INTEGRABILITY OF DISTRIBUTIONS IN CR-LIGHTLIKE SUBMANIFOLDS

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Abstract. In this paper, we study CR-lighlike submanifolds of an indefinite Kaehler manifold. Integrability of distributions on CR-lightlike submanifold investigated. We give some necessary and sufficient conditions on integrability of distibutions on CR-lightlike submanifolds in an indefinite Kaehler manifolds.

1. Introduction

The study of geometry of CR-submanifolds in a Kaehler manifold was initiated by Bejancu [1] and has been developed by [5], [6] and others. These submanifolds includes the other two classes of submanifolds as special cases.

The study of CR-submanifolds that mentioned above has been confined to differentiable manifolds with positive definite metric. In [8], K. L. Duggal initiated the geometry of CR-submanifolds with a Lorentz metric. The study of CR-submanifolds with a semi-Riemann metric continued in [9].

The general theory of a lightlike submanifold (or null)have been developed by Kupeli [7], Bejancu-Duggal, [10], [4]. In [4], the authors constructed the principal vector bundles related to a lightlike submanifold in a semi-Riemann manifold and obtained Gauss-Weingarten formulae as well as other properties of this submanifold.

On the other hand, K. L. Duggal A. Bejancu [12] introduced the notion of a CR-lightlike submanifolds in an indefinite Kaehler manifold. Moreover, they have investigated integrability of distributions on real lightlike hypersurfaces of an indefinite Kaehler manifold [11].

The purpose of this paper is to prove and show new results on integrability of distributions on CR-lightlike submanifolds in an indefinite Kaehler manifold.

This paper is organized as follows.

Some preliminary notations and results are given in Section 2. Particular, the basic definition and theorem for lightlike submanifolds and CR-lightlike submanifolds are given.

In Section 3, we consider CR-lightlike submannifold of an indefinite Kaehler manifold. Particular, we give some theorems on integrability of distributions on CR-lightlike submanifolds in an indefinite Kaehler manifolds.

2. Preliminaries

Let $(\overline{M}, \overline{g})$ be a real $(m+n)$-dimensional semi-Riemann manifold, $m, n > 1$ and $\overline{g}$ be a semi-Riemann metric on $\overline{M}$. We denote by q the constant index of $\overline{g}$ and we suppose that $\overline{M}$ is not Riemann manifold.

Let M be a lightlike submanifold of dimension m of $\overline{M}$. In this case there exists a smooth distribution on M, named a radical distribution such that $RadTM_p = TM_p \cap TM_p^\perp$, $\forall p \in M$. If the rank of $RadTM$ is $r > 0$, M is called an r-lightlike submanifold of $\overline{M}$. Then, there are four cases: i); $0 < r < \min\{m, n\}$; ii): $1 < r = n < m$ iii). $1 < r = m < n$ iv). $1 < r = m = n$. In the first case the submanifold is called an r-lightlike submanifold, in the second a coisotropic submanifold, in thethird an isotropic submanifold and in the fourth a totally lightlike submanifold.

Let M be an r-lightlike submanifold of $\overline{M}$. We consider a complementary distribution $S(TM)$ of $Rad(TM)$ on TM which is called a screen distribution. Then, we have the direct orthogonal sum

$$TM = RadTM \perp S(TM) \tag{2.1}$$

As $S(TM)$ is nondegenerate vector subbundle of $T\overline{M}\mid_M$, we put

$$T\overline{M}\mid_M = S(TM) \perp S(TM)^\perp \tag{2.2}$$

where $S(TM)^\perp$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\overline{M}\mid_M$. Moreover $S(TM)$, $S(TM)^\perp$ are non-degenerate we have the following orthogonal direct decomposition

$$S(TM)^\perp = S(TM^\perp) \perp S(TM^\perp)^\perp \tag{2.3}$$

Theorem 2.1.([4]) *Let* $(M, g, S(TM), S(TM^\perp))$ *be an r-lightlike submanifold of a semi-Riemannian manifold* $(\overline{M}, \overline{g})$. *Then, there exists a complementary vector bundle* $ltr(TM)$ *called a lightlike transversal bundle of* $Rad(TM)$ *in* $S(TM^\perp)^\perp$ *and basis of* $\Gamma(ltr(TM)\mid_U)$ *consist of smooth sections* $\{N_1, \ldots, N_r\}$ *of* $S(TM^\perp)^\perp\mid_U$ *such that*

$$\overline{g}(N_i, \xi_j) = \delta_{ij}, \qquad \overline{g}(N_i, N_j) = 0, \qquad i, j = 0, 1, \ldots, r$$

where $\{\xi_1, \ldots, \xi_r\}$ *is a basis of* $\Gamma(RadTM)\mid_U$.

