Submanifolds of Sasakian Manifolds with Concurrent Vector Field

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Abstract. The submanifolds of Sasakian manifolds with a concurrent vector field have been studied. Applications of such submanifolds to Ricci solitons and Yamabe solitons has also been showed.

1 Introduction

Sasakian manifold $\bar{M}$ is a $(2n + 1)$-dimensional almost contact metric manifold such that [1]

\begin{align}
(\bar{\nabla}_X \phi)Y &= g(X, Y)\xi - \eta(Y)X, \\
\bar{\nabla}_X \xi &= -\phi X,
\end{align}

(1.1, 1.2)

where $(\phi, \xi, \eta, g)$ is the almost contact metric structure and $\bar{\nabla}$ is the Riemannian connection on $\bar{M}$. A vector field $X$ on $\bar{M}$ is said to be conformal if

$$L_X g = 2\alpha g,$$

(1.3)

where $\alpha \in C^\infty(\bar{M})$ and $L_X$ denotes the Lie derivative along $X$. In particular, if $\alpha = 0$ then $X$ is Killing. And $X$ is said to be concurrent if

$$\bar{\nabla}_Z X = Z$$

(1.4)

for any $Z \in \chi(\bar{M})$.

Let $M$ be an $m$-dimensional submanifold of $\bar{M}$. A Ricci soliton on $M$ is a triplet $(g, W, \sigma)$ such that [12]

$$L_W g + 2S + 2\sigma g = 0,$$

(1.5)

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where $S$ is the Ricci tensor on $M$, $W$ is the potential vector field and $\sigma \in \mathbb{R}$. An Yamabe soliton on $M$ is a triplet $(g, W, \lambda)$ such that
\[
\frac{1}{2} \mathcal{L}_W g = (r - \lambda) g,
\]
where $r$ is the scalar curvature on $M$ and $\lambda \in \mathbb{R}$. If the dimension of $M$ is 2 then the notions of Ricci soliton and Yamabe soliton are equivalent. However, when the dimension of $M$ is greater than 2, they are different.

Chen and his co-author studied Euclidean submanifold whose canonical vector field are concurrent [4], concircular [11], conformal [10], torse-forming [9] and also in ([3], [5], [6]). Ricci soliton and Yamabe soliton whose canonical vector field are concurrent and conformal studied in ([2], [7], [8]).

The object of the present paper is to study of submanifolds of Sasakian manifolds with concurrent vector field. We also apply such submanifolds to Ricci solitons and Yamabe solitons.

2 Preliminaries

An odd dimensional smooth manifold $\bar{M}^{2n+1}$ is said to be an almost contact metric manifold if the following relations hold: [1]
\[
\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0,
\]
\[
g(X, \xi) = \eta(X), \quad \phi \circ \eta = 0,
\]
\[
g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)
\]
f for all $X, Y \in \chi(M)$, where $\phi$ is a tensor of type $(1, 1)$, $\xi$ is a vector field, $\eta$ is an 1-form and $g$ is a Riemannian metric on $\bar{M}$.

Let $\nabla$ and $\nabla^\perp$ be the induced connections on the tangent bundle $T M$ and the normal bundle $T^\perp M$ of $M$, respectively. Then we have
\[
\bar{\nabla}_X Y = \nabla_X Y + h(X, Y),
\]
\[
\bar{\nabla}_X V = -A_V X + \nabla^\perp_X V,
\]
where $h$ and $A_V$ are second fundamental form and shape operator respectively for the immersion of $M$ into $\bar{M}$ and they are related by the following equation, see [13]
\[
g(h(X, Y), V) = g(A_V X, Y)
\]
for any \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(T^\perp M) \). If \( h = 0 \), then \( M \) is said to be totally geodesic.

