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Pointwise (H, Φ) Strong Approximation by Fourier Series of L^{Ψ} Integrable Functions

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Abstract. We essentially extend and improve the classical result of G. H. Hardy and J. E. Littlewood on strong summability of Fourier series. We will present an estimation of the generalized strong mean (H, Φ) as an approximation version of the Totik type generalization of the result of G. H. Hardy, J. E. Littlewood, in case of integrable functions from L^{Ψ}. As a measure of such approximation we will use the function constructed by function Ψ complementary to Φ on the base of definition of the L^{Ψ} points. Some corollary and remarks will also be given.

1 Introduction

Let L^p $(1 \leq p < \infty)$ be the class of all 2π–periodic real–valued functions integrable in the Lebesgue sense with p–th power over $Q = [-\pi, \pi]$.

A mapping $\Phi : \mathbb{R} \to \mathbb{R}^+$ is termed an $N - function$ if (i) Φ is even and convex, (ii) $\Phi(u) = 0$ iff $u = 0$, (iii) $\lim\limits_{u\to 0} \frac{\Phi(u)}{u} = 0,\ \lim\limits_{u\to\infty} \frac{\Phi(u)}{u} = \infty$. The left derivative $p(t) = \Phi'(t)$ exists and is left continuous, nondecreasing on $(0, \infty)$, satisfies $0 < p(t) < \infty$ for $0 < t < \infty$, $p(0) = 0$ and $\lim_{t\to\infty} p(t) = \infty$. The left inverse q of p is, by definition, $q(s) = \inf\{t > 0 : p(t) > s\} = p^{-1}(s)$ for $s > 0$. Then Φ and Ψ given by $\Phi(u) = \int_0^{|u|} p(t)\, dt$ and $\Psi(v) = \int_0^{|v|} q(s)\, ds$ are called a pair of complementary $N - functions$ which satisfy the Young inequality: $|uv| \leq \Phi(u) + \Psi(v)$. The $N - function$ Ψ complementary to Φ can equally be defined by: $\Psi(v) = \sup\{u|v| - \Phi(u) : u \geq 0\}$, $v \in \mathbb{R}$. An example of complementary pair of $N - functions$ is following one: $\Phi(u) = e^{|u|} - |u| - 1$, $\Psi(v) = (1 + |v|)\log(1 + |v|) - |v|$. Using function Ψ we can define the Orlicz space $L^{\Psi} = \left\{ f : \int_Q \Psi(|f(x)|)\, dx < \infty \right\}$ (see [7]).

Consider the trigonometric Fourier series

$$Sf(x) = \frac{a_o(f)}{2} + \sum_{k=1}^{\infty} (a_k(f)\cos kx + b_k(f)\sin kx) = \sum_{k=0}^{\infty} c_k(f)\exp ikx$$

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and denote by $S_\nu f$ the partial sums of Sf. Then,

$$H_n^{\Phi} f(x) := \Phi^{-1}\left\{\frac{1}{n+1}\sum_{\nu=0}^{n}\Phi\left(\left|S_\nu f(x)-f(x)\right|\right)\right\}$$

($H_n^{\Phi} f = H_n^q f$ when $\Phi(u) = u^q$).

As a measure of the above deviation we will use the pointwise characteristic $(G_{p,\Psi} - points)$

$$G_x f(\delta)_{p,\Psi} := \Psi^{-1}\left\{\sum_{k=1}^{[\pi/\delta]}\Psi\left[\left(\frac{1}{k\delta}\int_{(k-1)\delta}^{k\delta}\left|\varphi_x(t)\right|^p dt\right)^{1/p}\right]\right\}, \quad p \geq 1,$$

where $\varphi_x(t) := f(x+t) + f(x-t) - 2f(x)$, constructed by function Ψ complementary to Φ on the base of the following definition of the Gabisonia points $(G_{p,s} - points)$ [1]

$$\lim_{n\to\infty} G_x f\left(\frac{\pi}{n+1}\right)_{p,s} = 0,$$

where

$$G_x f(\delta)_{p,s} := \left\{\sum_{k=1}^{[\pi/\delta]}\left(\frac{1}{k\delta}\int_{(k-1)\delta}^{k\delta}\left|\varphi_x(t)\right|^p dt\right)^{s/p}\right\}^{1/s}, \quad s > p \geq 1,$$