We consider the vector bundle

$$tr(TM) = ltr(TM) \perp S(TM^\perp). \tag{2.4}$$

Thus

$$T\overline{M} = TM \oplus tr(TM) = S(TM) \perp S(TM^\perp) \perp (Rad(TM) \oplus ltr(TM)) \tag{2.5}$$

Now, let $\overline{\nabla}$ be the Levi-Civita connection on $\overline{M}$, we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \forall X, Y \in \Gamma(TM)$$

and

$$\overline{\nabla}_X V = -A_V X + \nabla_X^\perp V, \qquad \forall X \in \Gamma(TM) \tag{2.6}$$

for any $V \in \Gamma(tr(TM))$. Using the projectors $L : tr(TM) \to ltr(TM)$, $S : tr(TM) \to S(TM^\perp)$. From [12], we have

$$\overline{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y) \tag{2.7}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^l N + D^s(X, N) \tag{2.8}$$

$$\overline{\nabla}_X W = -A_W X + \nabla_X^s W + D^l(X, W) \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$. Where $h^l(X, Y) = Lh(X, Y)$, $h^s(X, Y) = Sh(X, Y)$, $\nabla_X^l N$, $D^l(X, W) \in \Gamma(ltr(TM))$, $\nabla_X^s W$, $D^s(X, N) \in \Gamma(S(TM^\perp))$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$.

Denote by P the projection morphism of TM to the screen distribution, we consider the decomposition

$$\nabla_X PY = \nabla_X^* PY + h^*(X, PY) \tag{2.10}$$

$$\triangle_X \xi = -A_\xi^* X + \nabla_X^{*^t} \xi \tag{2.11}$$

for any $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$. Then we have the following equations

$$\overline{g}(h^l(X, PY), \xi) = g(A_\xi^* X, PY), \quad \overline{g}(h^*(X, PY), N) = g(A_N X, PY) \tag{2.12}$$

$$g(A_\xi^* PX, PY) = g(PX, A_\xi^* PY), \qquad A_\xi^* \xi = 0 \tag{2.13}$$

Let $(\overline{M}, \overline{J}, \overline{g})$ be a real $2m$-dimensional indefinite almost Hermitian manifold and M be a real n-dimensional submanifold of $\overline{M}$.

Definition 2.1.([12]) A submanifold M of an indefinite almost Hermitian manifold $\overline{M}$ is said to be a CR-lightlike submanifold if the following two conditions are fullfilled:

i) $\overline{J}(Rad(TM))$ is a distribution on M such that

$$Rad(TM) \cap \overline{J}Rad(TM) = \{0\}$$

ii) There exist vector bundles $S(TM), S(TM^\perp), ltr(TM), D_0$ and D' over M such that

$$S(TM) = \{\overline{J}(RadTM) \oplus D'\} \perp D_0, \quad \overline{J}D_0 = D_0, \quad \overline{J}D' = L_1 \perp L_2$$

where D_0 is a nondegenarete distribution on M and L_1, L_2 are vector bundles of $ltr(TM)$ and $S(TM^\perp)$, respectively.

From by the definition of CR-lightlike submanifold, we have

$$TM = D \oplus D'$$

where

$$D = RadTM \perp \overline{J}RadTM \perp D_0$$

We denote by S ve Q the projections on D and D', respectively. Then we have

$$\overline{J}X = fX + \omega X \qquad (2.14)$$

for any $X, Y \in \Gamma(TM)$. Where $fX = \overline{J}SX$ and $\omega X = \overline{J}QX$. On the other hand, we set

$$\overline{J}V = BV + CV \qquad (2.15)$$

for any $V \in \Gamma(tr(TM))$. Where $BV \in \Gamma(TM)$ and $CV \in \Gamma(tr(TM))$.

Unless otherwise stated, M_1 and M_2 are supposed to as $\overline{J}L_1$ and $\overline{J}L_2$, respectively.

3. Integrability of Distributions

Definition 3.1. Let $\overline{M}$ be an indefinite Kaehler manifold and M be CR-lightlike submanifold of $\overline{M}$. If $h(X,Y) = 0 \; \forall X, Y \in \Gamma(D)$ then M is called D-geodesic CR-lightlike submanifold.

