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On strongly starlike functions related to the Bernoulli lemniscate

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Abstract. Let $\mathcal{S}^*_L(\lambda)$ be the class of functions f, analytic in the unit disc $\Delta = \{z : |z| < 1\}$, with the normalization $f(0) = f'(0) - 1 = 0$, which satisfy the condition

$$\frac{zf'(z)}{f(z)} \prec (1+z)^{\lambda},$$

where $\prec$ is the subordination relation. The class $\mathcal{S}^*_L(\lambda)$ is a subfamily of the known class of strongly starlike functions of order λ. In this paper, the relations between $\mathcal{S}^*_L(\lambda)$ and other classes geometrically defined are considered. Also, we obtain some characteristics such as, bounds for coefficients, radius of convexity, the Fekete-Szegö inequality, logarithmic coefficients and the second Hankel determinant inequality for functions belonging to this class. The univalent functions f which satisfy the condition

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < 1 + \frac{\lambda}{2}, \qquad (z \in \Delta)$$

are also considered here.

Keywords. Univalent functions, Subordination, Strongly starlike, Strongly convex

1 Introduction and preliminary

Let $\mathcal{H}$ denote the class of *holomorphic functions* in the open unit disc $\Delta = \{z\colon\ z \in \mathbb{C} \text{ and } |z| < 1\}$ on the complex plane $\mathbb{C}$, and let $\mathcal{A}$ denote the subclass of functions $f \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta)\,. \tag{1.1}$$

The subclass of $\mathcal{A}$ consisting of all *univalent* functions f in Δ, is denoted by $\mathcal{S}$. Robertson [14], Brannan and Kirwan [4], introduced the classes $\mathcal{ST}(\beta)$, $\mathcal{CV}(\beta)$, and $\mathcal{SS}(\alpha)$ of *starlike*

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and convex functions of order $0 \leq \beta < 1$, and *strongly starlike function of order* $0 < \alpha \leq 1$, respectively, which are defined by

$$\mathcal{ST}(\beta) = \left\{ f \in \mathcal{A} \colon \Re\left\{ \frac{zf'(z)}{f(z)} \right\} > \beta, \quad z \in \Delta \right\},$$
$$\mathcal{CV}(\beta) = \left\{ f \in \mathcal{A} \colon \Re\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \beta, \quad z \in \Delta \right\},$$

and

$$\mathcal{SS}(\alpha) = \left\{ f \in \mathcal{A} \colon \left| \mathrm{Arg} \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2}\alpha, \quad z \in \Delta \right\}.$$

We also note that $\mathcal{SS}(1) = \mathcal{ST}(0) = \mathcal{ST}$ and $\mathcal{CV}(0) = \mathcal{CV}$ are the well-known classes of all normalized starlike and convex functions in Δ, respectively. Let $\mathcal{S}(a, b)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the inequality

$$a < \Re \left\{ \frac{zf'(z)}{f(z)} \right\} < b \qquad (z \in \Delta)\,,$$

for some real number a; $(0 \leq a < 1)$ and some real number b; $(b > 1)$ (See [8]).

Definition 1 ([5])**.** Let f and g be analytic in Δ. Then the function f is said to be *subordinate* to g in Δ, written by

$$f(z) \prec g(z), \tag{1.2}$$

if there exists a function $\omega(z) \in \mathcal{B}$ such that $f(z) = g(\omega(z))$; $(z \in \Delta)$, where $\mathcal{B}$ is the family of all Schwarz functions

$$\omega(z) = \sum_{n=1}^{\infty} w_n z^n \qquad (|\omega(z)| < 1,\ z \in \Delta)\,. \tag{1.3}$$

From the definition of subordinations, it is easy to show that the subordination (1.2) implies that

$$f(0) = g(0) \qquad \text{and} \qquad f(\Delta) \subset g(\Delta). \tag{1.4}$$

In particular, if $g(z)$ is univalent in Δ, then the subordination (1.2) is equivalent to the condition(1.4).

Definition 2 ([12])**.** Let $\mathcal{G}(\alpha)$ denote the class of locally univalent normalized analytic functions f in Δ satisfying the condition

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < 1 + \frac{\alpha}{2} \qquad (z \in \Delta)\,,$$

for some $0 < \alpha \leq 1$.

