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Submanifolds of Sasakian Manifolds with Concurrent Vector Field

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Abstract. The submanifolds of Sasakian manifolds with a concurrent vector field have been studied. Applications of such submanifolds to Ricci solitons and Yamabe solitons has also been showed.

1 Introduction

Sasakian manifold $\bar{M}$ is a $(2n+1)$-dimensional almost contact metric manifold such that [1]

$$(\bar{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \tag{1.1}$$
$$\bar{\nabla}_X \xi = -\phi X, \tag{1.2}$$

where (ϕ, ξ, η, g) is the almost contact metric structure and $\bar{\nabla}$ is the Riemannian connection on $\bar{M}$. A vector field X on $\bar{M}$ is said to be conformal if

$$\mathcal{L}_X g = 2\alpha g, \tag{1.3}$$

where $\alpha \in C^\infty(\bar{M})$ and $\mathcal{L}_X$ denotes the Lie derivative along X. In particular, if $\alpha = 0$ then X is Killing. And X is said to be concurrent if

$$\bar{\nabla}_Z X = Z \tag{1.4}$$

for any $Z \in \chi(\bar{M})$.

Let M be an m-dimensional submanifold of $\bar{M}$. A Ricci soliton on M is a triplet (g, W, σ) such that [12]

$$\mathcal{L}_W g + 2S + 2\sigma g = 0, \tag{1.5}$$

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where S is the Ricci tensor on M, W is the potential vector field and $\sigma \in \mathbb{R}$. An Yamabe soliton on M is a triplet (g, W, λ) such that

$$\frac{1}{2}\mathcal{L}_W g = (r - \lambda)g, \tag{1.6}$$

where r is the scalar curvature on M and $\lambda \in \mathbb{R}$. If the dimension of M is 2 then the notions of Ricci soliton and Yamabe soliton are equivalent. However, when the dimension of M is greater than 2, they are different.

Chen and his co-author studied Euclidean submanifold whose canonical vector field are concurrent [4], concircular [11], conformal [10], torse-forming [9] and also in ([3], [5], [6]). Ricci soliton and Yamabe soliton whose canonical vector field are concurrent and conformal studied in ([2], [7], [8]).

The object of the present paper is to study of submanifolds of Sasakian manifolds with concurrent vector field. We also apply such submanifolds to Ricci solitons and Yamabe solitons.

2 Preliminaries

An odd dimensional smooth manifold $\bar{M}^{2n+1}$ is said to be an almost contact metric manifold if the following relations hold: [1]

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \tag{2.1}$$

$$g(X, \xi) = \eta(X), \quad \phi \circ \eta = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.3}$$

for all $X, Y \in \chi(\bar{M})$, where ϕ is a tensor of type $(1, 1)$, ξ is a vector field, η is an 1-form and g is a Riemannian metric on $\bar{M}$.

Let ∇ and $\nabla^{\perp}$ be the induced connections on the tangent bundle TM and the normal bundle $T^{\perp}M$ of M, respectively. Then we have

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.4}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.5}$$

where h and A_V are second fundamental form and shape operator respectively for the immersion of M into $\bar{M}$ and they are related by the following equation, see [13]

$$g(h(X, Y), V) = g(A_V X, Y) \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$. If $h = 0$, then M is said to be totally geodesic. Let $\{e_i : 1 \leq i \leq m\}$ be an orthonormal basis to the tangent space at any point of M. Then the mean curvature of M is

$$H = \frac{1}{m}\sum_{i=1}^{m} h(e_i, e_i). \tag{2.7}$$

And M is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H. \tag{2.8}$$

Again M is said to be umbilical with respect to $V \in T^{\perp}M$ if

$$g(h(X, Y), V) = \mu g(X, Y) \tag{2.9}$$

for some function μ. In particular if $g(h(X, Y), H) = \mu g(X, Y)$ holds then M is said to be pseudo-umbilical. Consider

$$\phi X = PX + FX, \tag{2.10}$$

where PX and FX are the tangential and normal components of ϕX. And M is called generalized self-similar submanifold of $\bar{M}$ if

$$FX = fH, \tag{2.11}$$

where $f \in C^{\infty}(M)$.

3 Results

We now prove the followings:

Theorem 3.1. *Let M be a submanifold of $\bar{M}$ with a concurrent vector field X such that ξ is normal to M. Then PX is conformal if and only if M is umbilical with respect to FX.*

Proof. Since X is concurrent vector field of $\bar{M}$, we have from (1.4) that

$$\begin{aligned} \phi Z &= \phi \bar{\nabla}_Z X \\ &= \bar{\nabla}_Z \phi X - (\bar{\nabla}_Z \phi)X. \end{aligned} \tag{3.1}$$

Using (1.1), (2.4), (2.5) and (2.10) in (3.1) we have

$$\begin{aligned} PZ + FZ &= \bar{\nabla}_Z(PX + FX) - g(X, Z)\xi \\ &= \nabla_Z PX + h(Z, PX) + \nabla^{\perp}_Z FX - A_{FX}Z - g(X, Z)\xi \end{aligned} \tag{3.2}$$

Comparing the tangential component of (3.2) we have

$$\nabla_Z PX \quad = \quad PZ + A_{FX}Z. \tag{3.3}$$

Now we have

$$\begin{aligned} &(\mathcal{L}_{PX}g)(Y,Z) \\ &= g(\nabla_Y PX, Z) + g(Y, \nabla_Z PX) \\ &= g(PY + A_{FX}Y, Z) + g(Y, PZ + A_{FX}Z) \\ &= g(A_{FX}Y, Z) + g(A_{FX}Z, Y). \end{aligned} \tag{3.4}$$

Using (2.6) in (3.4) we have

$$(\mathcal{L}_{PX}g)(Y,Z) = 2g(h(Y,Z), FX). \tag{3.5}$$

Suppose PX is conformal. Then from (1.3) and (3.5) we have

$$g(h(Y,Z), FX) = \alpha g(Y,Z), \tag{3.6}$$

which implies that M is umbilical with respect to FX.