Theorem 3.1. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be CR-lightlike submanifold of $\overline{M}$. If M is D-geodesic then each leaf of holomorphic distribution is total geodesic.*

Proof. For $X, Y \in \Gamma(D)$ we have

$$\begin{aligned}
h(X,Y) &= \overline{\nabla}_X Y - \nabla_X Y \\
&= -\overline{\nabla}_X \overline{J}^2 Y - \nabla_X Y \\
&= -(\overline{\nabla}_X \overline{J})\overline{J}Y - \overline{J}\overline{\nabla}_X \overline{J}Y - \nabla_X Y.
\end{aligned}$$

Since $\overline{M}$ is a Kaehler, we get

$$h(X,Y) = -\overline{J}\overline{\nabla}_X \overline{J}Y - \nabla_X Y.$$

From (2.7) and (2.14), we derive

$$\begin{aligned}
h(X,Y) &= -\overline{J}\nabla_X \overline{J}Y - \overline{J}h(X,\overline{J}Y) - \nabla_X Y \\
&= -f\nabla_X \overline{J}Y - \omega\nabla_X \overline{J}Y - Bh(X,\overline{J}Y) - Ch(X,\overline{J}Y) - \nabla_X Y.
\end{aligned}$$

Thus we get

$$h(X,Y) = -\omega\nabla_X \overline{J}Y - Ch(X,\overline{J}Y). \qquad (3.1)$$

Hence we obtain

$$\omega \nabla_X \overline{J} Y = 0.$$

Thus the proof is complete.

From (3.1) we have the following corollary

Corollary 3.2. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be CR-lightlike submanifold of $\overline{M}$. Then holomorphic distribution is parallel if and only if $h(X, Y)$ has no component in $\overline{J} D'$ for any $X, Y \in \Gamma(D)$.*

Lemma 3.3. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be CR-lightlike submanifold of $\overline{M}$. Then we have*

$$f[X, Y] = \nabla_Y \overline{J} X - \nabla_X \overline{J} Y$$

for any $X, Y \in \Gamma(D)$.

Proof. For any $X, Y \in \Gamma(D)$ we have

$$h(X, \overline{J} Y) = \overline{\nabla}_X \overline{J} Y - \nabla_X \overline{J} Y \tag{3.2}$$

and

$$h(\overline{J} X, Y) = \overline{\nabla}_Y \overline{J} X - \nabla_Y \overline{J} X. \tag{3.3}$$

From (3.2) and (3.3) we derive

$$\begin{aligned}
h(X, \overline{J} Y) - h(\overline{J} X, Y) &= \overline{J} \overline{\nabla}_X Y - \overline{J} \overline{\nabla}_Y X - \nabla_X \overline{J} Y + \nabla_Y \overline{J} X \\
&= \overline{J} [X, Y] - \nabla_X \overline{J} Y + \nabla_Y \overline{J} X \\
&= f[X, Y] + \omega [X, Y] - \nabla_X \overline{J} Y + \nabla_Y \overline{J} X
\end{aligned}$$

Taking tangential parts of this equation, we have the proof of lemma.

Theorem 3.4. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold. Then D' is integrable if and only if*

$$A_{\omega W} Z - A_{\omega Z} W \in \Gamma(D')$$

for any $Z, W \in \Gamma(D')$.

Proof. Since $\overline{M}$ is an indefinite Kaehler manifold, we have

$$\overline{\nabla}_Z \overline{J} W = \overline{J} \overline{\nabla}_Z W$$

for any $Z, W \in \Gamma(D')$. Thus from (2.7), (2.10) and (2.14) we get

$$\begin{aligned}
\overline{\nabla}_Z f W + \omega W &= \overline{J} \nabla_Z W + \overline{J} h(Z, W) \\
-A_{\omega W} Z + \nabla_Z^\perp \omega W &= f \nabla_Z W + \omega \nabla_Z W + B h(Z, W) + C h(Z, W).
\end{aligned}$$

In similarly we get

$$-A_{\omega Z}W + \nabla^{\perp}_W \omega Z = f\nabla_W Z + \omega \nabla_W Z + Bh(Z,W) + Ch(Z,W)$$

Thus we have

$$-A_{\omega W}Z + A_{\omega Z}W + \nabla^{\perp}_Z \omega W - \nabla^{\perp}_W \omega Z = f[Z,W] + \omega[Z,W],$$

which is proves theorem.

Theorem 3.5. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold. Then D_0 is integrable if and only if*

i) $\overline{g}(h^*(Z_0, Z_0'), N) = \overline{g}(h^*(Z_0', Z_0), N)$
ii) $\overline{g}(h^*(Z_0, \overline{J}Z_0'), N) = \overline{g}(h^*(Z_0', \overline{J}Z_0), N)$
iii) $h^s(Z_0, \overline{J}Z_0') = h^s(\overline{J}Z_0, Z_0')$
iv) $g(\nabla^*_{Z_0} Z_0', \overline{J}\xi) = g(\nabla^*_{Z_0'} Z_0, \overline{J}\xi)$. *for any* $Z_0, Z_0' \in \Gamma(D_0)$, $N \in \Gamma(ltr(TM))$ *and* $\xi \in \Gamma(Rad(TM))$.