Definition 3. In 1976, Noonan and Thomas [11] defined the q^{th} Hankel determinant of the Taylor's coefficients of function $f \in \mathcal{A}$ of the from (1.1) for natural numbers n and q, as follows

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \qquad (a_1 = 1)\,. \tag{1.5}$$

The Hankel determinants $H_2(1) = a_3 - a_2^2$ and $H_2(2) = a_2a_4 - a_3^2$ are well-known as *Fekete-Szegö* and *second Hankel determinant functionals* respectively. Further, Fekete and Szegö introduced the generalized functional $a_3 - \delta a_2^2$, where δ is some real number. We will give the sharp upper bound for the second Hankel determinant $|H_2(2)|$, when f has lemniscate of Bernoulli domain.

Definition 4. Let $\mathcal{P}$ be a class of the analytic functions p of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \qquad (z \in \Delta)\,. \tag{1.6}$$

satisfying $\Re\{p(z)\} > 0$ in the unit discΔ.

Lemma 1.1 ([15]). *Let $q(z) = \sum_{n=1}^{\infty} B_n z^n$ be analytic and convex univalent in Δ. If $p(z) = \sum_{n=1}^{\infty} A_n z^n$ is analytic in Δ and satisfies the subordination $p(z) \prec q(z)$, then*

$$|A_n| \le |B_1| \qquad (n = 1, 2, \dots)\,.$$

Lemma 1.2. *[6, p.254] If the function $\omega \in \mathcal{B}$ given by (1.3). Then*

$$\begin{aligned} w_2 &= \xi\left(1 - w_1^2\right), \\ w_3 &= \left(1 - w_1^2\right)\left(1 - |\xi|^2\right)\zeta - w_1\left(1 - w_1^2\right)\xi^2, \end{aligned}$$

for some complex number ξ, ζ with $|\xi| \le 1$ and $|\zeta| \le 1$.

Lemma 1.3. *[7, p.10] If the function $\omega \in \mathcal{B}$ given by (1.3), then*

$$\left|w_2 - \mu w_1^2\right| \le \max\{1, |\mu|\}\,.$$

Let us denote by $\mathcal{Q}$ the class of functions f that are analytic and injective on $\overline{\Delta} \setminus E(f)$, where

$$E(f) = \left\{\zeta \colon \zeta \in \partial\Delta \quad \text{and} \quad \lim_{z \to \zeta} f(z) = \infty\right\},$$

and are such that

$$f'(\zeta) \ne 0 \qquad \text{for} \quad \zeta \in \partial\Delta \setminus E(f).$$

Lemma 1.4. *[10, p.24] Let $q \in \mathcal{Q}$ with $q(0) = 1$ and let $p(z) = 1 + p_1 z + \cdots$ be analytic in Δ with $p(z) \ne 1$. If $p \not\prec q$ in Δ then there exits points $z_0 \in \Delta$ and $\zeta \in \partial\Delta \setminus E(q)$ and there exits a real number $m \ge 1$ for which*

$$p(|z| < |z_0|) \subset q(\Delta), \qquad p(z_0) = q(\zeta), \qquad z_0 p'(z_0) = m\zeta q'(\zeta).$$

The purpose of this work is to define a new subfamily of $\mathcal{P}$ related to a domain bounded by

$$LB(\lambda) = \left\{\rho e^{i\varphi} \colon \quad \rho = \left(2\cos\frac{\varphi}{\lambda}\right)^{\lambda}, \quad -\frac{\lambda\pi}{2} < \varphi \le \frac{\lambda\pi}{2}\right\}.$$

We present a new resolution to get the univalence from class functions $LB(\lambda)$. The curve $LB(\lambda)$ is composed of a base pattern symmetrical about real axis obtained for $-\lambda\pi/2 < \varphi \le \lambda\pi/2$. The classes $\mathcal{S}_L^*(\lambda)$ is introduced and its properties and its relevance to other classes presented. In the sequel, we get the extremal functions of class $\mathcal{S}_L^*(\lambda)$. Also, some examples are presented.

2 The class $\mathcal{S}^*_L(\lambda)$ and its properties

This section provides a detailed exposition of an analytic function that maps the unit disk onto a domain bounded by a lemniscate of Bernoulli and contained in a right halfplane.