and the characteristic $(L^{\Psi} - points)$

$$w_x f(\delta)_{\Psi} := \Psi^{-1}\left\{\frac{1}{\delta}\int_0^{\delta}\Psi\left(\left|\varphi_x(t)\right|\right)dt\right\}$$

constructed on the base of definition of the Lebesgue points $(L^p - points)$ defined as usually by the formula

$$\lim_{n\to\infty} w_x f\left(\frac{\pi}{n+1}\right)_p = 0,$$

where

$$w_x f(\delta)_p := \left\{\frac{1}{\delta}\int_0^{\delta}\left|\varphi_x(t)\right|^p dt\right\}^{1/p}.$$

It is well-known that $H_n^q f(x) -$ means tend to 0 (as $n \to \infty$) at the $L^p - points$ x of $f \in L^p$ $(1 < p < \infty)$ and at the $G_{1,s} - points$ x of $f \in L^1$ $(s > 1)$. These facts were proved as a generalization of the Fejér classical result on the convergence of the $(C, 1)$ -means of Fourier series by G. H. Hardy and J. E. Littlewood in [4, 5] and by O. D. Gabisoniya in [1], respectively. In case L^1 and convergence almost everywhere the first results on this area were due to J. Marcinkiewicz [11] and A. Zygmund [16, 17]. It is also clear, as it shown L. Gogoladze [3] and W. A. Rodin [13], that $H_n^{\Phi} f(x) -$means, with $\Phi(u) = \exp u - 1$ also tend to 0 almost everywhere (as $n \to \infty$). The estimates of $H_n^{\Phi} f(x) -$means, for $f \in L^p$ $(1 < p < \infty)$ was obtained in [9] but in the case

$f \in L^1$ the estimates of $H_n^{\Phi} f(x) -$ means with $\Phi(u) = u^q$ was obtained in [8, 12]. Finally, the estimation of the $H_n^{\Phi} f(x) -$ mean for $f \in L^1$ by $G_x f\left(\frac{\pi}{n+1}\right)_{1,\Psi}$ as approximation version of the Totik type (see [14, 15]) generalization of the mentioned results of J. Marcinkiewicz and A. Zygmund was obtained in [10] as follows:

Theorem *Let $\Phi, \Psi \in \digamma$ are complementary pair of $N - functions$, such that $\Psi(x)/x$ be non-decreasing and $\Psi(x)/x^2$ non-increasing, and let p be convex, its left inverse q be non-negative, continuous, and strictly increasing such that $\frac{q(s)}{s}$ is non-increasing. If $f \in L^1$, then*

$$H_n^{\Phi} f(x) \ll G_x f\left(\frac{\pi}{n+1}\right)_{1,\Psi},$$

for $n = 0, 1, 2, .,$ where

$$\digamma = \{\Phi : \Phi \text{ is } N - function \text{ with convex or concave left derivative } p\}.$$

In this paper we will consider the function $f \in L^{\Psi}$ and the quantity $H_n^{\Phi} f(x)$, but as a measure of approximation of this quantity we will use the function constructed by function Ψ complementary to Φ on the base of definition of the L^{Ψ} points. We note that O. D. Gabisoniya in [2] shows that for $f \in L^p$ $(1 < p < \infty)$ the relation

$$G_x f(\delta)_{1,p} = o_x(1)$$

holds at every $L^p - points$ x of f. Here we will show that for $f \in L^{\Psi}$ the relation

$$G_x f\left(\frac{\pi}{n+1}\right)_{1,\Psi} = o_x(1)$$

and thus the relation

$$H_n^{\Phi} f(x) = o_x(1)$$

hold at every $L^{\Psi} - points$ x of f. More precisely, we will prove the estimate of the quantity $H_n^{\Phi} f(x)$ by the characteristic constructed with L^{Ψ} pointwise modulus of continuity. Such estimate is a significant improvement and extension of the results of G. H. Hardy and J. E. Littlewood from [4, 5]. Considered here function Φ can be an exponential function but the space L^{Ψ} can be in between L^1 and L^p with $p > 1$. We also give some corollary with some example of such functions Φ and Ψ. This a very sharpened form of the conjecture of G. H. Hardy and J. E. Littlewood from [6] was proved by Wang, Fu Traing in [18]. Additionally, a remark on the mentioned results of G. H. Hardy and J. E. Littlewood from [4, 5] will be formulated. Finally, we formulate a remark on the conjugate Fourier series.