Proof. We note that to show D_0 is integrable it is sufficies to show that $[Z_0, Z_0'] \in \Gamma(D_0)$ for any $Z_0, Z_0' \in \Gamma(D_0)$. By the definition CR-lightlike submanifold D_0 is integrable if and only if

$$\overline{g}([Z_0, Z_0'], N) = \overline{g}([Z_0, Z_0'], \overline{J}N) = \overline{g}([Z_0, Z_0'], \overline{J}W) = \overline{g}([Z_0, Z_0'], \overline{J}\xi) = 0$$

for any $Z_0, Z_0' \in \Gamma(D_0)$, $N \in \Gamma(ltr(TM))$ and $\xi \in \Gamma(Rad(TM))$. First, From (2.7) and (2.10) we have

$$\overline{g}([Z_0, Z_0'], N) = \overline{g}(h^*(Z_0, Z_0'), N) - \overline{g}(h^*(Z_0', Z_0), N). \tag{3.4}$$

On the other hand we get

$$\overline{g}([Z_0, Z_0'], \overline{J}N) = \overline{g}(\overline{\nabla}_{Z_0} Z_0', \overline{J}N) - \overline{g}(\overline{\nabla}_{Z_0'} Z_0, \overline{J}N).$$

Using the indefinite Kaehler character of $\overline{M}$, we obtain

$$\overline{g}([Z_0, Z_0'], \overline{J}N) = -\overline{g}(\overline{\nabla}_{Z_0} \overline{J}Z_0', N) + \overline{g}(\overline{\nabla}_{Z_0'} \overline{J}Z_0, N)$$

From (2.10) we have

$$\overline{g}([Z_0, Z_0'], \overline{J}N) = -\overline{g}(h^*(Z_0, \overline{J}Z_0'), N) + \overline{g}(h^*(Z_0', \overline{J}Z_0), N). \tag{3.5}$$

From (2.7) and using the indefinite Kaehler character of $\overline{M}$, we derive

$$\overline{g}([Z_0, Z_0'], \overline{J}W) = \overline{g}(\overline{\nabla}_{Z_0} Z_0', \overline{J}W) - \overline{g}(\overline{\nabla}_{Z_0'} Z_0, \overline{J}W)$$
$$= -\overline{g}(\overline{\nabla}_{Z_0} \overline{J}Z_0', W) + \overline{g}(\overline{\nabla}_{Z_0'} \overline{J}Z_0, W)$$

and again (2.7) we get

$$\overline{g}([Z_0, Z_0'], \overline{J}W) = -\overline{g}(h^s(Z_0, \overline{J}Z_0'), W) + \overline{g}(h^s(\overline{J}Z_0, Z_0'), W). \tag{3.6}$$

Finally From equation of (2.10) we obatin

$$\overline{g}([Z_0, Z_0'], \overline{J}\xi) = \overline{g}(\nabla^*_{Z_0} Z_0', \overline{J}\xi) - \overline{g}(\nabla^*_{Z_0'} Z_0, \overline{J}\xi). \tag{3.7}$$

Thus (3.4),(3.5),(3.6) and (3.7) proof is complete.

Corollary 3.6. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold. Then D_0 is integrable if and only if*

i) $g(Z_0', A_N Z_0) = g(Z_0, A_N Z_0')$
ii) $g(\overline{J}Z_0', A_N Z_0) = g(\overline{J}Z_0, A_N Z_0')$
iii) $h^s(Z_0, \overline{J}Z_0') = h^s(\overline{J}Z_0, Z_0')$
iv) $\overline{g}(h^l(Z_0, \overline{J}Z_0'), \xi) = \overline{g}(h^l(Z_0, \overline{J}Z_0'), \xi)$

for any $Z_0, Z_0' \in \Gamma(D_0)$, $N \in \Gamma(ltr(TM))$ and $\xi \in \Gamma(Rad(TM))$

Theorem 3.7. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a CR-lightlike submanifold of $\overline{M}$. Then $RadTM$ is integrable if and only if*

i) $\overline{g}(h^l(\xi, \overline{J}\xi''), \xi') = \overline{g}(h^l(\xi', \overline{J}\xi''), \xi)$
ii) $\overline{g}(h^l(\xi, Z_0), \xi') = \overline{g}(h'(\xi', Z_0), \xi)$
iii) $h^s(\xi', \overline{J}\xi) = h^s(\overline{J}\xi', \xi)$
iv) $\overline{g}(h^*(\xi, \overline{J}\xi'), N) = \overline{g}(h^*(\xi', \overline{J}\xi), N)$.

for any $Z_0 \in \Gamma(D_0)\xi$, $\xi', \xi'' \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. By the definition CR-lightlike submanifold, $RadTM$ is integrable if and only if

$$\overline{g}([\xi, \xi'], \overline{J}\xi'') = \overline{g}([\xi, \xi'], Z_0) = \overline{g}([\xi, \xi'], \overline{J}W) = \overline{g}([\xi, \xi'], \overline{J}N) = 0.$$