Let

$$\mathfrak{q}_\lambda(z) = (1+z)^\lambda \equiv \mathrm{e}^{\lambda \ln(1+z)} \qquad (0<\lambda<1)\,,$$

where the branch of the power is chosen to be $\mathfrak{q}_\lambda(0)=1$; more explicitly,

$$\begin{aligned}\mathfrak{q}_\lambda(z) &= 1+\sum_{k=1}^{\infty}\frac{\lambda(\lambda-1)\cdots(\lambda-k+1)}{k!}z^k = 1+\sum_{k=1}^{\infty}B_k z^k \\ &= 1+\lambda z+\frac{\lambda(\lambda-1)}{2}z^2+\frac{\lambda(\lambda-1)(\lambda-2)}{6}z^3+\cdots \quad (z\in\Delta)\,.\end{aligned} \tag{2.1}$$

We note that the set $\mathfrak{q}_\lambda(\Delta)$ lies in the region bounded by the right loop of the lemniscate of Bernoulli given by

$$LB(\lambda)=\left\{\rho e^{i\varphi}\colon \rho=\left(2\cos\frac{\varphi}{\lambda}\right)^\lambda,\quad -\frac{\lambda\pi}{2}\varphi\le\frac{\lambda\pi}{2}\right\}.$$

Since by take $z=e^{i\theta}$; $(\theta\in(-\pi,\pi))$, we have

$$\mathfrak{q}_\lambda\big(e^{i\theta}\big)=\big(1+e^{i\theta}\big)^\lambda=\left(2\cos\frac{\theta}{2}\right)^\lambda e^{i\frac{\lambda\theta}{2}}=\left(2\cos\frac{\theta}{2}\right)^\lambda\left(\cos\frac{\lambda\theta}{2}+i\sin\frac{\lambda\theta}{2}\right).$$

Hence

$$\Re\big\{\mathfrak{q}_\lambda\big(e^{i\theta}\big)\big\}=\left(2\cos\frac{\theta}{2}\right)^\lambda\cos\frac{\lambda\theta}{2}=Q(\theta)\qquad(-\pi<\theta<\pi)\,.$$

So we can see that $Q(\theta)$ is well defined also for $\theta=\pi$. The function $Q(\theta)$; $(-\pi<\theta\le\pi)$ attains its minimal value when $\theta=\pi$, and maximum value when $\theta=0$.

If we take $\mathfrak{q}_\lambda\big(e^{i\theta}\big)=\rho e^{i\varphi}$, simple calculations show that $\varphi=\lambda\theta/2$ and $\rho=\big(2\cos\frac{\theta}{2}\big)^\lambda$. Therefore its boundary $\mathfrak{q}_\lambda\big(e^{i\theta}\big)$ in the polar coordinates will be as follows

$$\mathfrak{q}_\lambda\big(e^{i\theta}\big)=\left\{w=\rho e^{i\varphi}\colon\quad \rho=\left(2\cos\frac{\varphi}{\lambda}\right)^\lambda,\quad -\frac{\lambda\pi}{2}<\varphi\le\frac{\lambda\pi}{2}\right\}. \tag{2.2}$$

Thus from (2.2) we have $\big|\mathsf{Arg}\,\big\{\mathfrak{q}_\lambda\big(e^{i\theta}\big)\big\}\big|<\lambda\pi/2$. Additionally, the right loop of the lemniscate of Bernoulli $LB(\lambda)$ is a boundary of the domain $\mathfrak{q}_\lambda(\Delta)$. Also note that $\mathfrak{q}_\lambda(\mathbb{D})$ is a domain which is symmetric about the real axis, starlike with respect to the point $\mathfrak{q}_\lambda(0)=1$, and satisfies $\mathfrak{q}'_\lambda(0)=\lambda>0$. Also, $LB(\lambda)$ has tangential radial vector $\varphi=\pm\lambda\pi/2$ (see Fig. 1.).

