^*) > 0$. As a matter of fact, we have $\lambda_s^+(a) = \lambda^* + \lambda_+$ and $\lambda_s^-(a) = \lambda^* - \lambda_-$, where $\lambda_\pm = \inf_{\{u \in H_0^s(\Omega) | I_\pm(u) = 1\}} J_{\lambda^*} u$, and

$$J_{\lambda^*}(u) = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} dy dx + \int_\Omega [c(x) - \lambda^* a(x)] u^2 dx.$$

We next consider the asymptotic behavior of $\mu(\lambda)$ as $\lambda \to \infty$ under the conditions that $a(x) \lneq 0$ in Ω and that $a(x)$ vanishes in a sub-domain Ω_0 of Ω.

Theorem 2.6. *Let $\mu(\lambda)$ be the principal eigenvalue of (2.4). Suppose that $c \in C^\alpha(\overline{\Omega})$ with some $\alpha \in (0, 1)$ and $a \in L^\infty(\Omega)$ with $a \lneq 0$ in Ω. Assume that $\Omega_0 := int\, a^{-1}(0)$ is connected and of class C^2 and $\overline{\Omega}_0 \subset \Omega$. Moreover, $\inf_{\overline{\Omega} \setminus \Omega_\delta} a(x) < 0$ for sufficiently small $\delta > 0$, where $\Omega_\delta := \{x \in \overline{\Omega} \mid dist(x, \Omega_0) < \delta\}$. Then $\lim_{\lambda \to \infty} \mu(\lambda) = \mu_{\Omega_0}$, where μ_{Ω_0} is the principal eigenvalue of $\{L_c, \mathcal{D}, \Omega_0\}$.*

Proof. Notice that $\mu(\lambda) < \mu_{\Omega_0}$. Thus, suffices it to show that for each $\epsilon > 0$, there exists $\lambda_\epsilon > 0$ for which $\mu(\lambda) \geq \mu_{\Omega_0} - \epsilon$ provided that $\lambda \geq \lambda_\epsilon$. In view of Theorem 2.1, this can be done by showing that $\{L_c - \lambda a - (\mu_{\Omega_0} - \epsilon), \mathcal{D}, \Omega\}$ has a strict super-solution. To obtain a strict super-solution, we first observe that $\lim_{\delta \to 0} \mu_{\Omega_\delta} = \mu_{\Omega_0}$, which is guaranteed by Theorem 2.2. Here μ_{Ω_δ} is the principal eigenvalue of $\{L_c, \mathcal{D}, \Omega_\delta\}$. Therefore, $\mu_{\Omega_\delta} > \mu_{\Omega_0} - \epsilon$ as long as δ is sufficiently

small. Given that $\epsilon > 0$, we chose $\delta > 0$ such that $\mu_{\Omega_\delta} \geq \mu_{\Omega_0} - \frac{\epsilon}{2}$. Fix $R > 0$, and let u be the unique solution to the following boundary value problem

$$\begin{cases} (L_c - \mu_{\Omega_0} + \epsilon)u = 1 - c(x)R + (\mu_{\Omega_0} - \epsilon)R, & x \in \Omega_\delta, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega_\delta. \end{cases}$$

Then Remark 1 implies that $\|u\|_{L^2(\Omega_\delta)} \leq \epsilon^{-1}2\|\tilde{f}\|_{L^2(\Omega_\delta)}$, where $\tilde{f} = 1 - c(x)R + (\mu_{\Omega_0} - \epsilon)R$. Hence Proposition 2.1 and Proposition 1.1 of [21] imply that $\|u\|_{C^s(\mathbb{R}^n)} \leq \epsilon^{-1}K$, where $K > 0$ is a constant depending on $n, s, R, c, \mu_{\Omega_0}$, and Ω_0 only. Note that $(-\Delta)^s u = (\mu_{\Omega_0} - \epsilon - c)u + \tilde{f}$ in Ω_δ and $u = 0$ in $\mathbb{R}^n \setminus \Omega_\delta$. Let $d(x) = d(x, \partial\Omega_\delta)$. As shown in [22], there holds that $\|u/d^s\|_{C^{s+\min\{s,\alpha\}/2}(\overline{\Omega_\delta})} \leq C\|(\mu_{\Omega_0} - \epsilon - c)u + \tilde{f}\|_{C^{\min\{s,\alpha\}}(\overline{\Omega_\delta})}$ for some positive constant C depending on n, s, and Ω. Let $w = u/d^s$ in Ω_δ, and $\overline{u} = u + R$. Clearly, we have that $(L_c - \mu_{\Omega_0} + \epsilon)\overline{u} = 1$ in Ω_δ, and $\overline{u} = R$ in $\mathbb{R}^n \setminus \Omega_\delta$. In particular, Theorem 2.1 implies that $\overline{u} > 0$ in Ω_δ. We next show that $\overline{u}$ is a desired strict supersolution of $\{L_c - \lambda a - (\mu_{\Omega_0} - \epsilon), \mathcal{D}, \Omega\}$ if $\lambda > 0$ is sufficiently large. To this end, we first show that $(-\Delta)^s\overline{u}$ is bounded in $\mathbb{R}^n \setminus \Omega_\delta$. Given $x \in \mathbb{R}^n \setminus \Omega_\delta$, if $x \notin \partial\Omega_\delta$, then let x' be the project of x on $\partial\Omega_\delta$. Then, for every $\varepsilon > 0$, we have

$$\int_{B_\varepsilon^c(x)} \frac{\overline{u}(x) - \overline{u}(y)}{|x - y|^{n+2s}} dy$$

$$= \int_{B_\varepsilon^c(x) \cap \Omega_\delta} \frac{\overline{u}(x) - \overline{u}(y)}{|x - y|^{n+2s}} dy$$

$$= \int_{B_\varepsilon^c(x) \cap \Omega_\delta} \frac{d^s(x')w(x') - d^s(y)w(y)}{|x - y|^{n+2s}} dy$$

$$= w(x') \int_{B_\varepsilon^c(x) \cap \Omega_\delta} \frac{d^s(x') - d^s(y)}{|x - y|^{n+2s}} dy - \int_{B_\varepsilon^c(x) \cap \Omega_\delta} \frac{[d^s(x') - d^s(y)][w(x') - w(y)]}{|x - y|^{n+2s}} dy$$

$$= I_{1,\varepsilon}(x) - I_{2,\varepsilon}(x).$$

Here $B_\varepsilon^c(x) = \mathbb{R}^n \setminus B_\varepsilon(x)$ and we used the fact that

$$d^s(z)w(z) - d^s(y)w(y) = $$
$$d^s(z)[w(z) - w(y)] + w(z)[d^s(z) - d^s(y)] - [d^s(z) - d^s(y)][w(z) - w(y)]$$

for $z, y \in \mathbb{R}^n$ (see [21]). Lemma 3.1 of [21] shows that $(-\Delta)^s d^s \in C^\beta(\overline{\Omega_{\delta,\rho_0}})$ for some $\beta \in (0, 1)$, where $\Omega_{\delta,\rho_0} = \Omega_\delta \cap \{d < \rho_0\}$, and $\rho_0 > 0$ depends on Ω_δ only. In fact, a careful look into the proof of Lemma 3.1 of [21] shows that ρ_0 can be so chosen that it is independent of δ if δ is sufficiently small. Since $|x - y| \geq |x' - y|$, and $w \in C^{s+\min\{s,\alpha\}/2}(\overline{\Omega_\delta})$, it follows that $|I_{1,\varepsilon}|$ are uniformly bounded for $x \in \Omega \setminus \Omega_\delta$ and $\varepsilon > 0$. Additionally, as $|d^s(x') - d^s(y)||w(x') - w(y)| \leq \epsilon^{-1}C|x' - y|^{2s+\gamma}$, where $\gamma = \min\{1-s, \min\{s,\alpha\}/2\}$, and C is the constant specified above, we have $|d^s(x') - d^s(y)||w(x') - w(y)||x' - y|^{-(n+2s)} \in L^1(\Omega_\delta)$, and so $|I_{2,\varepsilon}|$ are uniformly bounded

as well. Hence, $|(-\Delta)^s \overline{u}|$ is bounded in $x \in \Omega \setminus \Omega_\delta$, and $|(-\Delta)^s \overline{u}| \leq \epsilon^{-1} C$, where $C > 0$ is a constant depending on $n, s, R, c, \mu_{\Omega_0}, \Omega_0$, and Ω. Consequently, $(L_c - (\mu_{\Omega_0} - \epsilon)\overline{u} \in L^\infty(\Omega)$. Since $\inf_{\overline{\Omega} \setminus \Omega_\delta} a < 0$, there exists λ_ϵ such that $(L_c - \lambda a - \mu_{\Omega_0} - \epsilon)\overline{u} \geq 0$ in Ω provided that $\lambda \geq \lambda_\epsilon$. It then follows from Theorem 2.1 that $\mu(\lambda) > \mu_{\Omega_0} - \epsilon$ if $\lambda \geq \lambda_\epsilon$. The proof is completed. $\qquad \square$

We now conclude section 2 with a brief discussion on an optimization problem for positive principal eigenvalues of (2.4). Our discussion is motivated by [8] and provides a answer for the minimization of positive principal eigenvalues of (2.4), which is parallel to Theorem 3.9 of [8].