Let \( \{ e_i : 1 \leq i \leq m \} \) be an orthonormal basis to the tangent space at any point of \( M \). Then the mean curvature of \( M \) is

\[
H = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i).
\]

(2.7)

And \( M \) is said to be totally umbilical if

\[
h(X, Y) = g(X, Y)H.
\]

(2.8)

Again \( M \) is said to be umbilical with respect to \( V \in T^\perp M \) if

\[
g(h(X, Y), V) = \mu g(X, Y)
\]

(2.9)

for some function \( \mu \). In particular if \( g(h(X, Y), H) = \mu g(X, Y) \) holds then \( M \) is said to be pseudo-umbilical. Consider

\[
\phi X = PX + FX,
\]

(2.10)

where \( PX \) and \( FX \) are the tangential and normal components of \( \phi X \). And \( M \) is called generalized self-similar submanifold of \( \bar{M} \) if

\[
FX = fH,
\]

(2.11)

where \( f \in C^\infty(M) \).

3 Results

We now prove the followings:

**Theorem 3.1.** Let \( M \) be a submanifold of \( \bar{M} \) with a concurrent vector field \( X \) such that \( \xi \) is normal to \( M \). Then \( PX \) is conformal if and only if \( M \) is umbilical with respect to \( FX \).

**Proof.** Since \( X \) is concurrent vector field of \( \bar{M} \), we have from (1.4) that

\[
\phi Z = \phi \bar{\nabla}_Z X
\]

\[
= \bar{\nabla}_Z \phi X - (\bar{\nabla}_Z \phi) X.
\]

(3.1)

Using (1.1), (2.4), (2.5) and (2.10) in (3.1) we have

\[
PZ + FZ = \bar{\nabla}_Z (PX + FX) - g(X, Z)\xi
\]

\[
= \nabla_Z PX + h(Z, PX) + \nabla^\perp_Z FX - AFXZ - g(X, Z)\xi
\]

(3.2)
Comparing the tangential component of (3.2) we have
\[ \nabla_Z PX = PZ + A_{FX}Z. \] (3.3)

Now we have
\[
(\mathcal{L}_{PX}g)(Y, Z) = g(\nabla_Y PX, Z) + g(Y, \nabla_Z PX)
\]
\[
= g(PY + A_{FX}Y, Z) + g(Y, PZ + A_{FX}Z)
\]
\[
= g(A_{FX}Y, Z) + g(A_{FX}Z, Y).
\] (3.4)

Using (2.6) in (3.4) we have
\[
(\mathcal{L}_{PX}g)(Y, Z) = 2g(h(Y, Z), FX).\] (3.5)

Suppose \( PX \) is conformal. Then from (1.3) and (3.5) we have
\[
g(h(Y, Z), FX) = \alpha g(Y, Z),\] (3.6)
which implies that \( M \) is umbilical with respect to \( FX \).

Conversely, assume that \( M \) is umbilical with respect to \( FX \). Then from (2.9) and (3.5) we have
\[
(\mathcal{L}_{PX}g)(Y, Z) = 2\mu g(Y, Z),\] (3.7)
which means that \( PX \) is conformal.

**Theorem 3.2.** Let \( M \) be a submanifold of \( \bar{M} \) with a concurrent vector field \( X \). Then \( X \) is a homothetic vector field.

**Proof.** Since \( X \) is a concurrent vector field, so we have from (1.4) and (2.4) that
\[
\nabla_Z X + h(X, Z) = Z. \] (3.8)

Equating tangential and normal components of (3.8) we get
\[
\nabla_Z X = Z, \quad h(X, Z) = 0. \] (3.9)

Now we have
\[
(\mathcal{L}_{X}g)(Y, Z) = g(\nabla_Y X, Z) + g(Y, \nabla_Z X).\] (3.10)

Using (3.9) in (3.10) we have
\[
(\mathcal{L}_{X}g)(Y, Z) = 2g(Y, Z),\] (3.11)
which implies that \( X \) is conformal vector field of \( M \) with constant function \( \alpha = 1 \), i.e. \( X \) is homothetic. \( \square \)
Theorem 3.3. Let $M$ be a submanifold of $\bar{M}$ with a concurrent vector field $X$. If $(g, X, \sigma)$ is a Ricci soliton on $M$ then $M$ is Einstein and such a soliton is shrinking.