Lemma 2.1. *The functions $\mathfrak{q}_\lambda(z)$ are convex univalent in Δ for each $0<\lambda<1$. Moreover $g_\lambda(z)=(\mathfrak{q}_\lambda(z)-1)/\lambda\in\mathcal{CV}((1+\lambda)/2)$. Also, if $|z|=r<1$, then*

$$\min_{|z|=r}|\mathfrak{q}_\lambda(z)|=\mathfrak{q}_\lambda(-r)\qquad\textit{and}\qquad\max_{|z|=r}|\mathfrak{q}_\lambda(z)|=\mathfrak{q}_\lambda(r).$$

Proof. Let us consider

$$g_\lambda(z)=(\mathfrak{q}_\lambda(z)-1)/\lambda\qquad(z\in\Delta)\,.$$

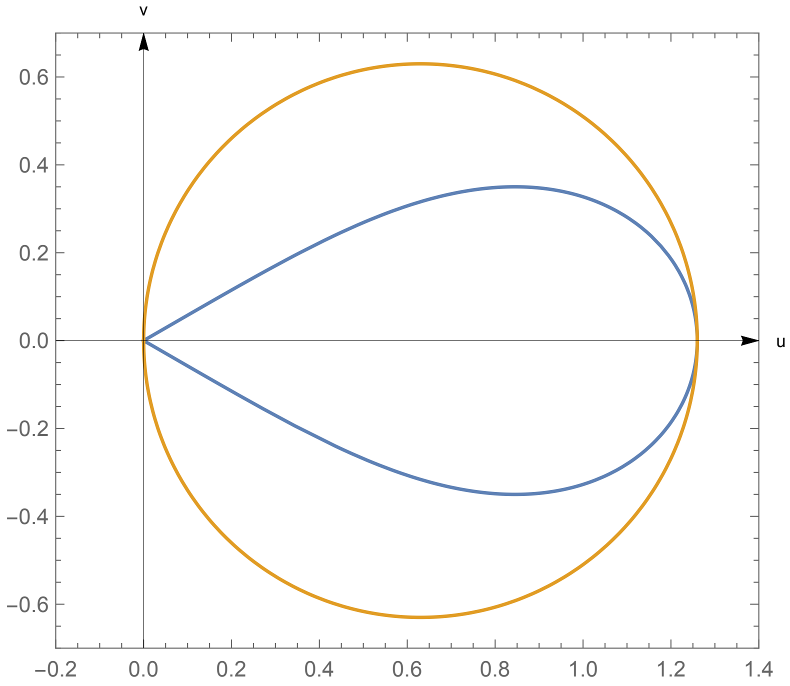

Figure 1: image of unit circle under $\mathfrak{q}_\lambda(z)$ for $\lambda = \frac{1}{2}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Then, we have

$$\Re\left\{1 + \frac{zg_\lambda''(z)}{g_\lambda'(z)}\right\} = \Re\left\{\frac{1+\lambda z}{1+z}\right\} > \frac{\lambda+1}{2},$$

so $g_\lambda \in \mathcal{CV}((\lambda+1)/2) \subset \mathcal{ST}$, so $\mathfrak{q}_\lambda(z)$ are convex univalent too for each $0 < \lambda < 1$. In order to prove the second part of lemma, let $\theta \in [0, 2\pi)$, then the function

$$Q(\theta) = \left|\mathfrak{q}_\lambda(re^{i\theta})\right| = \left|1 + re^{i\theta}\right|^\lambda = \left(1 + r^2 + 2r\cos\theta\right)^{\frac{\lambda}{2}} \quad (0 < r < 1),$$

attains its minimum at $\theta = \pi$ and maximum at $\theta = 0$. This ends the proof. □

Theorem 2.1. *Let $\mathfrak{p}(z) \in \mathcal{H}$ with $\mathfrak{p}(0) = 1$. If*

$$\mathfrak{p}(z) \prec \mathfrak{q}_\lambda(z), \qquad (z \in \Delta),$$

then

$$|\mathrm{Arg}\,\{\mathfrak{p}(z)\}| < \frac{\lambda\pi}{2}, \qquad 0 < \Re\{\mathfrak{p}(z)\} < 2^\lambda, \quad (z \in \Delta), \tag{2.3}$$

and

$$\left|\mathfrak{p}^{\frac{1}{\lambda}}(z) - 1\right| < 1, \qquad (z \in \Delta). \tag{2.4}$$

Conversely, if $\mathfrak{p} \in \mathcal{P}$ with $|\mathrm{Arg}\,\{\mathfrak{p}\}| < (\lambda\pi)/2$ and $\mathfrak{p}$ satisfies (2.4), then $\mathfrak{p} \prec \mathfrak{q}_\lambda$ in Δ.