Theorem 2.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $\mu(0)$ be the principal eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$. Let $\mathcal{M} = \{a(x) \in L^\infty(\Omega) \mid -a_2 \leq a(x) \leq a_2; \overline{a} = a_0; a(x) \gneqq 0 \text{ in } \Omega_a\}$, where $a_0, a_1,$ and a_2 are constants with $a_1, a_2 > 0, -a_2|\Omega| < a_0 \leq a_1|\Omega|, \Omega_a \subset \Omega$ is a nonempty open subset, and $\overline{a} = |\Omega|^{-1} \int_\Omega a dx$. Let $\lambda_s^+(a)$ be the positive principal eigenvalue of (2.4). Suppose that $\mu(0) > 0$. Then there exist a measurable set $E \subseteq \Omega$ and a simple function $a^* = a_1\chi_E - a_2\chi_{\Omega \setminus E} \in \mathcal{M}$ such that $\lambda_s^+(a^*) = \inf_{a \in \mathcal{M}} \lambda_s^+(a)$.*

Proof. The proof is similar to that for Theorem 3.9 of [8]. Only part of it needs elaboration for the sake of clarity. As shown in [8], for each $w \in L^2(\Omega)$, there exists a measurable set $E_w \subset \Omega$ such that $a_1\chi_{E_w} - a_2\chi_{E_w^c}$ solves $\sup_{a \in \mathcal{M}} \int_\Omega aw^2 dx$. Let $w \in L^2(\Omega)$, we write $a_w^* = a_1\chi_{E_w} - a_2\chi_{E_w^c}$. Given that $a \in \mathcal{M}$, let φ be the eigenfunction for $\lambda_s^+(a)$. Note that $\int a_\varphi^* \varphi^2 dx \geq \int a\varphi^2 dx$. Hence, we have

$$\lambda_s^+(a_\varphi^*) = \inf_{u \in H_0^s, \int au^2 dx > 0} \frac{J(u)}{\displaystyle\int_\Omega a_\varphi^* u^2 dx} \leq \frac{J(\varphi)}{\displaystyle\int_\Omega a_\varphi^* \varphi^2 dx} \leq \frac{J(\varphi)}{\displaystyle\int_\Omega a\varphi^2 dx} \leq \lambda_s^+(a).$$

This implies that $\lambda_s^0 = \inf_{a \in \mathcal{M}} \lambda_s^+(a) = \inf\{\lambda_s^+(a_1\chi_E - a_2\chi_{\Omega \setminus E})\}$. To complete the proof, we only need to show λ_s^0 is attainable. Choose a sequence $\{a_n, \lambda_s^+(a_n)\}$ with $a_n \in \mathcal{M}$ such that $\lim_{n \to \infty} \lambda_s^+(a_n) = \lambda_s^0$. Let φ_n be the positive eigenfunction associated with $\lambda_s^+(a_n)$. Since $J(\varphi_n)$ are bounded, there exists a subsequence, still denoted by $\{\varphi_n\}$, such that φ_n weakly converge to φ_* in $H_0^s(\Omega)$ for some $\varphi_* \in H_0^s(\Omega)$, and $\lim_{n \to \infty} \|\varphi_n - \varphi_*\|_{L^2(\Omega)} = 0$. In particular, $\lim_{n \to \infty} J(\varphi_n) = \lambda_s^0 \geq J(\varphi_*) \geq \mu(0)\|\varphi_*\|_{L^2(\Omega)}$. Observe that $\lim_{n \to \infty} \int_\Omega a_n(\varphi^2 - \varphi_*^2)dx = 0$, and $\int_\Omega a_n \varphi_n^2 dx = 1$. Thus, $\varphi^* \neq 0$. Now let $\phi_n = (J(\varphi_n))^{-\frac{1}{2}}\varphi_n$ and $\phi_* = (J(\varphi_*))^{-\frac{1}{2}}\varphi_*$. Then, we have

$$\frac{1}{\lambda_s^0} \geq \frac{1}{\lambda_s^+(a_{\phi_*}^*)} \geq \int_\Omega a_{\phi_*}^* \phi_*^2 dx \geq \int_\Omega a_n \phi_*^2 dx = \frac{\lambda_s^0}{J(\varphi_*)} \int_\Omega a_n \left(\frac{\varphi_*}{\sqrt{\lambda_s^0}}\right)^2 dx,$$

$$\int_\Omega a_n \phi_n^2 dx = \frac{1}{\lambda_s^+(a_n)}.$$

Hence

$$\frac{1}{\lambda_s^0} \geq \frac{1}{\lambda_s^+(a_{\phi_*}^*)} \geq \int_\Omega a_{\phi_*}^* \phi_*^2 dx \geq \frac{\lambda_s^0}{J(\varphi_*)} \left\{ \int_\Omega a_n \phi_n^2 dx + \int_\Omega a_n \left[\left(\frac{\varphi_*}{\sqrt{\lambda_s^0}} \right)^2 - \phi_n^2 \right] dx \right\}.$$

By passing the limits as $n \to \infty$, we find that $\lambda_s^0 = J(\varphi_*)$ and $\lambda_s^0 = \lambda_s^+(a_{\phi_*}^*)$. The proof is completed. □

3   Cooperative systems

This section is focused on the principal eigenvalues and a maximum principle for cooperative systems involving fractional Laplace operators. The results established in the section are a counterpart of those given in [18] for cooperative elliptic systems. We again assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary and first consider boundary value problem

$$\begin{cases} (-\Delta)^{s_i} u_i + \sum_{j=1}^k c_{i,j} u_j = f_i(x) & \text{in } \Omega \quad (1 \leq i \leq k) \\ u_i = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3.1}$$

where $c_{i,j} \in L^\infty(\Omega)$, $1 \leq i, j \leq k$, and $s_i \in (0,1)$, $k \geq 2$ stands for an integer. Throughout this section, we shall assume without loss of generality that $0 < s_1 \leq \cdots \leq s_i \leq s_{i+1} \leq \cdots \leq s_k < 1$.

Definition 6. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Given $f = (f_1, \cdots, f_k)$ with $f_i \in L^2(\Omega)$, a function $w = (w_1, \cdots, w_k)$ with $w_i \in H_0^{s_i}(\Omega)$ is said to be a weak solution to the boundary value problem (3.1) if

$$\frac{c_{n,s_i}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[w_i(x) - w_i(y)][\phi_i(x) - \phi_i(y)]}{|x - y|^{n+2s}} dy dx + \int_\Omega \sum_{j=1}^k c_{i,j} w_j \phi_i dx = \int_\Omega f_i \phi_i dx$$

for any $\phi = (\phi_1, \cdots, \phi_k)$ with $\phi_i \in H_0^{s_i}(\Omega)$ for $i \in \{1, \cdots, k\}$.

Proposition 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $w = (w_1, \cdots, w_k)$ with $w_i \in H_0^{s_i}(\Omega)$ $(i = 1, \cdots, k)$ be a weak solution of (3.1). Namely,*

$$\sum_{i=1}^k \left\{ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[w_i(x) - w_i(y)][\phi_i(x) - \phi_i(y)]}{|x - y|^{n+2s_i}} dy dx + \sum_{j=1}^k \int_\Omega c_{i,j} w_j \phi_i dx \right\} = \sum_{i=1}^k \int_\Omega f_i \phi_i dx$$

for any $\phi = (\phi_1, \cdots, \phi_k)$ with $\phi_i \in H_0^{s_i}(\Omega)$. Assume that $c_{i,j} \in L^\infty(\Omega)$ $(i \neq j, i, j = 1, \cdots, k)$, $c_{ii} \in L^p(\Omega)$, and $f_i \in L^p(\Omega)$ $(i = 1, \cdots, k)$, $n > 2s_k$, where $p > \max\{n/2s_1, 2/[1 - \frac{2(s_k - s_1)}{n - 2s_k}]\}$. Then there exists a constant $C > 0$ such that

$$\|w_i\|_{L^\infty(\Omega)} \leq C, \qquad 1 \leq i \leq k$$

where C depends only on $|\Omega|, n, p, s_i, \|c_{i,j}\|_{L^\infty(\Omega)}$ $(i \neq j), \|w_i\|_{L^2(\Omega)} \|c_{ii}\|_{L^p(\Omega)}$, and $\|f_i\|_{L^p(\Omega)}$.