Conversely, assume that M is umbilical with respect to FX. Then from (2.9) and (3.5) we have

$$(\mathcal{L}_{PX}g)(Y,Z) = 2\mu g(Y,Z), \tag{3.7}$$

which means that PX is conformal. □

Theorem 3.2. *Let M be a submanifold of $\bar{M}$ with a concurrent vector field X. Then X is a homothetic vector field.*

Proof. Since X is a concurrent vector field, so we have from (1.4) and (2.4) that

$$\nabla_Z X + h(X,Z) = Z. \tag{3.8}$$

Equating tangential and normal components of (3.8) we get

$$\nabla_Z X = Z, \;\; h(X,Z) = 0. \tag{3.9}$$

Now we have

$$(\mathcal{L}_X g)(Y,Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X). \tag{3.10}$$

Using (3.9) in (3.10) we have

$$(\mathcal{L}_X g)(Y,Z) = 2g(Y,Z), \tag{3.11}$$

which implies that X is conformal vector field of M with constant function $\alpha = 1$, i.e. X is homothetic. □

Theorem 3.3. *Let M be a submanifold of $\bar{M}$ with a concurrent vector field X. If (g, X, σ) is a Ricci soliton on M then M is Einstein and such a soliton is shrinking.*

Proof. Since (g, X, σ) is a Ricci soliton on M, we have the equation (1.5). Using (3.11) in (1.5) we get $S(Y, Z) = -(\sigma + 1)g(Y, Z)$, which implies that M is Einstein.
By virtue of (3.9) we get

$$R(Y, Z)X = \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X = 0,$$

and hence $S(Y, X) = 0$. So, $\sigma + 1 = 0$, i.e., $\sigma = -1$. Hence the given Ricci soliton is shrinking. □

Theorem 3.4. *Let M be a submanifold of $\bar{M}$ with a concurrent vector field X. If (g, X, λ) is an Yamabe soliton on M then such soliton is shrinking, steady and expanding according as $r < 1$, $r = 1$ and $r > 1$ respectively.*

Proof. Since (g, X, λ) is an Yamabe soliton on M, we have the equation (1.6). Using (3.11) in (1.6) we get $\lambda = r - 1$. Hence the result. □

Theorem 3.5. *Let M be a submanifold of $\bar{M}$ with a concurrent vector field X such that ξ is normal to M. If (g, PX, λ) is an Yamabe soliton on M, then PX is conformal.*

Proof. Let (g, PX, λ) be an Yamabe soliton on M. Then from the equation (1.6), we get

$$\frac{1}{2}(\mathcal{L}_{PX} g)(Y, Z) = (r - \lambda)g(Y, Z). \tag{3.12}$$

From (3.5) and (3.12) we have

$$g(h(Y, Z), FX) = (r - \lambda)g(Y, Z) \tag{3.13}$$

for all $Y, Z \in \Gamma(TM)$, which implies that M is umbilical with respect to FX. Then by virtue of Theorem 3.1, it follows that PX is conformal. □

Theorem 3.6. *Let M be a generalized self-similar submanifold of $\bar{M}$ with a concurrent vector field X such that ξ is normal to M. Then PX is conformal vector field if and only if M is pseudo-umbilical.*

Proof. Let M be a generalized self-similar submanifold of $\bar{M}$, then we have the equation (2.11). If PX is conformal vector field, then we have the equation (3.6). From (2.11) and (3.6) we can say that M is pseudo-umbilical.

Conversely, if M is pseudo umbilical submanifold then from equation (2.11) we say that M is umbilical with respect to FX. So, by virtue of Theorem 3.1 it follows that PX is conformal vector field. □

Theorem 3.7. *Let M be a submanifold of $\bar{M}$ with a concurrent vector field X such that ξ is normal to M. Then (g, PX, σ) is a Ricci soliton on M if and only if the following condition holds:*

$$S(Y, Z) = -\sigma g(Y, Z) - g(h(Y, Z), FX) \tag{3.14}$$

for any Y, Z tangent to M.

Proof. Using (3.5) in (1.5), we get the equation (3.14). □

Theorem 3.8. *Let M be a submanifold of $\bar{M}$ with a concurrent vector field X such that ξ is normal to M and (g, PX, σ) is a Ricci soliton on M. Then PX is conformal if and only if M is umbilical.*

Proof. Since (g, PX, σ) is a Ricci soliton on M, then we have (3.14). Also since PX is conformal, using (3.7) in (1.5) we have

$$S(Y, Z) = -\sigma g(Y, Z) - \mu g(Y, Z). \tag{3.15}$$

From (3.14) and (3.15) we can say that M is umbilical.

Conversely, suppose M is umbilical. Then we have the equation (2.9). Using (2.9) in (3.14) we get

$$S(Y, Z) = -\sigma g(Y, Z) - \mu g(Y, Z). \tag{3.16}$$

Using (3.16) in (1.5), we obtain

$$(\mathcal{L}_{PX} g)(Y, Z) = 2\mu g(Y, Z), \tag{3.17}$$

which means that PX is conformal. □

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