Proof. Since $(g, X, \sigma)$ is a Ricci soliton on $M$, we have the equation (1.5). Using (3.11) in (1.5) we get $S(Y, Z) = -(\sigma + 1)g(Y, Z)$, which implies that $M$ is Einstein. By virtue of (3.9) we get

$$R(Y, Z)X = \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y, Z]} X = 0,$$

and hence $S(Y, X) = 0$. So, $\sigma + 1 = 0$, i.e., $\sigma = -1$. Hence the given Ricci soliton is shrinking.

Theorem 3.4. Let $M$ be a submanifold of $\bar{M}$ with a concurrent vector field $X$. If $(g, X, \lambda)$ is a Yamabe soliton on $M$ then such soliton is shrinking, steady and expanding according as $r < 1$, $r = 1$ and $r > 1$ respectively.

Proof. Since $(g, X, \lambda)$ is a Yamabe soliton on $M$, we have the equation (1.6). Using (3.11) in (1.6) we get $\lambda = r - 1$. Hence the result.

Theorem 3.5. Let $M$ be a submanifold of $\bar{M}$ with a concurrent vector field $X$ such that $\xi$ is normal to $M$. If $(g, PX, \lambda)$ is an Yamabe soliton on $M$, then $PX$ is conformal.

Proof. Let $(g, PX, \lambda)$ be an Yamabe soliton on $M$. Then from the equation (1.6), we get

$$\frac{1}{2}(\mathcal{L}_{PX} g)(Y, Z) = (r - \lambda)g(Y, Z). \quad (3.12)$$

From (3.5) and (3.12) we have

$$g(h(Y, Z), FX) = (r - \lambda)g(Y, Z) \quad (3.13)$$

for all $Y, Z \in \Gamma(TM)$, which implies that $M$ is umbilical with respect to $FX$. Then by virtue of Theorem 3.1, it follows that $PX$ is conformal.

Theorem 3.6. Let $M$ be a generalized self-similar submanifold of $\bar{M}$ with a concurrent vector field $X$ such that $\xi$ is normal to $M$. Then $PX$ is conformal vector field if and only if $M$ is pseudo-umbilical.

Proof. Let $M$ be a generalized self-similar submanifold of $\bar{M}$, then we have the equation (2.11). If $PX$ is conformal vector field, then we have the equation (3.6). From (2.11) and (3.6) we can say that $M$ is pseudo-umbilical.

Conversely, if $M$ is pseudo umbilical submanifold then from equation (2.11) we say that $M$ is umbilical with respect to $FX$. So, by virtue of Theorem 3.1 it follows that $PX$ is conformal vector field.
Theorem 3.7. Let $M$ be a submanifold of $\bar{M}$ with a concurrent vector field $X$ such that $\xi$ is normal to $M$. Then $(g, PX, \sigma)$ is a Ricci soliton on $M$ if and only if the following condition holds:

$$S(Y, Z) = -\sigma g(Y, Z) - g(h(Y, Z), FX)$$ (3.14)

for any $Y, Z$ tangent to $M$.

Proof. Using (3.5) in (1.5), we get the equation (3.14). □

Theorem 3.8. Let $M$ be a submanifold of $\bar{M}$ with a concurrent vector field $X$ such that $\xi$ is normal to $M$ and $(g, PX, \sigma)$ is a Ricci soliton on $M$. Then $PX$ is conformal if and only if $M$ is umbilical.

Proof. Since $(g, PX, \sigma)$ is a Ricci soliton on $M$, then we have (3.14). Also since $PX$ is conformal, using (3.7) in (1.5) we have

$$S(Y, Z) = -\sigma g(Y, Z) - \mu g(Y, Z).$$ (3.15)

From (3.14) and (3.15) we can say that $M$ is umbilical.

Conversely, suppose $M$ is umbilical. Then we have the equation (2.9). Using (2.9) in (3.14) we get

$$S(Y, Z) = -\sigma g(Y, Z) - \mu g(Y, Z).$$ (3.16)

Using (3.16) in (1.5), we obtain

$$(\mathcal{L}_{PX} g)(Y, Z) = 2\mu g(Y, Z),$$ (3.17)

which means that $PX$ is conformal. □

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