Thus

$$\begin{aligned}
\overline{g}([\xi, \xi'], \overline{J}\xi'') &= \overline{g}(\overline{\nabla}_\xi \xi', \overline{J}\xi'') - \overline{g}(\overline{\nabla}_{\xi'} \xi, \overline{J}\xi'') \\
&= -\overline{g}(\xi', \overline{\nabla}_\xi \overline{J}\xi'') + \overline{g}(\xi, \overline{\nabla}_{\xi'} \overline{J}\xi'') \\
&= -\overline{g}(h^l(\xi, \overline{J}\xi''), \xi') + \overline{g}(h^l(\xi', \overline{J}\xi''), \xi)
\end{aligned}$$

and

$$\begin{aligned}
\overline{g}([\xi, \xi'], Z_0) &= \overline{g}(\overline{\nabla}_\xi \xi', Z_0) - \overline{g}(\overline{\nabla}_{\xi'} \xi, Z_0) \\
&= -\overline{g}(A^*_{\xi'} \xi, Z_0) + \overline{g}(A^*_\xi \xi', Z_0) \\
&= -\overline{g}(h^l(\xi, Z_0), \xi') + \overline{g}(h^l(\xi', Z_0), \xi).
\end{aligned}$$

On the other hand we get

$$\begin{aligned}
\overline{g}([\xi,\xi'],\overline{J}W) &= \overline{g}(\nabla_\xi\xi',\overline{J}W) - \overline{g}(\overline{\nabla}_{\xi'}\xi\,\overline{J}W) \\
&= -\overline{g}(W,\overline{\nabla}_\xi\overline{J}\xi') + \overline{g}(W,\overline{\nabla}_{\xi'},\overline{J}\xi) \\
&= -\overline{g}(h^s(\xi,\overline{J}\xi'),W) + \overline{g}(h^s(\xi',\overline{J}\xi),W)
\end{aligned}$$

and

$$\begin{aligned}
\overline{g}([\xi,\xi'],\overline{J}N'') &= \overline{g}(\overline{\nabla}_{\xi'}\xi,\overline{J}N) - \overline{g}(\overline{\nabla}_{\xi'}\xi),(\overline{J}N) \\
&= -\overline{g}(N,\overline{\nabla}_\xi\overline{J}\xi') + \overline{g}(W,\overline{\nabla}_{\xi'}\overline{J}\xi) \\
&= -\overline{g}(h^*(\xi,\overline{J}\xi'),N) + \overline{g}(h^*(\xi',\overline{J}\xi),N).
\end{aligned}$$

Thus the proof is complete

Corollary 3.8. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a CR-lightlike submanifold of $\overline{M}$. Then $RadTM$ is integrable if and only if*

  i) $A_\xi^*\xi' \notin \Gamma(M_1)$
 ii) $g(\nabla_{\xi'}^*\overline{J}Z_0,\overline{J}\xi) = g(\nabla_\xi^*\overline{J}Z_0,\overline{J}\xi')$
iii) $g(A_W\xi',\overline{J}\xi) = g(A_W\xi,\overline{J}\xi')$
iv) $g(A_N\xi',\overline{J}\xi) = g(A_N\xi,\overline{J}\xi')$

for any $Z_0 \in \Gamma(D_0)$ $\xi,\xi',\xi'' \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

Theorem 3.9. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a CR-lightlike submanifold of $\overline{M}$. Then each leaf of Radical distribution is totally geodesic in M if and only if*

  i) $A_\xi^*\xi' \notin \Gamma(M_1 \perp D_0)$
 ii) $\overline{g}(h^*(\xi',\overline{J}\xi'),N) = 0$
iii) $\overline{g}(h^s(\xi,\overline{J}\xi'),W) = 0$

where $M_1 = \overline{J}L_1$.

Proof. By the definition CR-lightlike submanifold, each leaf of Radical distribution is totally geodesic in M if and only if

$$\overline{g}(\nabla_X Y,\overline{J}\xi) = \overline{g}(\nabla_X Y,\overline{J}N) = \overline{g}(\nabla_X Y,\overline{J}W) = \overline{g}(\nabla_X Y,Z_0) = 0$$