Proof. The subordination $\mathfrak{p} \prec \mathfrak{q}_\lambda$ with $\mathfrak{p}(0) = \mathfrak{q}_\lambda(0)$, and the geometric properties of $\mathfrak{q}_\lambda(\Delta)$ from Section 1, yield (2.3).

In order to prove the second part of theorem, since $p(z) \prec \mathfrak{q}_\lambda(z)$; $(z \in \Delta)$, then

$$p(z) = (1 + \omega(z))^\lambda, \qquad (z \in \Delta), \tag{2.5}$$

where $\omega \in \mathcal{B}$. From (2.5), we get

$$\omega(z) = p^{\frac{1}{\lambda}}(z) - 1, \quad |\omega(z)| < 1, \quad (z \in \Delta),$$

and finally assertion (2.4) as follows.

Conversely, for $\mathfrak{p} \in \mathcal{P}$ satisfy the condition (2.4), then we easily show that $\mathfrak{p} = \rho e^{\imath\varphi}$ lies in a domain bounded by lemniscate of Bernoulli $LB(\lambda)$. It completes the proof. $\square$

Definition 5. Let $\mathcal{S}^*_L(\lambda)$ denote the class of analytic functions $f \in \mathcal{A}$ satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec \mathfrak{q}_\lambda(z), \qquad (z \in \Delta). \tag{2.6}$$

Geometrically, the condition (2.6) means that the quantity $zf'(z)/f(z)$ lies in the region bounded by the right loop of the lemniscate of Bernoulli $LB(\lambda)$. Since a domain $\mathfrak{q}_\lambda(\Delta)$ is contained in a right half-plane, we deduce that $\mathcal{S}^*_L(\lambda)$ is a proper subset of a class of a starlike functions $\mathcal{ST}$. Additional properties of $\mathfrak{q}_\lambda(\Delta)$ yield:

$$\mathcal{S}^*_L(\lambda) \subset \mathcal{SS}(\alpha) \quad \textit{for} \quad \lambda \le \alpha \le 1,$$
$$\mathcal{S}^*_L(\lambda) \subset \mathcal{S}(0, b) \quad \textit{for} \quad b \ge 2^\lambda.$$

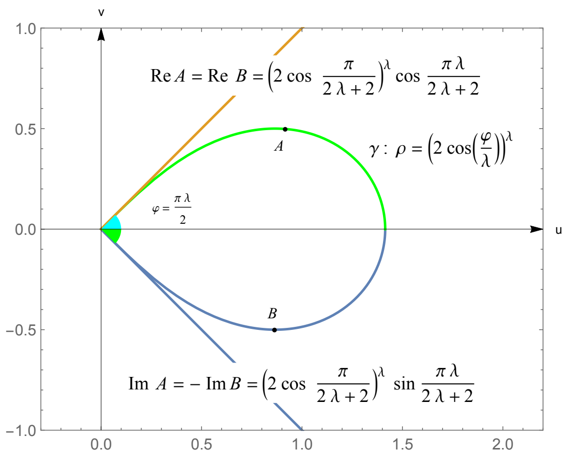

Figure 2: The lemniscate of Bernoulli $\rho = \left(2\cos\frac{\varphi}{\lambda}\right)^\lambda$ and the circle $\rho = 2^\lambda \cos\varphi$ for $\lambda = \frac{1}{3}$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Also, we have $LB(\lambda) \subset \left\{w \colon \left|w - 2^{\lambda-1}\right| < 2^{\lambda-1}\right\}$. The right-half of the lemniscate of Bernoulli γ_1 and the circle $\gamma_2\colon \left(x - 2^{\lambda-1}\right)^2 + y^2 = 4^{\lambda-1}$ are presented in **Fig. 2**. Thus for $M \ge 2^{\lambda-1}$, we have

$$(1+z)^\lambda \prec \frac{M + Mz}{M - (M-1)z}, \qquad (z \in \Delta).$$

Since the function $\frac{M+Mz}{M-(M-1)z}$ is univalent in Δ, then

$$\mathcal{S}^*_L(\lambda) \subset \left\{ f \in \mathcal{A} \colon \left| \frac{zf'(z)}{f(z)} - M \right| < M, \quad \text{for all } z \in \Delta \right\}.$$