Proof. We fix $q_1 = \frac{p}{p-1}$ and set $q_i > 1$ ($2 \leq i \leq k$) such that $\frac{n}{(n-2s_1)q_1} = \frac{n}{(n-2s_2)q_2} = \cdots = \frac{n}{(n-2s_k)q_k}$. Let $\chi = \frac{n}{(n-2s_1)q_1}$. Clearly $\chi > 1$, and $2 > q_k \geq \cdots \geq q_i \geq \cdots \geq q_1$. The Moser iteration is again utilized to deduce the conclusion. To implement each iteration, we always chose an ordered pair $(\beta_1, \cdots, \beta_k)$ so that $(1 + \beta_1)q_1 = (1 + \beta_2)q_2 = \cdots = (1 + \beta_k)q_k$. Let $w_{i,m} = \min\{|w_i|, m\}$, where $m > 0$ is a given number. Similar to the proof of Proposition 2.1, Multiplying the ith equation of (3.1) by $(w_{i,m} + 1)^{\beta_i} - 1$ and integrating the resulting equation over Ω give

$$\left(\int_\Omega (|w_{i,m}| + 1)^{\frac{(1+\beta_i)n}{n-2s_i}} dx \right)^{\frac{n-2s_i}{n}} \leq C(n, s_i)(\frac{\beta_i + 1}{2\sqrt{\beta_i}})^2 \sum_{j=1}^k \int_\Omega |c_{i,j}||w_j|(|w_{i,m}| + 1)^{\beta_i} dx$$

$$+ C(n, s_i)(\frac{\beta_i + 1}{2\sqrt{\beta_i}})^2 \left\{ \int_\Omega |f_i|(|w_{i,m}| + 1)^{\beta_i} dx + |\Omega|^{\frac{n-2s_i}{n} - \frac{1}{q_i}} \int_\Omega (|w_{i,m}| + 1)^{(1+\beta_i)q_i} dx \right\}.$$

Let $r_i = \frac{(1+\beta_i)q_i}{\beta_i}$ and $t_i = \frac{(1+\beta_i)q_i}{(1+\beta_i)q_i - \beta_i}$. For $j \neq i$, using Hölder's inequality and Cauchy inequality yield that

$$\int_\Omega |c_{i,j}||w_j|(|w_{i,m}| + 1)^{\beta_i} dx \leq \sup |c_{i,j}| \|w_j\|_{L^{t_i}(\Omega)} \|(|w_{i,m}| + 1)^{\beta_i}\|_{L^{r_i}(\Omega)}$$

$$\leq \sup |c_{i,j}| \frac{1}{\beta_i + 1} \left(\int_\Omega |w_j|^{\frac{(1+\beta_i)q_i}{(1+\beta_i)q_i - \beta_i}} dx \right)^{\frac{(1+\beta_i)q_i - \beta_i}{q_i}} + \frac{\beta_i}{\beta_i + 1} \left(\int_\Omega (|w_{i,m}| + 1)^{(1+\beta_i)q_i} dx \right)^{\frac{1}{q_i}}$$

$$\leq \sup |c_{i,j}| |\Omega|^{\frac{(1+\beta_i)(q_i - 1)}{q_i}} \left(\int_\Omega |w_j|^{(1+\beta_i)q_i} dx \right)^{\frac{1}{q_i}} + \left(\int_\Omega (|w_{i,m}| + 1)^{(1+\beta_i)q_i} dx \right)^{\frac{1}{q_i}}.$$

Let $p_i = \frac{q_i}{q_i - 1}$. Note that $p_i \leq p$. Then, we find that

$$\left(\int_\Omega (|w_{i,m}| + 1)^{\frac{(1+\beta_i)n}{n-2s_i}} dx \right)^{\frac{n-2s_i}{n}}$$

$$\leq C(n, s_i)(\frac{1 + \beta_i}{2\sqrt{\beta_i}})^2 \left\{ \|c_{ii}\|_{L^{p_i}(\Omega)} \left(\int_\Omega [(|w_i| + 1)(|w_{i,m}| + 1)^{\beta_i}]^{q_i} dx \right)^{\frac{1}{q_i}} \right.$$

$$+ \sup_{i \neq j} |c_{i,j}| |\Omega|^{\frac{(1+\beta_i)(q_i - 1)}{q_i}} \sum_{j=1,j\neq i}^k \left(\int_\Omega (|w_j| + 1)^{(1+\beta_i)q_i} dx \right)^{\frac{1}{q_i}}$$

$$+ \left[k - 1 + |\Omega|^{\frac{n-2s_i}{n} - \frac{1}{q_i}} + \|f_i\|_{L^{p_i}(\Omega)} \right] \left(\int_\Omega (|w_{i,m}| + 1)^{(1+\beta_i)q_i} dx \right)^{\frac{1}{q_i}} \right\}.$$

Let $\gamma_{q_k} > 0$ be the number such that $\frac{1+\beta}{2\sqrt{\beta}} \leq \sqrt{\gamma_{q_k}(1 + \beta)}$ provided that $\beta \geq \frac{2-q_k}{q_k}$. Write $C_i' = C(n, s_i)\gamma_{q_k}(1+\beta_i) \left[\sup_{i\neq j} |c_{i,j}| |\Omega|^{\frac{(1+\beta_i)(q_i - 1)}{q_i}} + k - 1 + |\Omega|^{\frac{n-2s_i}{n} - \frac{1}{q_i}} + \|c_{ii}\|_{L^{p_i}(\Omega)} + \|f_i\|_{L^{p_i}(\Omega)} \right]$. Note that $\beta_i \geq \frac{2-q_k}{q_k}$ for all i whenever $\beta_k \geq \frac{2-q_k}{q_k}$. By passing the limits as $m \to \infty$, we have

$$\left(\int_\Omega (|w_i| + 1)^{\frac{(1+\beta_i)n}{n-2s_i}} dx \right)^{\frac{n-2s_i}{n}} \leq C_i' \left(\sum_{j=1}^k \int_\Omega (|w_j| + 1)^{(1+\beta_i)q_i} dx \right)^{\frac{1}{q_i}}.$$

Let again $\theta = (1 + \beta_i)q_i$ and $\chi = \frac{n}{n-2s_i}\frac{1}{q_i}$, and assume without loss of generality that $|\Omega| > 1$, then

$$\left(\int_\Omega (|w_i| + 1)^{\chi\theta} dx\right)^{\frac{1}{\chi\theta}} \le \left(\frac{c_1\theta}{q_1}\right)^{\frac{q_k}{\theta}} \left[|\Omega|c_3^{\frac{q_k}{\theta}} + c_2^{\frac{q_k}{\theta}}\right]\left(\sum_{j=1}^k \int_\Omega (|w_j| + 1)^\theta dx\right)^{\frac{1}{\theta}},$$

where $c_1 = \max_{1\le i\le k} C(n, s_i)\gamma_{q_k}$, $c_2 = k-1+|\Omega|^{\frac{n-2s_1}{n}} + \max_{1\le i\le k}(\|c_{ii}\|_{L^{p_i}(\Omega)} + \|f_i\|_{L^{p_i}(\Omega)})$, and $c_3 = \sup_{i\ne j}|c_{i,j}|_{L^\infty}$. In view of the fact that $\left(\sum_{i=1}^k |a_i|\right)^t \le \sum_{i=1}^k |a_i|^t$ for any $0 < t < 1$ and $a_i \in \mathbb{R}$, we have

$$\left(\sum_{i=1}^k \int_\Omega (|w_i| + 1)^{\chi\theta} dx\right)^{\frac{1}{\chi\theta}} \le k\left(\frac{c_1\theta}{q_1}\right)^{\frac{q_k}{\theta}} \left[|\Omega|c_3^{\frac{q_k}{\theta}} + c_2^{\frac{q_k}{\theta}}\right]\left(\sum_{i=1}^k \int_\Omega (|w_i| + 1)^\theta dx\right)^{\frac{1}{\theta}}.$$

Set $\theta_{m+1} = \chi\theta_m$ and $\theta_0 = 2$. Then by induction, we have

$$\|(|w_i| + 1)\|_{L^{\theta_{m+1}}(\Omega)} \le \prod_{j=0}^m k\left(\frac{c_1\theta_j}{q_1}\right)^{\frac{q_k}{\theta_j}} \left[|\Omega|c_3^{\frac{q_k}{\theta}} + c_2^{\frac{q_k}{\theta_j}}\right]\left(\sum_{j=1}^k \|(|w_j| + 1)\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

Since $\chi > 1$, there exists m such that $\theta_{m+1} > p$. Namely, after finite number of iterations, we find that $w_i \in L^p(\Omega)$ $(1 \le i \le k)$. Write that $\widehat{f_i} = f_i + \sum_{j=1, j\ne i}^k c_{i,j}w_j$, clearly, $\widehat{f_i} \in L^p(\Omega)$. Thus, Proposition 2.1 implies that $w_i \in L^\infty(\Omega)$ $(1 \le i \le k)$. The proof is completed. $\qquad\square$

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Suppose that $c_{i,j} \in L^\infty(\Omega)$ $(1 \le i, j \le k)$. Given $f_i \in L^2(\Omega)$ $(1 \le i \le k)$, then the boundary value problem*

$$\begin{cases} (-\Delta)^{s_i}u_i + \mu u_i + \sum_{j=1}^m c_{i,j}u_j = f_i & \text{in } \Omega \\ u_i = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{3.2}$$

has a unique weak solution $w = (w_1, \cdots, w_k)$ with $w_i \in H_0^{s_i}(\Omega)$ provided that $\mu \in \mathbb{R}$ is sufficiently large. In particular, if $c_{i,j} \le 0$ whenever $i \ne j$ and $f_i \ge 0$ for each i, then $w_i \ge 0$.