We shall write $I_1 \ll I_2$ if there exists a positive constant K, sometimes depended on some parameters, such that $I_1 \leq K I_2$.

2 Statement of the results

Our main theorem has the following form:

Theorem 2.1. *Let* $\Phi, \Psi \in \digamma$ *are complementary pair of* $N - functions$*, such that* $\Psi(u)$ *and* u^2 *are equivalent for small* $u \geq 0$*,* $\Psi(x)/x$ *be non-decreasing and* $\Psi(x)/x^2$ *non-increasing, and let* p *be convex, its left inverse* q *be continuous, and strictly increasing such that* $\frac{q(s)}{s}$ *is non-increasing and series* $\sum_{k=0}^{\infty} \frac{1}{(k+1)^{1/2}} q\left(\frac{1}{k+1}\right)$ *is convergent. If* $f \in L^{\Psi}$*, then*

$$\begin{aligned} H_n^{\Phi} f(x) \quad &\ll \quad \Psi^{-1}\left[\sum_{k=0}^{n} \frac{\Psi\prime\left(\frac{1}{k+1}\right)}{(k+1)^{1/2}} \Psi\left(w_x f\left(\frac{\pi(k+1)}{n+1}\right)_{\Psi}\right)\right. \\ &\left. + (n+1)\Psi\left(\frac{1}{n+1}\right)\Psi\left(w_x f(\pi)_{\Psi}\right)\right], \end{aligned}$$

for $n = 0, 1, 2, ...$

From this result we can derive the following corollary.

Corollary 2.2. *Let* $\Phi(t) = e^{|t|} - |t| - 1$ *and* $\Psi(t) = (1 + |t|)\log(1 + |t|) - |t|$*. If* $f \in L^{\Psi}$*, then*

$$\begin{aligned} &\sum_{k=0}^{n} \frac{\Psi\prime\left(\frac{1}{k+1}\right)}{(k+1)^{1/2}} \Psi\left(w_x f\left(\frac{\pi(k+1)}{n+1}\right)_{\Psi}\right) + (n+1)\Psi\left(\frac{1}{n+1}\right)\Psi\left(w_x f(\pi)_{\Psi}\right) \\ = \quad & o_x(1) \ \text{a.e. (at } L^{\Psi} - points\ x\text{)} \end{aligned}$$

and thus also

$$H_n^{\Phi} f(x) = o_x(1) \ \text{a.e. (at } L^{\Psi} - points\ x\text{).}$$

Finally we have also two remarks.

Remark 1. Let $\Phi(t) = t^{\alpha}$ and $\Psi(t) = t^{\alpha/(\alpha-1)}$ $(\alpha \geq 2)$. For such functions the assumptions of Theorem 2.1 are fulfilled. If $f \in L^{\Psi}$, then relations of the before corollary hold evidently. Thus we have the mentioned results of G. H. Hardy and J. E. Littlewood.

Remark 2. We can observe that in the light of the O. D. Gabisoniya [2] and I. Ya. Novikov, W. A. Rodin [12] results our pointwise results remain true for the conjugate Fourier series too.

3 Auxiliary result

Here we present the following lemma:

Lemma 3.1. *If a function Ψ satisfies the conditions $\Psi(u) \gg u^2$ for small $u \geq 0$ and $\Psi(u) \ll u^2$ for all $u \geq 0$, then*

$$\Psi\left(\frac{u}{n+1}\right) \ll \Psi\left(\frac{1}{n+1}\right)\Psi(u)$$

for small $u \geq 0$ and $n = 0, 1, 2, \ldots$

Proof. Our inequality follows at once from the following inequalities

$$\Psi\left(\frac{u}{n+1}\right) \ll \left(\frac{u}{n+1}\right)^2 \ll \left(\frac{1}{n+1}\right)^2 \Psi(u), \text{ for } u \ll 1.$$