$X,Y \in \Gamma(Rad(TM))$. Thus, from (2.7) and (2.11) we have

$$\begin{aligned}
\overline{g}(\nabla_X Y\overline{J}\xi) &= \overline{g}(\overline{\nabla}_X Y,\overline{J}\xi) - g(h(X,Y),\overline{J}\xi) \\
&= \overline{g}(\overline{\nabla}_X Y,\overline{J}\xi) \\
&= -\overline{g}(Y,\overline{\nabla}_X\overline{J}\xi) \\
&= \overline{g}(\overline{J}Y,\overline{\nabla}_X\xi) \\
&= -\overline{g}(\overline{J}Y,A_\xi^*X)
\end{aligned}$$

and from (2.7) and (2.10) we get

$$
\begin{aligned}
\overline{g}(\nabla_X Y, \overline{J}N) &= \overline{g}(\overline{\nabla}_X Y, \overline{J}N) - g(h(X,Y), \overline{J}N) \\
&= \overline{g}(\overline{\nabla}_X Y, \overline{J}N) \\
&= \overline{g}(\overline{J}Y, \overline{\nabla}_X N) \\
&= -\overline{g}(\overline{J}Y, A_N X) \\
&= -\overline{g}(h^*(X, \overline{J}Y), N)
\end{aligned}
$$

In similarly we derive

$$
\begin{aligned}
\overline{g}(\nabla_X Y, \overline{J}W) &= \overline{g}(\overline{\nabla}_X Y, \overline{J}W) - g(h(X,Y), \overline{J}W) \\
&= -\overline{g}(\overline{\nabla}_X \overline{J}Y, W) - g(h(X,Y), \overline{J}W) \\
&= \overline{g}(\overline{J}Y, \overline{\nabla}_X W) - g(h(X,Y), \overline{J}W) \\
&= -\overline{g}(h^s(X, \overline{J}Y), W)
\end{aligned}
$$

and

$$
\overline{g}(\nabla_X Y, Z_0) = -\overline{g}(A_Y^* X, Z_0)
$$

from these equations, we have assertion of theorem.

Theorem 3.10. *Let $\overline{M}$ be an inndefinite Kaehler manifold and M be a CR-lightlike submanifold of $\overline{M}$. Then $\overline{J}RadTM$ is integrable if and only if*

i) $g(\overline{J}\xi, A_N \overline{J}\xi') = g(\overline{J}\xi', A_N \overline{J}\xi)$
ii) $h^s(\overline{J}\xi, \xi') = h^s(\overline{J}\xi', \xi)$
iii) $g(h^l(\overline{J}\xi, \xi'), \xi'') = g(h^l(\xi, \overline{J}\xi'), \xi'')$
iv) $g(Z_0, A_\xi^* \overline{J}\xi') = g(Z_0, A_\xi^* \overline{J}\xi)$.

Proof. $\overline{J}RadTM$ is integrable if and only if

$$
\overline{g}([\overline{J}\xi', \overline{J}\xi], N) = \overline{g}([\overline{J}\xi' \overline{J}\xi], \overline{J}W) = \overline{g}([\overline{J}\xi', \overline{J}\xi], \overline{J}\xi'') = \overline{g}([\overline{J}\xi', \overline{J}\xi], Z_0) = 0
$$

for any $\xi, \xi' \in \Gamma(RadTM)$, $N \in \Gamma(ltrTM)$, $W \in \Gamma(S(TM)^\perp)$ and $Z_0 \in \Gamma(D_0)$. Thus from (2.8) we have

$$
\begin{aligned}
\overline{g}([\overline{J}\xi', \overline{J}\xi], N) &= \overline{g}(\overline{\nabla}_{\overline{J}\xi'} \overline{J}\xi, N) - \overline{g}(\overline{\nabla}_{\overline{J}\xi} \overline{J}\xi', N) \\
&= -\overline{g}(\overline{J}\xi, \overline{\nabla}_{\overline{J}\xi'} N) + \overline{g}(\overline{J}\xi', \overline{\nabla}_{\overline{J}\xi} N) \\
&= \overline{g}(\overline{J}\xi, A_N \overline{J}\xi') - \overline{g}(\overline{J}\xi', A_N \overline{J}\xi)
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{g}([\overline{J}\xi', \overline{J}\xi], \overline{J}W) &= \overline{g}(\overline{\nabla}_{\overline{J}\xi'} \overline{J}\xi, \overline{J}W) - \overline{g}(\overline{\nabla}_{\overline{J}\xi} \overline{J}\xi', \overline{J}W) \\
&= \overline{g}(\overline{\nabla}_{\overline{J}\xi'} \xi, W) - \overline{g}(\overline{\nabla}_{\overline{J}\xi} \xi', W)
\end{aligned}
$$

or from (2.7) we get

$$\overline{g}([\overline{J}\xi', \overline{J}\xi], \overline{J}W) = \overline{g}(h^s(\overline{J}\xi', \xi), W) - \overline{g}(h^s(\overline{J}\xi, \xi'), W)$$