Proof. The existence and uniqueness of a weak solution shall be established via the Lax-Milgram theorem. To this end, we set

$$X = \prod_{j=1}^k H_0^{s_j}(\Omega),$$

$$Y = \underbrace{L^2(\Omega) \times \cdots \times L^2(\Omega)}_{k}.$$

For $u = (u_1, \cdots, u_k)$, and $v = (v_1, \cdots, v_k) \in Y$ or X, define

$$(u \mid v) = \sum_{j=1}^k \int_\Omega u_j v_j dx,$$

$$\langle u \mid v \rangle = \sum_{j=1}^{k} \left[\int_{\mathbb{R}^{2n}} \frac{[u_j(x) - u_j(y)][v_j(x) - v_j(y)]}{|x-y|^{n+2s_i}} dy dx + \int_{\Omega} u_j v_j dx \right],$$

and $\|u\|_X = \sqrt{\langle u \mid u \rangle}, \|u\|_Y = \sqrt{(u \mid u)}$. Clearly, both X and Y are Hilbert spaces. Now Let $a(u,v) : X \times X \to \mathbb{R}$ be defined by

$$a(u,v) = \sum_{i=1}^{k} \left\{ \frac{c_{n,s_i}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u_i(x) - u_i(y)][v_i(x) - v_i(y)]}{|x-y|^{n+2s_i}} dy dx + \right.$$
$$\left. \sum_{j=1}^{k} \int_{\Omega} c_{i,j} u_j v_i dx \right\}. \tag{3.3}$$

It follows from Sobolve embedding inequality and Hölder inequality that

$$|a(u,v) + \mu(u \mid v)| \le C_{\mu,1} \|u\|_X \|v\|_X, \qquad a(u,u) + \mu(u \mid u) \ge C_{\mu,2} \|u\|_X^2$$

for some constants $C_{\mu,1} > 0$ and $C_{\mu,2}$. It is also easy to see that $C_{\mu,2} \ge 0$ provided that μ is sufficiently large. Following [16], we let $\mathcal{L}_c : D(\mathcal{L}_c) \to Y$ be the unbounded linear operator associated with $a(\cdot, \cdot)$ with $a(u,v) = (\mathcal{L}_c u \mid v)$ for $u, v \in X$. Here

$$D(\mathcal{L}_c) = \{u \in X \mid v \longmapsto a(u,v) \text{ is continuous on } X \text{ in terms of the topology induced by } Y\}.$$

Similarly, we let $\mathcal{L}_c + \mu I$ be the linear operator associated with $a(u,v) + \mu(u \mid v)$. Notice that $D(\mathcal{L}_c)$ is dense in X. As a matter of fact, $\underbrace{C_0^2(\Omega) \times \cdots \times C_0^2(\Omega)}_{k} \subset D(\mathcal{L}_c)$. Hence Theorem 3.6 of [16] implies that $(\mathcal{L}_c + \mu I)$ has bounded inverse as μ is sufficiently large. In other words, the boundary value problem (3.2) has a unique weak solution $(\mathcal{L}_c + \mu I)^{-1} f$, where $f \in Y$. Note that $\mathcal{L}_c$ is Fredholm due to the compactness of $(\mathcal{L}_c + \mu I)^{-1}$ since X is compactly embedded in Y. We now proceed to complete the last part of this lemma. Assume that $c_{i,j} \le 0$ if $i \ne j$. Let $\phi_i = w_i^-$, where $w_i^- = \min\{w_i, 0\}$. Then

$$\sum_{i=1}^{k} \left\{ \frac{c_{n,s_i}}{2} \int_{\mathbb{R}^{2n}} \frac{[w_i^-(x) - w_i^-(y)]^2 + 2w_i^+(x)w_i^-(y)}{|x-y|^{n+2s_i}} dy dx + \right.$$
$$\left. \mu \int_{\Omega} |w_i^-|^2 dx - \sum_{j=1,j \ne i}^{k} \int_{\Omega} c_{i,j} w_j^+ w_i^- dx \right\}$$
$$= - \sum_{i=1}^{k} \left\{ \sum_{j=1}^{k} \int_{\Omega} c_{i,j} w_j^- w_i^- dx + \int_{\Omega} f_i w_i^- dx \right\}.$$

In view of the fact that $c_{i,j} \le 0$ if $i \ne j$, and using the Hölder inequality, we find

$$\sum_{i=1}^{k} \left\{ \frac{c_{n,s_i}}{2} \int_{\mathbb{R}^{2n}} \frac{[w_i^-(x) - w_i^-(y)]^2}{|x-y|^{n+2s_i}} dydx + \mu \int_{\Omega} |w_i^-|^2 dx \right\}$$

$$\leq \sum_{i=1}^{k} \left\{ \sum_{j=1}^{k} \int_{\Omega} |c_{i,j}| |w_j^- w_i^-| dx - \int_{\Omega} f_i w_i^- dx \right\}$$

$$\leq 2 \max_{1 \leq i,j \leq k} |c_{i,j}|_{L^\infty} \sum_{j=1}^{k} \int_{\Omega} |w_j^-|^2 dx - \sum_{j=1}^{k} \int_{\Omega} f_i w_i^- dx.$$

Let $\lambda_{min} = \min_{1 \leq i \leq k} \{\lambda_{s_i}\}$, where λ_{s_i} is the principal eigenvalue of $(-\Delta)^{s_i}$ subject to Dirichlet boundary condition in Ω, then as long as $\mu > 2 \max_{1 \leq i,j \leq k} |c_{i,j}|_{L^\infty} - \lambda_{min}$, we have

$$\theta \sum_{i=1}^{k} \int_{\Omega} |w_i^-|^2 dx \leq - \sum_{j=1}^{k} \int_{\Omega} f_i w_i^- dx.$$

Here $\theta = \mu - 2 \max_{1 \leq i,j \leq k} |c_{i,j}|_{L^\infty} + \lambda_{min}$. It immediately follows that $w_i^- = 0$ $(1 \leq i \leq k)$. The proof is completed. $\qquad \square$

Theorem 3.1. *Let $\mathcal{L}_c$ be defined in Lemma 3.1. Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in \mathcal{C}^1$, and $n > 2s_k$. Assume that $c_{i,j} \in L^\infty(\Omega)$ and $c_{i,j} \leq 0$ if $i \neq j$. Moreover, there exists a permutation $p_1 p_2 \cdots p_k$ of $\{1, \cdots, k\}$ for which $c_{p_i p_{i+1}} \lneqq 0$ in Ω $(i = 1, \cdots, k-1)$, and $c_{p_k p_1} \lneqq 0$ in Ω. Then $\mathcal{L}_c$ possesses exactly one real eigenvalue $\lambda_\mathbf{s}$ that is equal to $\mathfrak{s}(\mathcal{L}_c)$, where $\mathfrak{s}(\mathcal{L}_c) = \inf\{\Re\lambda \mid \lambda \in \sigma(\mathcal{L}_c)\}$. In particular, $\lambda_\mathbf{s}$ is a simple eigenvalue. Assume further that $c_{i,j} = c_{j,i}$ provided that $i \neq j$, then $\lambda_\mathbf{s} = \inf_{u \in X, \|u\|_X = 1} a(u,u)$, where $a(u,v)$ is defined by (3.3).*