□

4 Proofs of the results

4.1 Proof of Theorem 2.1

In view of Theorem we have to prove that

$$\begin{aligned}\Psi\left[G_x f\left(\frac{\pi}{n+1}\right)_{1,\Psi}\right] \ll & \sum_{k=0}^{n-1}\frac{1}{(k+1)^2}\Psi\left(w_x f\left(\frac{\pi(k+1)}{n+1}\right)_{\Psi}\right) \\ & +(n+1)\Psi\left(\frac{1}{n+1}\right)\Psi\left(w_x f(\pi)_{\Psi}\right).\end{aligned}$$

Let $\Delta_{\nu}^{n} = \left(\frac{\pi\nu}{n+1}, \frac{\pi(\nu+1)}{n+1}\right)$ for $\nu = 0, 1, 2, \ldots, n$ and

$$\sup_{\substack{n.k \\ 0\leq k\leq n}}\left[\frac{n+1}{\pi(k+1)}\int_0^{\frac{\pi(k+1)}{n+1}}\Psi\left(\left|\varphi_x(t)\right|\right)dt\right] = M(x) < \infty \text{ (at } L^{\Psi} - points\ x).$$

Then, for any $\epsilon > 0$ there exists a natural number $n_\epsilon = n_\epsilon(x)$ such that $M(x) < \epsilon\sqrt{n_\epsilon}$, whence by the assumptions on Ψ and Ψ', convexity of Ψ and the Abel transformation

$$\begin{aligned}& \sum_{k=0}^{n}\Psi\left(\frac{1}{k+1}\right)\Psi\left(\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\right) \\ \leq & \sum_{k=0}^{n}\Psi\left(\frac{1}{k+1}\right)\left(\frac{n+1}{\pi}\int_{\Delta_k^n}\Psi\left(\left|\varphi_x(t)\right|\right)dt\right)\end{aligned}$$

$$= \frac{n+1}{\pi}\sum_{k=0}^{n-1}\left[\Psi\left(\frac{1}{k+1}\right) - \Psi\left(\frac{1}{k+2}\right)\right]\sum_{\nu=0}^{k}\int_{\Delta_\nu^n}\Psi\left(\left|\varphi_x(t)\right|\right)dt$$

$$+\frac{n+1}{\pi}\Psi\left(\frac{1}{n+1}\right)\sum_{\nu=0}^{n}\int_{\Delta_\nu^n}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt$$

$$\begin{aligned}
&\le \quad \frac{n+1}{\pi}\sum_{k=0}^{n-1}\Psi\prime\left(\frac{1}{k+1}\right)\left(\frac{1}{k+1}\right)^2\int_0^{\frac{\pi(k+1)}{n+1}}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt \\
&\quad +\left(n+1\right)\Psi\left(\frac{1}{n+1}\right)\frac{1}{\pi}\int_0^{\pi}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt
\end{aligned}$$

$$\begin{aligned}
&= \quad \sum_{k=1}^{n-1}\frac{1}{k+1}\Psi\prime\left(\frac{1}{k+1}\right)\left[\frac{n+1}{\pi\left(k+1\right)}\int_0^{\frac{\pi(k+1)}{n+1}}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt\right] \\
&\quad +\left(n+1\right)\Psi\left(\frac{1}{n+1}\right)\frac{1}{\pi}\int_0^{\pi}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt,
\end{aligned}$$

and by the consideration similar to that of O. D. Gabisoniya in [2, p. 925]

$$\begin{aligned}
&\sum_{k=0}^{n}\frac{1}{\left(k+1\right)^{1/2}}\Psi\prime\left(\frac{1}{k+1}\right)\left[\frac{n+1}{\pi\left(k+1\right)^{3/2}}\int_0^{\frac{\pi(k+1)}{n+1}}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt\right] \\
\ll\ &\max_{0\le k\le n}\left[\frac{n+1}{\pi\left(k+1\right)^{3/2}}\int_0^{\frac{\pi(k+1)}{n+1}}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt\right] \\
\ll\ &\left(\max_{0\le k\le n_\epsilon}+\max_{n_\epsilon\le k\le n}\right)\left[\frac{n+1}{\pi\left(k+1\right)^{3/2}}\int_0^{\frac{\pi(k+1)}{n+1}}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt\right] \\
\ll\ &\left[\frac{n+1}{\pi}\int_0^{\frac{\pi(n_\epsilon+1)}{n+1}}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt\right]+\frac{M\left(x\right)}{\sqrt{n_\epsilon}} \\
=\ &o_x\left(1\right)+\epsilon\ (\text{at } L^\Psi - points\ x).
\end{aligned}$$