In similarly we get

$$\begin{aligned}
\overline{g}([\overline{J}\xi', \overline{J}\xi], \overline{J}\xi'') &= \overline{g}(\overline{\nabla}_{J\xi'}\overline{J}\xi, \overline{J}\xi'') - \overline{g}(\overline{\nabla}_{\overline{J}\xi}\overline{J}\xi', \overline{J}\xi'') \\
&= \overline{g}(\overline{\nabla}_{\overline{J}\xi'}\xi, \xi'') - \overline{g}(\overline{\nabla}_{\overline{J}\xi}\xi', \xi'') \\
&= \overline{g}(h^l(\overline{J}\xi', \xi), \xi'') - \overline{g}(h^l(\overline{J}\xi, \xi'), \xi'')
\end{aligned}$$

and

$$\begin{aligned}
\overline{g}([\overline{J}\xi', \overline{J}\xi], Z_0) &= \overline{g}(\overline{\nabla}_{\overline{J}\xi'}\overline{J}\xi, Z_0) - \overline{g}(\overline{\nabla}_{\overline{J}\xi}\overline{J}\xi', Z_0) \\
&= -\overline{g}(\overline{\nabla}_{\overline{J}\xi'}\xi, \overline{J}Z_0) + \overline{g}(\overline{\nabla}_{\overline{J}\xi}\xi', \overline{J}Z_0) \\
&= \overline{g}(\overline{J}Z_0, A^*_\xi \overline{J}\xi') - \overline{g}(\overline{J}Z_0, A_{\xi'}J\xi).
\end{aligned}$$

from these equations, proof is complete.

Lemma 3.11. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a CR-lightlike submanifold of $\overline{M}$. If f is parallel then*

$$Bh(X, Y) = 0$$

for any $Y \in \Gamma(D)$.

Proof. For any $X, Y \in \Gamma(TM)$, form (2.7) and (2.6) we obtain

$$(\overline{\nabla}_X f)Y = A_{\omega Y}X + Bh(X, Y). \tag{3.8}$$

From hypothesis of lemma proof is complete.

Theorem 3.12. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a CR-lightlike submanifold of $\overline{M}$. If f is parallel then each leaf of holomorphic distribution is totally geodesic in M.*

Proof. Since f is parallel, from lemma we have $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$. Thus we derive

$$\overline{J}h(X, Y) - Ch(X, Y) = 0$$
$$\overline{J}\overline{\nabla}_X Y - \overline{J}\nabla_X Y - Ch(X, Y) = 0.$$

Since $\overline{M}$ is an indefinite Kaehler manifold we get

$$\overline{\nabla}_X \overline{J}Y - \overline{J}\nabla_X Y - Ch(X, Y) = 0$$
$$\nabla_X \overline{J}Y + h(X, \overline{J}Y) - f\nabla_X Y - \omega\nabla_X Y - Ch(X, Y) = 0,$$

Hence we obtain $\nabla_X \overline{J}Y = f\nabla_X Y$, which is proves our assertion.

From theorem we have the following corollary.

Corollary 3.13. *Let $\overline{M}$ be an indefinite Kaehler manifold and M be a CR-lightlike submanifold of $\overline{M}$. Then holomorphic distribution is parallel if and only if $Bh(X,Y) = 0$ $X,Y \in \Gamma(D)$.*

Theorem 3.14. *Let M be a CR-lightlike submanifold of an indefinite Kaehler manifold. Then anti-invariant distribution is parallel if and only if $A_{\overline{J}Y}X$ has no components in holomorphic distribution for any $X,Y \in \Gamma(D')$.*

Proof. From (3.8) we have

$$-f\nabla_X Y = A_{\omega Y}X + Bh(X,Y)$$

for any $X,Y \in \Gamma(D')$. Hence proof is complete.

Example 3.1. We consider a submanifold M codimension two in IR_2^8 given by the equations

$$x_7 = x_1 \cos\alpha - x_2 \sin\alpha - x_5 x_6 \tan\alpha$$
$$x_8 = x_1 \sin\alpha + x_2 \cos\alpha + x_5 x_6,$$

where $\alpha \in IR - \{\frac{\pi}{2} + k\pi, k \in Z\}$. Then TM is spanned by

$$U_1 = (1,0,0,0,0,0,\cos\alpha,\sin\alpha)$$
$$U_2 = (0,1,0,0,0,0,-\sin\alpha,\cos\alpha)$$
$$U_3 = (0,0,1,0,0,0,0,0)$$
$$U_4 = (0,0,0,1,0,0,0,0)$$
$$U_5 = (0,0,0,0,1,0,-x_6\tan\alpha,x_6)$$
$$U_6 = (0,0,0,0,0,1,-x_5\tan\alpha,x_5).$$