Proof. As shown in the proof of Lemma 3.1, $(\mathcal{L}_c + \mu I)^{-1}$ is positive if μ is sufficiently large. Let r_μ be the spectral radius of $(\mathcal{L}_c + \mu I)^{-1}$. To show that $r_\mu > 0$ provided that μ is sufficiently large, let $\Omega' \subset \Theta \subset \Omega$ be a bounded domain, where Θ is a compact subset of Ω. Let $f = (f_1, \cdots, f_k)$ be so chosen that $f_i \gneqq 0$ and $f_i \in C_c(\Omega')$. Write $\psi = (\psi_1, \cdots, \psi_k) = (\mathcal{L}_c + \mu I)^{-1} f$. Thus, Lemma 3.1 shows that $\psi_k \in H_0^{s_i}(\Omega)$ and $\psi_k \geq 0$. Moreover, Proposition 3.1 and Theorem 1.2 of [21] imply that $\psi_k \in C^{s_i}(\mathbb{R}^n)$. Note that $(-\Delta)^{s_i}\psi_k + c_{ii}\psi_k \geq 0$ in Ω, and $\psi_k = 0$ in Ω^c, thus, it follows from the Proposition 2.3 that either $\psi_i > 0$ or $\psi_i = 0$ in Ω for each i. Due to the assumption, if $\psi_i = 0$ for some i, then $\psi_i = 0$ for all i. As $f_i \gneqq 0$, we see that $\psi_i > 0$ in Ω for all i. Hence there exists $\theta > 0$ such that $\psi \geq \theta f$, that is, $(\mathcal{L}_c + \mu I)^{-1} f \geq \theta f$, and hence $\sigma(\mathcal{L}_c + \mu I)^{-1}) \setminus \{0\} \neq \varnothing$. Then the Krein-Rutman theorem implies that $1/r_\mu - \mu$ is an eigenvalue of $\mathcal{L}_c$ with a positive eigenfunction $(\phi_1, \cdots, \phi_k) \geq 0$. Write $\lambda_\mathbf{s} = 1/r_\mu - \mu$. It follows from the same reasoning as above that $\phi_i \in C^{s_i}(\mathbb{R}^n)$, and $(-\Delta)^{s_i}\phi_i + \sum_{i=1}^{k} c_{i,j}\phi_j = \lambda_\mathbf{s}\phi_i$ in Ω, and $\phi_i = 0$ in Ω^c. In particular, $\phi_i > 0$ in Ω for all i. Meanwhile, as $\mathcal{L}_c$ is resolvent positive, $\mathfrak{s}(\mathcal{L}_c)$ is also an eigenvalue of $\mathcal{L}_c$. Obviously,

$\mathfrak{s}(\mathcal{L}_c) \leq \lambda_{\mathbf{s}}$. Next we show that $\mathfrak{s}(\mathcal{L}_c) = \lambda_{\mathbf{s}}$. To this end, we let $(\phi_1, \cdots, \phi_k) \in \ker(\mathfrak{s}(\mathcal{L}_c)I - \mathcal{L}_c)$, and assume without loss of generality that $\varphi_1(x_1) < 0$ for some $x_1 \in \Omega$. In view of Theorem 1.2 of [21], there exists $C > 0$ such that $|\varphi_i(x)| \leq Cd(x, \partial\Omega)^{s_i}$. Now assume to the contrary that $\mathfrak{s}(\mathcal{L}_c) < \lambda_{\mathbf{s}}$. Note that there exists $t > 0$ such that $t(\phi_1, \cdots, \phi_k) + (\varphi_1, \cdots, \varphi_k) \geq 0$ in $\overline{\Omega}$. Let $\bar{t} = \inf\{t \in \mathbb{R}^+ \mid t(\phi_1, \cdots, \phi_k) + (\varphi_1, \cdots, \varphi_k) \geq 0\}$. Apparently, $\bar{t} > 0$ and $\bar{t}\phi_i + \varphi_i \geq 0$ in $\overline{\Omega}$ for all i. Notice that

$$(-\Delta)^{s_i}(t\phi_i + \varphi_i) + [c_{i,i} - \lambda_{\mathbf{s}}](t\phi_i + \varphi_i) \geq 0 \ \text{ in } \Omega, \quad t\phi_i + \varphi_i = 0 \ \text{ in } \mathbb{R}^n \setminus \Omega.$$

It follows from the same reasoning that either $(\bar{t}\phi_1 + \varphi_1, \cdots, \bar{t}\phi_k + \varphi_k) > 0$ in Ω or $(\bar{t}\phi_1 + \varphi_1, \cdots, \bar{t}\phi_k + \varphi_k) = 0$ in Ω. If the former is true, then Lemma 2.2 implies that $\bar{t}\phi_i(x) + \varphi_i(x) \geq \beta_i d(x, \partial\Omega)^{s_i}$ in Ω for some $\beta_i > 0$. Let $\beta = \min_{1 \leq i \leq k}\{\beta_i\}$. Since $\phi_i(x) \leq Cd(x, \partial\Omega)^{s_i}$, there exists $t' \in (0, \bar{t})$ for which $t'\phi_i(x) \leq \frac{\beta}{2}d(x, \partial\Omega)^{s_i}$. Now let $\tau = \bar{t} - t'$. It is clear that $\tau \in (0, \bar{t})$ and $\tau\phi_i + \varphi_i \geq \frac{\beta}{2}d(x, \partial\Omega)^{s_i}$ for all $x \in \Omega$, which contradicts the definition of $\bar{t}$. Consequently, $(\varphi_1, \cdots, \varphi_k) = -\bar{t}(\phi_1, \cdots, \phi_k)$. However, this also gives rise to a contradiction if $\mathfrak{s}(\mathcal{L}_c) < \lambda_{\mathbf{s}}$, which confirms that $\mathfrak{s}(\mathcal{L}_c) = \lambda_{\mathbf{s}}$. The above argument also shows that no other eigenvalue of $\mathcal{L}_c$ has a corresponding positive eigenfunction. In addition, the same argument leads to the fact that $\ker(\lambda_{\mathbf{s}}I - \mathcal{L}_c) = \text{span}\{(\phi_1, \cdots, \phi_k)\}$. To show that $\ker(\lambda_{\mathbf{s}}I - \mathcal{L}_c)^2 = \ker(\lambda_{\mathbf{s}}I - \mathcal{L}_c)$, we let $\mathcal{L}_c^*$ be the linear operator associated with $a^*(u, u)$, where $a^*(u, u) : X \times X \to \mathbb{R}$ is given by

$$a^*(u, v) = \sum_{j=1}^k \left\{ \frac{c_{n,s_j}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{[u_j(x) - u_j(y)][v_j(x) - v_j(y)]}{|x - y|^{n + 2s_j}} dydx + \sum_{i=1}^k \int_\Omega c_{j,i} u_i v_j dx \right\}.$$

Namely, $\mathcal{L}_c^*$ is the adjoint operator of $\mathcal{L}_c$. Hence $\lambda_{\mathbf{s}}$ is an eigenvalue of $\mathcal{L}_c^*$, and there exists a nonnegative eigenfunction $(\phi_1^*, \cdots, \phi_k^*)$ of $\mathcal{L}_c^*$ corresponding to $\lambda_{\mathbf{s}}$ with $\phi_i^* \in C^{s_i}(\mathbb{R}^n)$. The same argument given above implies that $\phi_i^* > 0$ for each i. Thus, the Fredholm alternative yields that $\ker(\lambda_{\mathbf{s}}I - \mathcal{L}_c)^2 = \ker(\lambda_{\mathbf{s}}I - \mathcal{L}_c)$. To conclude the proof, we let $\lambda' = \inf_{u \in X, \|u\|_X = 1} a(u, u)$ and show that $\lambda' = \lambda_{\mathbf{s}}$. In light of [9], λ' is attained by some $w \in X$, and $\inf_{u \in X, \|u\|_X = 1} a(u, u)$ is an eigenvalue of $\mathcal{L}_c$. Then, from the fact that $a(|w|, |w|) \leq a(w, w)$, we infer that $|w|$ is an eigenfunction associated with $\inf_{u \in X, \|u\|_X = 1} a(u, u)$. Since $\lambda_{\mathbf{s}}$ is the only eigenvalue of $\mathcal{L}_c$ with positive eigenfunctions, we must have $\lambda_{\mathbf{s}} = \inf_{u \in X, \|u\|_X = 1} a(u, u)$. The proof is completed. $\quad\square$

Thereafter, given $\mathbf{c} = \{c_{i,j}\}_{1 \leq i,j \leq k}$ with $c_{i,j} \in L^\infty(\Omega)$. Suppose that $\mathbf{c} = \{c_{i,j}\}_{1 \leq i,j \leq k}$ satisfies the condition given in Theorem 3.1, we shall write

$$\begin{cases} (\mathbf{L_c}u)(x) := (-\Delta)^{s_i}u_i + \sum_{j=1}^k c_{i,j}(x)u_j, & x \in \Omega \ (1 \leq i \leq k) \\ \mathbf{D}u := u, & x \in \mathbb{R}^n \setminus \Omega. \end{cases} \tag{3.4}$$

Let $\lambda_{\mathbf{s}}$ be given in Theorem 3.1, then $\lambda_{\mathbf{s}}$ is referred to as the principal eigenvalue of $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$.

Definition 7. Let $u = (u_1, \cdots, u_k) \in C(\overline{\Omega}, \mathbb{R}^k) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^k)$, u is said to be a classical super-solution of $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$ if $\mathbf{L_c} u \in L^\infty(\Omega, \mathbb{R}^k)$, $(-\Delta)^{s_i} u_i + \sum_{j=1}^k c_{i,j} u_j \geq 0$ in Ω, and $u_i \geq 0$ in $\mathbb{R}^n \setminus \Omega$ for all $i \in \{1, \cdots, k\}$. In addition, u is said to be a strict super-solution of $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$ provided that u is a (classical) super-solution of $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$ while u is not classical solution of $\mathbf{L_c} u = 0$ in Ω and $u = 0$ in $\mathbb{R}^n \setminus \Omega$.