Since $\lim_{u\to 0}\frac{\Psi(u)}{u}=0$, we have

$$\left(n+1\right)\Psi\left(\frac{1}{n+1}\right)\frac{1}{\pi}\int_0^{\pi}\Psi\left(\left|\varphi_x\left(t\right)\right|\right)dt=o_x\left(1\right),$$

and therefore

$$\sum_{k=0}^{n}\Psi\left(\frac{1}{k+1}\right)\Psi\left(\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x\left(t\right)\right|dt\right)=o_x\left(1\right).$$

This relation with the evident estimate

$$\sum_{k=0}^{\infty}\Psi\left(\frac{1}{k+1}\right)\ge\Psi\left(1\right)>0$$

yields

$$\Psi\left(\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\right)=o_x(1)\,,\text{ for }k=1,2,3,...,n,$$

and

$$\Psi\left(\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\right)\ll 1,\text{ for }k=1,2,3,...,n,$$

as well

$$\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\ll 1,\text{ for }k=1,2,3,...,n.$$

Contrary, if we assume

$$\Psi\left(\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\right)\gg 1,\text{ for }k=1,2,3,...,n,$$

then

$$\sum_{k=0}^{n}\Psi\left(\frac{1}{k+1}\right)\Psi\left(\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\right)\gg\sum_{k=0}^{n}\Psi\left(\frac{1}{k+1}\right)>0$$

but it is impossible, whence the conjecture $\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\ll 1$ is true.

Hence, by the assumption, Lemma 3.1 gives

$$\Psi\left(\frac{n+1}{\pi(k+1)}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\right)\ll\Psi\left(\frac{1}{k+1}\right)\Psi\left(\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\right)$$

and consequently

$$\begin{aligned}\Psi\left[G_xf\left(\frac{\pi}{n+1}\right)_{1,\Psi}\right]&=\sum_{k=0}^{n}\Psi\left(\frac{n+1}{\pi(k+1)}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\right)\\&\ll\sum_{k=0}^{n}\Psi\left(\frac{1}{k+1}\right)\Psi\left(\frac{n+1}{\pi}\int_{\Delta_k^n}\left|\varphi_x(t)\right|dt\right).\end{aligned}$$

Applying the above calculation we obtain

$$\begin{aligned}&\Psi\left[G_xf\left(\frac{\pi}{n+1}\right)_{1,\Psi}\right]\\\ll&\sum_{k=0}^{n-1}\frac{1}{(k+1)^{1/2}}\Psi\prime\left(\frac{1}{k+1}\right)\left[\frac{n+1}{\pi(k+1)}\int_0^{\frac{\pi(k+1)}{n+1}}\Psi\left(\left|\varphi_x(t)\right|\right)dt\right]\\&+(n+1)\,\Psi\left(\frac{1}{n+1}\right)\frac{1}{\pi}\int_0^{\pi}\Psi\left(\left|\varphi_x(t)\right|\right)dt.\end{aligned}$$

Finally, by the definition of the $w_xf\left(\cdot\right)_{\Psi}$ the desired at the begin estimate follows. □

4.2 Proof of Corollary 2.2

At the begin, we note that if $\Phi(t) = e^{|t|} - |t| - 1$, $\Psi(t) = (1 + |t|)\log(1 + |t|) - |t|$, then $\frac{\Psi(t)}{t}$ and $q\,(t) = \Psi'(t) = \log(1 + |t|)$ increase, series $\sum_{k=0}^{\infty} \frac{1}{(k+1)^{1/2}} q\left(\frac{1}{k+1}\right)$ is convergence, $\frac{\Psi(t)}{t^2}$ and $\frac{q(t)}{t}$ decrease and $\Psi\,(t) \leq t^2$ for all $t \geq 0$ as well $\Psi\,(t) \gg t^2$ for small $t \geq 0$. Therefore, by Theorem 2.1 and its proof, the results follow immediately. □

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