This example has been given in [12] for IR_2^6. Then its easy check that submanifold 1-lightlike submanifold of IR_2^8 such that $Rad(TM) = Sp\{U_1\}$. Moreover, using the canonical complex structure of IR_2^8, we see that $U_2 = JU_1$. Thus $JRad(TM)$ is a distribution on M. Hence M is a CR-lightlike submanifold. Then we have vector field of $S(TM^\perp)$ as follows

$$W = (\sin\alpha, \cos\alpha, 0, 0, -x_6, -x_5, 0, 1).$$

Now we take a vector field of $S(TM^\perp)^\perp$ as follows

$$V = (t, v, 0, 0, \frac{x_6\sin\alpha + x_5\cos\alpha}{x_5^2 + x_6^2}, \frac{x_5\sin\alpha - x_6\cos\alpha}{x_5^2 + X_6^2},$$
$$(1+t)\cos\alpha - v\sin\alpha, (1+t)\sin\alpha + v\cos\alpha),$$

where t, v are constants. Then it is easily check that $\overline{g}(V, U_1) = 1$. Thus, by the using

$$N = \frac{1}{\overline{g}(V, U_1)}\left\{V - \frac{\overline{g}(V, V)}{\overline{g}(V, U_1)}U_1\right\}$$

we have

$$N = \left(-\frac{1}{2}\left(\frac{1}{x_5^2 + x_6^2} + 1\right), v, 0, 0, \frac{x_6 \sin\alpha + x_5 \cos\alpha}{x_5^2 + x_6^2}, \frac{x_5 \sin\alpha - x_6 \cos\alpha}{x_5^2 + x_6^2},\right.$$
$$\left.\frac{1}{2}\cos\alpha - \frac{\cos\alpha}{2(x_5^2 + x_6^2)} - v\sin\alpha, \frac{1}{2}\sin\alpha - \frac{\sin\alpha}{2(x_5^2 + x_6^2)} + v\cos\alpha\right).$$

Hence the distribution D' is spanned by JN and JW. On the other hand we describe that $JU_3 = U_4$. Hence D_0 is spanned by $\{U_3, U_4\}$. Thus we have $D = Sp\{U_1, JU_1, U_3, U_4\}$. Now, we can investigate integrability of distributions on M. Firstly, its easily obtained that D_0 and D are integrable. The bracket of JN and JW is obtained

$$[JN, JW] = \frac{(x_6^2 - x_5^2)}{(x_5^2 + x_6^2)^2}\frac{\partial}{\partial x_2} + \frac{4x_5^2 x_6 \cos\alpha + 2x_6^2 x_5 \sin\alpha - 2x_5^3 \sin\alpha}{(x_5^2 + x_6^2)^2}\frac{\partial}{\partial x_5}$$
$$+ \frac{-4x_6^2 x_5 \cos\alpha + 2x_5^2 x_6 \sin\alpha - 2x_6^3 \sin\alpha}{(x_5^2 + x_6^2)^2}\frac{\partial}{\partial x_6}$$
$$+ \frac{(x_5^2 - x_6^2)\sin\alpha}{(x_5^2 + x_6^2)^2}\frac{\partial}{\partial x_7} + \frac{(x_6^2 - x_5^2)\cos\alpha}{(x_5^2 + x_6^2)^2}\frac{\partial}{\partial x_8}$$

On the other hand, by the definition of CR-lightlike submanifold, D' is integrable if and only if

$$\overline{g}([JN, JW], N) = \overline{g}([JN, JW], JN) = \overline{g}([JN, JW], U_4) = \overline{g}([JN, JW], U_3) = 0.$$

It is easily obtained that $\overline{g}([JN, JW], U_4) = \overline{g}([JN, JW], U_3) = 0$. On the other hand we have

$$\overline{g}([JN, JW], N) = \frac{2x_5 x_6 \cos 2\alpha + 2x_5 x_6 + (x_6^2 - x_5^2)\sin 2\alpha}{(x_5^2 + x_6^2)^2} \tag{3.9}$$

and

$$\overline{g}([JN, JW], JN) = \frac{-2x_5 x_6 \sin 2\alpha + (x_6^2 - x_5^2)\cos 2\alpha}{(x_5^2 + x_6^2)^2} \tag{3.10}$$

Thus, from (3.9) and (3.10) we see that D' isn't integrable. Otherwise $x_5 = 0$ and $x_6 = 0$. This is impossible.

Remark 3.1. In the case of the real hypersurfaces of an indefinite Kaehler manifold, integrability of the distributions $Rad(TM)$, $J(Rad(TM))$ and D' are trivial. Because, the dimensions of them are one. However, these distributions may not be integrable in any CR-lightlike submanifold. Indeed, we see that the distribution D' is not integrable in the Example 3.1. Therefore our results about the integrability of distributions in CR-lightlike submanifold more general then the results about the integrability of distributions in real lightlike hypersurfaces of an indefinite Kaehler manifold.

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