Definition 8. Let $\mathbf{L_c}$ and $\mathbf{D}$ be defined in (3.4). Then $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$ is said to satisfy the strong maximum principle if $u_i > 0$ in Ω for each $i \in \{1, \cdots, k\}$ whenever $u = (u_1, \cdots, u_k) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^k)$ with $u_i \in C^{s_i}(\overline{\Omega})$ for each i is a nontrivial super-solution of $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$.

Theorem 3.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain and is of class $\mathcal{C}^1$. Let $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$ be defined in (3.4). Assume that $\{c_{i,j}\}_{1 \leq i,j \leq k}$ satisfies the condition given in Theorem 3.1. Let $\lambda_\mathbf{s}$ be the principal eigenvalue of $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$. Then the following statements are equivalent*

(i) $\lambda_\mathbf{s} > 0$,

(ii) $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$ has a positive strict super-solution $u \in C(\overline{\Omega}, \mathbb{R}^k) \cap L^\infty_{loc}(\mathbb{R}^n, \mathbb{R}^k)$,

(iii) $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$ satisfies the strong maximum principle;

(iv) the linear boundary value problem

$$\begin{cases} (-\Delta)^{s_i} u_i + \sum_{j=1}^k c_{i,j}(x) u_j = f_i & in \ \Omega, \ (1 \leq i \leq k) \\ u_i = 0 & in \ \mathbb{R}^n \setminus \Omega \end{cases}$$

has an unique solution $u \in X$ for each $f = (f_1, \cdots, f_k)$ with $f_i \in L^2(\Omega)$. In particular, $u_i \geq 0$ for all i provided that $f_i \geq 0$ for each $i \in \{1, \cdots, k\}$.

Proof. The proof is very similar to the one for Theorem 2.1. We just omit details. □

Corollary 3.3. *Let $\Omega_i \ (i = 1, 2) \subset \mathbb{R}^n$ be two bounded open domains with $\partial \Omega_i \in \mathcal{C}^1$ such that $\Omega_2 \subseteq \Omega_1$. Denote the principal eigenvalues of $\{\mathbf{L_c}, \mathbf{D}, \Omega_i\}$ by λ_{Ω_i}, respectively. Then $\lambda_{\Omega_1} \leq \lambda_{\Omega_2}$. In case that $\Omega_2 \subsetneq \Omega_1$, then $\lambda_{\Omega_1} < \lambda_{\Omega_2}$. Let $\lambda_{\mathbf{c}_I}$ be the principal eigenvalue of $\{\mathbf{L}_{\mathbf{c}_I}, \mathbf{D}, \Omega\}$ with $\mathbf{c}_I = \{c^I_{i,j}\}_{1 \leq i,j \leq k}$ and $c^I_{i,j} \in L^\infty(\Omega); (I = 1, 2)$. Assume that $c^1_{i,j} \leq c^2_{i,j}$. Then $\lambda_{\mathbf{c}_1} \leq \lambda_{\mathbf{c}_2}$. Moreover, if there exist i and j such that $c^1_{i,j} \lneq c^2_{i,j}$ in Ω, then $\lambda_{\mathbf{c}_1} < \lambda_{\mathbf{c}_2}$.*

Proof. In light of theorem 3.2, the proof is the same as the one for Proposition 2.5. The proof is completed. □

Corollary 3.4. *Suppose that all of the assumptions of Theorem 3.2 are satisfied. Let $u \in C(\mathbb{R}^n) \cap C^s(\overline{\Omega})$ be a strict super-solution of $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$. Then exists a positive constant ω, depending only on $n, s_i, c_{i,j}$, such that $u \geq 0$ whenever $|\Omega| \leq \omega$.*

Proof. Let $\lambda_{\mathbf{s}}$ be the principle eigenvalue of $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$. In view of Theorem 3.2, it is sufficient to show that $\lambda_{\mathbf{s}} > 0$ if $|\Omega| \leq \omega$. Let $\phi = (\phi_1, \cdots, \phi_k)$ be an eigenfunction corresponding to $\lambda_{\mathbf{s}}$ with $\|\phi\|_Y = 1$. By virtute of Proposition 2.4 and Theorem 3.1, we find that

$$\lambda_{\mathbf{s}} \geq \sum_{i=1}^{k} \frac{c_{n,s_i}}{2} \int_{\mathbb{R}^n} \frac{|\phi_i(x) - \phi_i(y)|^2}{|x-y|^{n+2s_i}} dydx - 2 \max_{1 \leq i,j \leq k} |c_{i,j}|_{L^\infty} \geq \sum_{i=1}^{k} \frac{K(n,s_i)}{r^{2s_i}} - 2 \max_{1 \leq i,j \leq k} |c_{i,j}|_{L^\infty},$$

where r is the positive constant such that $|B_r(0)| = |\Omega|$, and $K(n, s_i) > 0$ is a constant only depending on n and s_i. Therefore, there exists $\omega > 0$ such that $\lambda_{\mathbf{s}} > 0$ if $0 < r \leq \omega$. The proof is completed. $\square$

4   Applications

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathcal{C}^1$. Suppose that $u, v \in L_{loc}^\infty(\mathbb{R}^n) \cap C^s(\overline{\Omega})$ satisfy*

$$(-\Delta)^s u - f(x,u) \geq 0 \geq (-\Delta)^s v - f(x,v), \qquad x \in \Omega,$$
$$u \gneq 0 \text{ in } \mathbb{R}^n, \text{ and } \quad u \geq v \text{ in } \mathbb{R}^n \setminus \Omega.$$

Here $f \in C^{0,1}(\overline{\Omega} \times \mathbb{R})$ satisfies that $f(x,0) = 0$ for any $x \in \overline{\Omega}$. Assume further that $tf(x,z) \geq f(x,tz)$ in $\overline{\Omega}$ for any $t > 1$, $z > 0$, and there exist $x' \in \Omega$ and $\overline{B}_r(x') \subset \Omega$ such that $tf(x,z) > f(x,tz)$ for any $x \in \overline{B}_r(x)$, and any $t > 1$, $z > 0$, where $r > 0$. Then $u \geq v$ in $\mathbb{R}^n$.

Proof. The argument is similar to the one adopted in the proof of Theorem 2.1. We shall give a sketch. Let $(u-v)(x_0) = \min_{x \in \overline{\Omega}}(u-v)(x)$. Again we need to show that $(u-v)(x_0) \geq 0$. Note that $(u-v)(x) \geq 0$ for all $x \in \mathbb{R}^n \setminus \Omega$ and $u - v \in C^s(\overline{\Omega})$. Hence, there exists $\kappa > 0$ for which $(u-v)(x) \geq -\kappa d(x, \partial\Omega)^s$ for all $x \in \Omega$. On the other hand, the assumptions show that $(-\Delta)^s u + \frac{f(x,u)}{u}u \geq 0$ in Ω and $u \gneq 0$. Therefore, Lemma 2.2 implies that there is $\beta > 0$ for which $u \geq \beta d(x, \partial\Omega)^s$ for all $x \in \Omega$. Now assume that the desired conclusion is not true, that is, $(u-v)(x_0) < 0$. Then, as the proof of Theorem 2.1, let $\bar{t} = \inf\{t \in \mathbb{R} \mid (t+1)u(x) - v(x) \geq 0 \text{ in } \overline{\Omega}\}$. Clearly, $\bar{t} > 0$. Write $w = (\bar{t}+1)u - v$. Due the assumptions made on f, there holds

$$(-\Delta)^s (\bar{t}+1)u - f(x, (\bar{t}+1)u) \gneq 0, \quad (-\Delta)^s w - \frac{f(x, (\bar{t}+1)u) - f(x,v)}{(\bar{t}+1)u - v} w \gneq 0, \quad x \in \Omega.$$

Consequently, there exists $\gamma > 0$ such that $(\bar{t}+1)u - v \geq \gamma d(x, \partial\Omega)^s$ for all $x \in \overline{\Omega}$. Hence the rest proof follows from the same reasoning given in the proof of Theorem 2.1. The proof is completed. $\square$

Theorem 4.1. *Suppose that all the assumptions of Proposition 4.1 on Ω and f are satisfied. Assume further that $f \in C^{0,1+\alpha}(\overline{\Omega} \times \mathbb{R})$ for some $\alpha \in (0,1)$, and there exists $M > 0$ for which $f(x,u) \leq 0$ for all $x \in \Omega$ whenever $u \geq M$. Let λ_s be the principle eigenvalue of $\{L_c, \mathcal{D}, \Omega\}$, where $c(x) = -f_z(x,0)$. Then the following boundary value problem*

$$
\begin{cases}
(-\Delta)^s u = f(x,u), & x \in \Omega \\
u = 0, & x \in \mathbb{R}^n \setminus \Omega
\end{cases}
\tag{4.1}
$$

has an unique nontrivial bounded positive solution if and only if $\lambda_s < 0$. Here $0 < s < 1$.

Proof. We first prove the only if part by an old trick. Let $u \in L^\infty(\Omega)$ be the positive solution, and let φ be a positive eigenfunction associated with λ_s. Now multiplying the both sides of (4.1) by φ and integrating resulting equation over Ω yield that

$$
\begin{aligned}
0 &= \int_\Omega [(-\Delta)^s u \varphi - f(x,u)\varphi] dx \\
&= \int_\Omega [u(-\Delta)^s \varphi - f(x,u)\varphi] dx \\
&= \int_\Omega \left\{ \lambda_s + \left[f_z(x,0) - \frac{f(x,u)}{u} \right] \right\} u\varphi \, dx.
\end{aligned}
$$

Notice that $u, \varphi > 0$ in Ω, and $f_z(x,0) \geq f(x,u)/u$ in Ω. In particular, the assumption implies that there exist a $x' \in \Omega$ and an open neighborhood $B_{r'}(x')$ of x' such that $f_z(x,0) > f(x,u)/u$ for all $x \in B_{r'}(x')$. Consequently, $\lambda_s < 0$. We now proceed to show that (4.1) has a bounded positive solution provided that $\lambda_s < 0$. Since $f(x,t\varphi) - tf_z(x,0)\varphi = (\int_0^1 f_z(x,\tau t\varphi)d\tau - f_z(x,0))t\varphi$, there exists $t_1 > 0$, which is sufficiently small such that $f(x,t_1\varphi) \geq t_1 f_z(x,0)\varphi + \frac{\lambda_s}{2} t_1 \varphi$ and $t_1 \varphi < M$. Thus, $(-\Delta)^s t_1 \varphi - f(x,t_1\varphi) \leq 0$ in Ω and $t_1 \varphi = 0$ in $\mathbb{R}^n \setminus \Omega$.

Now set $X = \{u \in C(\mathbb{R}^n) \mid u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$, and $Y = \{u \in C(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$. Let $J : X \to Y$ be given by $Ju := u|_{\overline{\Omega}}$. Clearly, J is a linear bijection. Given $h \in Y$, we let $u = (-\Delta)^s + \rho I)^{-1} h$ be the unique (weak) solution of

$$
\begin{cases}
(-\Delta)^s u + \rho u = h, & x \in \Omega, \\
u = 0, & x \in \mathbb{R}^n \setminus \Omega.
\end{cases}
\tag{4.2}
$$

Here $\rho > 0$ is a constant. In view of Proposition 1.1 of [21], $(-\Delta)^s + \rho I)^{-1} h \in C^s(\mathbb{R}^n)$ is also a classic solution of (4.2). Next we let $\mathcal{F} : X \to X$ be defined by $\mathcal{F}u := (-\Delta)^s + \rho I)^{-1} F(x, Ju)$, where $F(x,z) = \rho z + f(x,z)$. We now show that $\mathcal{F}$ is continuous and compact. In fact, given that $h_1, h_2 \in X$, assume without loss of generality that $\|h_1 - h_2\|_X \leq 1$, and $\|h_1\| \leq M$, it follows from Theorem 2.1 that

$$
\|\mathcal{F}h_1 - \mathcal{F}h_2\|_{L^2(\Omega)} \leq C_\rho \|F(x, Jh_1) - F(x, Jh_2)\|_{L^2(\Omega)} \leq C(\rho, f, M)\|h_1 - h_2\|_X.
$$

Note that $(-\Delta)^s(\mathcal{F}h_1 - \mathcal{F}h_2) + \rho(\mathcal{F}h_1 - \mathcal{F}h_2) = F(x, h_1) - F(x, h_2)$. It follows from Proposition 2.1 and Proposition 1.1 of [21] that

$$\|\mathcal{F}h_1 - \mathcal{F}h_2\|_{C^s(\mathbb{R}^n)} \leq C'(\rho, f, M, \Omega)\|h_1 - h_2\|_X.$$

Therefore, $\mathcal{F}$ is continuous. In case that $h_2 = 0$, for any $h \in X$ with $\|h\|_X \leq M$, It is clear that $\|\mathcal{F}h\|_{C^s(\mathbb{R}^n)} \leq C'(\rho, f, M, \Omega)M$. Hence Ascoli-Arzelà theorem implies that $\mathcal{F}$ is also compact. Next we let $X_{\varphi,M} := \{u \in X \mid t_1\varphi \leq u \leq M\}$. Apparently, $X_{\varphi,M}$ is a close and convex subspace of X. Let ρ be sufficiently large such that $\rho > L(M)$, where $L(M)$ is the Lipschitz constant for $f(x, z)$ over $z \in [0, M]$. Then $F(x, z)$ is non-decreasing in z. Thanks to the fact that

$$(-\Delta)^s t_1\varphi + \rho(t_1\varphi) \leq F(x, t_1\varphi), \ x \in \Omega, \qquad (-\Delta)^s M + \rho M \geq F(x, M), \ x \in \Omega,$$

from Theorem 2.1, we infer that $t_1\varphi \leq \mathcal{F}h \leq M$ as long as $h \in X_{\varphi,M}$. Thus, it follows from Schauder's fixed point theorem that $\mathcal{F}$ has a fixed point in $X_{\varphi,M}$. Namely, (4.1) has a solution in $X_{\varphi,M}$. We now proceed to show the uniqueness of the bounded positive solution of (4.1). Let u_1 and u_1 be two bounded positive classic solutions of (4.1), in view of Proposition 4.1, by interchanging the role of u_1 and u_2 as a super-solution and a sub-solution, we find that $u_1 = u_2$. The proof is completed. $\qquad \square$

Theorem 4.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with $\partial\Omega \in \mathcal{C}^1$. Let $f = (f_1, \cdots, f_k) \in C^{0,2+\alpha}(\overline{\Omega} \times \mathbb{R}^k, \mathbb{R}^k)$ satisfies that $f(x, 0) = 0$ for all $x \in \overline{\Omega}$. Given $z = (z_1, \cdots, z_k)$, let $c_{i,j}(x, z) = -\partial_j f_i$, where $\partial_j f_i(x, z)$ stands for the partial derivative of f_i with respect to z_j. Suppose that there exist k positive constants M_i $(1 \leq i \leq k)$ such that $c_{i,j}(x, z) \leq 0$ for any $(x, z) \in \overline{\Omega} \times \prod_{i=1}^k [0, M_i]$ and $i \neq j$. Moreover, there exist a permutation $p_1 p_2 \cdots p_k$ of $\{1, \cdots, k\}$ and subsets $\Omega_{p_i p_{i+1}}$ $(1 \leq i \leq k-1)$ and $\Omega_{p_k p_1}$ such that $c_{p_i p_{i+1}} < 0$ for any $(x, z) \in \Omega_{p_i p_{i+1}} \times \prod_{i=1}^k [0, M_i]$, and $c_{p_k p_1} < 0$ for any $(x, z) \in \Omega_{p_k p_1} \times \prod_{i=1}^k [0, M_i]$. In particular, $c_{i,j}(x, 0) \leq c_{i,j}(x, z)$ for all $(x, z) \in \overline{\Omega} \times \prod_{i=1}^k [0, M_i]$, and there exist some i, j and $x_1 \in \Omega$ such that $c_{i,j}(x, 0) < c_{i,j}(x, z)$ for any $x \in \overline{B}_{r_1}(x_1) \times \prod_{i=1}^k (0, M_i]$, where $r_1 > 0$ and $B_{r_1}(x_2) \subset \Omega$. Assume further that for each i, $f_i \leq 0$ whenever $(z_1, \cdots, z_i, \cdots, z_k) \geq (0, \cdots, M_i, \cdots 0)$. Let $0 < s_i < 1$ $(1 \leq i \leq k)$. Suppose that $2\min_{1 \leq i \leq k} s_i \geq \max_{1 \leq i \leq k} s_i$. Then the following boundary value problem*

$$\begin{cases} (-\Delta)^{s_i} u_i = f_i(x, u), & x \in \Omega \\ u_i = 0, & x \in \mathbb{R}^n \setminus \Omega \end{cases} \tag{4.3}$$

has a bounded positive solution u with $u = (u_1, \cdots, u_k)$ and $u_i \in [0, M_i]$ if and only if $\lambda_{\mathbf{s}} < 0$. Here $\lambda_{\mathbf{s}}$ is the principal eigenvalue of $\{\mathbf{L_c}, \mathcal{D}, \Omega\}$ with $\mathbf{c} = \{c_{i,j}(x, 0)\}_{1 \leq i,j \leq k}$.

Proof. We first prove the non-existence of a bounded positive solution u with $u_i \in [0, M_i]$ provided that $\lambda_{\mathbf{s}} \geq 0$. Assume to the contrary that (4.3) has a bounded positive solution u with $u_i \in [0, M_i]$ when $\lambda_{\mathbf{s}} \geq 0$. Then Proposition 1.1 of [21] implies that $u_i \in C^{s_i}(\mathbb{R}^n)$. Let $x^i \in \Omega$ be a point such that $u_i(x^i) = \max_{x \in \mathbb{R}^n} u_i(x)$, then we see that $0 < (-\Delta)^{s_i} u_i(x^i) = f_i(x^i, u(x^i))$ unless $u_i(x) = 0$ for all $x \in \mathbb{R}^n$. Thus, the assumption implies that $0 \leq u_i(x) \leq u_i(x^i) < M_i$ for each i. In addition, since $f(x, 0) = 0$, we have

$$\widetilde{c}_{i,j}(x) := -\int_0^1 \partial_j f_i(x, tu_1(x), \cdots, tu_k(x))dt, \quad (-\Delta)^{s_i} u_i + \sum_{j=1}^k \widetilde{c}_{i,j}(x)u_j = 0, \quad x \in \Omega.$$

Hence the assumption shows that $u_i(x) > 0$ in Ω for each i, and consequently, 0 is the principal eigenvalue of $\{\mathbf{L}_{\widetilde{\mathbf{c}}}, \mathbf{D}, \Omega\}$. Here $\widetilde{\mathbf{c}} = \{\widetilde{c}_{i,j}\}_{1 \leq i,j \leq k}$. Due to the assumption, $\widetilde{c}_{i,j}(x) \geq c_{i,j}(x, 0)$ for $1 \leq i, j \leq k$. Moreover, it is easy to see that

$$\widetilde{c}_{i,j}(x) - c_{i,j}(x, 0) \geq \int_{\frac{1}{2}}^1 [\partial_j f_i(x, 0) - \partial_j f_i(x, tu_1(x), \cdots, tu_k(x))]dt.$$

In view of the assumption, there exist some i, j such that $\widetilde{c}_{i,j}(x) \gneq c_{i,j}(x, 0)$ in Ω. It then follows from Corollary 3.3 that $\lambda_{\mathbf{s}} < 0$, which is a contradiction. We now proceed to show the existence of a bounded positive solution to (4.3) given that $\lambda_{\mathbf{s}} < 0$. The argument is almost the same as the one for Theorem 4.1. Only a sketch is needed. Let $\varphi = (\varphi_1, \cdots, \varphi_k)$ be a positive eigenfunction for $\lambda_{\mathbf{s}}$ with $|\varphi|_{L^\infty} = 1$. Note that $(-\Delta)^{s_i} \varphi_i + c_{ii}(x, 0)\varphi_i \geq 0$ in Ω, it follows from Lemma 2.2 that $\varphi_i \geq \beta_i d^{s_i}(x, \partial\Omega)$ for some $\beta_i > 0$, where $d^{s_i}(x) = d^{s_i}(x, \partial\Omega)$. Meanwhile, since $\varphi_i \in C^{s_i}(\mathbb{R}^n)$ and $\varphi_i = 0$ in Ω^c. there exists $C_i > 0$ for which $\varphi_i \leq C_i d^{s_i}(x)$. Given $t \in (0, 1)$, for each $x \in \Omega$, applying Taylor's formula gives

$$\left| f_i(x, t\varphi) - \sum_{j=1}^k \partial_j f_i(x, 0)t\varphi_j \right| = \left| \sum_{\tau, j=1}^k t^2 \varphi_j \varphi_\tau \int_0^1 [(1-\theta)\partial_j \partial_\tau f_i(x, \theta t\varphi)]d\theta \right|$$

$$\leq \sum_{\tau, j=1}^k \left[\eta_i t^2 \varphi_j \varphi_\tau \right],$$

where $\eta_i = \max_{1 \leq \tau, j \leq k} \{\sup_{(x,z) \in \overline{\Omega} \times [0,1]^k} |\partial_\tau \partial_j f_i|\}$. In view of the assumption, we may assume without loss of generality that $0 < s_1 \leq s_2 \cdots \leq s_i \leq \cdots \leq s_k$, so $2s_1 \geq s_k$, and we have

$$0 < \eta_i \frac{\sum_{\tau, j=1}^k t^2 \varphi_\tau \varphi_j}{t\varphi_i} \leq t\eta_i \frac{\sum_{\tau, j=1}^k C_\tau C_j d^{s_\tau + s_j}(x)}{\beta_i d^{s_i}(x)}.$$

Since for any $i, j, \tau \in \{1, \cdots, k\}$, $s_\tau + s_j \geq 2s_1 \geq s_k \geq s_i$, it follows that

$$\lim_{t \to 0} t\eta_i \frac{\sum_{\tau, j=1}^k C_\tau C_j d^{s_\tau + s_j}(x)}{\beta_i d^{s_i}(x)} = 0 \quad \text{uniformly in } \Omega.$$

Thus, there exists $t_1 > 0$ sufficiently small such that $f_i(x, t_1\varphi) \geq t_1 \sum_{j=1}^{k} \partial_j f_i(x, 0)\varphi_j + t_1 \frac{\lambda_s}{2}\varphi_i$ in Ω, and $t_1\varphi_i < M_i$ for each i. Namely, $(-\Delta)^{s_i} t_1\varphi_i - f_i(x, t_1\varphi) \leq 0$ in Ω for each i. On the other hand, we have $(-\Delta)^{s_i} M_i - f_i(x, M_1, \cdots, M_k) \geq 0$ in Ω for each i. Now let $\mathbf{F} : \prod_{i=1}^{k} X_i \to \prod_{i=1}^{k} X_i$ be defined by

$$\mathbf{F}u := ((-\Delta)^{s_1} + \rho I)^{-1} F_1(x, Ju), \cdots, (-\Delta)^{s_i} + \rho I)^{-1} F_i(x, Ju),$$
$$\cdots, (-\Delta)^{s_k} + \rho I)^{-1} F_k(x, Ju)),$$

where $X_i = X$, which is specified in the proof of Theorem 4.1, and J is the linear bijection given in the proof of Theorem 4.1. In addition, $F_i(x, z) = \rho z_i + f_i(x, z)$, and

$$\rho \geq \max_{1 \leq i \leq k} \sup_{(x,z) \in \overline{\Omega} \times \prod_{i=1}^{k}[0, M_i]} |\partial_i f_i|.$$

Clearly, $F_i(x, z)$ is nondecreasing in z. The same argument as that given in the proof of Theorem 4.1 yields that $\mathbf{F}$ is continuous and compact. Moreover, $t_1\varphi \leq \mathbf{F}h \leq M$ as long as $h \in \Xi_{\varphi, M}$, where $\Xi_{\varphi, M} = \{u \in \prod_{i=1}^{k} X_i \mid t_1\varphi_1 \leq u_i \leq M_i, \ 1 \leq i \leq k\}$, and $M = (M_1, \cdots, M_k)$. Thus, Schauder's fixed point theorem shows that (4.3) has a solution u with $0 \leq u_i < M_i$. The proof is completed. $\square$

The assumption of Theorem 4.2 on s_i seems to be a bit restrictive, and could be relaxed via degree theory approach and should be dealt with in the near future. Nonetheless, it always holds if $s_i \in [\frac{1}{2}, 1)$ for all i. Theorem 4.2 may be used to study the existence and non-existence of non-negative equilibria of a nonlocal Ross-Macdonald model as follows:

$$\begin{cases} w_t + (-\Delta)^{s_1} w = a(x)b(x)m(x)(1 - w)v - r_1(x)w, & (t, x) \in \mathbb{R}^+ \times \Omega, \\ v_t + (-\Delta)^{s_2} v = a(x)c(x)(1 - v)w - r_2(x)v, & (t, x) \in \mathbb{R}^+ \times \Omega, \\ w = v = 0, & (t, x) \in \mathbb{R}^+ \times \Omega^c. \end{cases}$$

Here $a \gneq 0, b \gneq 0, c \gneq 0; m \gneq 0, r_1 \gneq 0,$ and $r_2 \gneq 0$ in Ω. The biological meanings of these parameters can be found in [25], w and v are the density of infected vectors and the density of infectious mosquitoes at (t, x), respectively. Suppose that $0 < s_i < 1$ $(i = 1, 2)$, and either $2s_1 \geq s_2, s_1 \leq s_2$ or $2s_2 \geq s_1, s_2 \leq s_1$. Then all of the assumptions of Theorem 4.2 on f and s_i are satisfied. Let $\lambda_\mathbf{s}$ be the principal eigenvalue of $\{\mathbf{L_c}, \mathbf{D}, \Omega\}$, where $\mathbf{L_c}$ is given by

$$\mathbf{L_c}(w, v) = ((-\Delta)^{s_1} w + r_1 w - abmv, (-\Delta)^{s_2} v + r_2 v - acw).$$

Then Theorem 4.2 suggests that $\lambda_\mathbf{s}$ plays a role analogous to that of basic reproduction number, a crucial threshold in the Ross-Macdonald theory of mosquito-borne pathogen transmission. Thanks to Corollary 3.4, $\lambda_\mathbf{s} > 0$ if $|\Omega|$ is sufficiently small. Hence, a plausible biological implication of Theorem 4.2 is that spatial environments of small size can not sustain disease spreading since 0 is the only non-negative equilibrium in $[0, 1] \times [0, 1]$, which is linearly stable.

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Guangyu Zhao   Department of Mathematics and Statistics, Auburn University, Auburn, AL 36849

E-mail: gzz0021@auburn.edu