The structural formula for functions in the class $\mathcal{S}^*_L(\lambda)$ is as follows:

$$g \in \mathcal{S}^*_L(\lambda) \Longleftrightarrow g(z) = z \exp\left(\int_0^z \frac{p(t)-1}{t} \,\mathrm{d}t \right) \quad \textit{for some } p \prec \mathfrak{q}_\lambda. \tag{2.7}$$

This above representation gives many examples of functions in class $\mathcal{S}^*_L(\lambda)$. The function $F_{\lambda,n}$ with definition

$$\begin{aligned} F_{\lambda,n}(z) &= z \exp\left(\int_0^z \frac{\mathfrak{q}_\lambda(t^n)-1}{t} \,\mathrm{d}t \right) \\ &= z + \frac{\lambda}{n} z^{n+1} + \frac{\lambda^2(n+2)-n\lambda}{4n^2} z^{2n+1} \\ &+ \frac{\lambda\big(\left(2n^2+9n+6\right)\lambda^2 - \left(6n^2+9n\right)\lambda + 4n^2\big)}{36n^3} z^{3n+1} + \cdots \ (z \in \Delta), \end{aligned} \tag{2.8}$$

for $n = 1, 2, \ldots$ is extremal function for several problems in the class $\mathcal{S}^*_L(\lambda)$. For $n = 1$ we have

$$\begin{aligned} F_\lambda(z) = F_{\lambda,1}(z) &= z \exp\left(\int_0^z \frac{\mathfrak{q}_\lambda(t)-1}{t} \,\mathrm{d}t \right) \\ &= z + \lambda z^2 + \left(\frac{3\lambda^2-\lambda}{4} \right) z^3 + \left(\frac{17\lambda^3 - 15\lambda^2 + 4\lambda}{36} \right) z^4 + \cdots. \end{aligned} \tag{2.9}$$

Theorem 2.2. *If a function f belongs to the class $\mathcal{G}(\lambda)$, then $f' \prec \mathfrak{q}_\lambda$ in Δ. Also, f is univalent function in Δ.*

Proof. Suppose that $f'(z) \not\prec \mathfrak{q}_\lambda(z)$ in Δ. Then by Lemma 1.4 there exist $z_0 \in \Delta$ and $\zeta \in \partial\Delta$; $(\zeta \neq -1)$ such that

$$f'(z_0) = \mathfrak{q}_\lambda(\zeta), \qquad z_0 f''(z_0) = m\zeta \mathfrak{q}'_\lambda(\zeta),$$

for some $m \geq 1$. Hence

$$\Re\left\{ 1 + \frac{z_0 f''(z_0)}{f'(z_0)} \right\} = 1 + m\lambda \Re\left\{ \frac{\zeta}{1+\zeta} \right\} = 1 + \frac{m\lambda}{2} \geq 1 + \frac{\lambda}{2},$$

which contradicts the hypothesis $f \in \mathcal{G}(\lambda)$. Thus, we conclude that $f'(z) \prec \mathfrak{q}_\lambda(z)$ for all $z \in \Delta$. From condition (2.3) we have $\Re\{f'(z)\} > 0$. Therefore f is univalent. □

From (2.7) and from Theorem 2.2 and , we get the following corollary.

Corollary 2.3. *Let $f \in \mathcal{G}(\lambda)$ for $0 < \lambda < 1$. Then the function*

$$g(z) = z \exp\left(\int_0^z \frac{f'(t)-1}{t} \,\mathrm{d}t \right)$$

belongs to $\mathcal{S}^*_L(\lambda)$.

Example 1. The function $f(z) = z\exp(-Az)$ belongs in class $\mathcal{S}^*_L(\lambda)$ if $|A| \leq \frac{\lambda}{2+\lambda}$.

From the results in [9], equation (2.9), and Lemma 2.1, we have the following sharp estimates for function $f \in \mathcal{S}^*_L(\lambda)$.

Theorem 2.4. *If $f \in \mathcal{S}^*_L(\lambda)$ and $|z| = r < 1$, then*

$$-F_\lambda(-r) \le |f(z)| \le F_\lambda(r),$$
$$F'_\lambda(-r) \le |f'(z)| \le F'_\lambda(r),$$
$$|\mathsf{Arg}\,\{f(z)/z\}| \le \max_{|z|=r} \mathsf{Arg}\,\{F_\lambda(z)/z\}\,.$$

Equality holds for some $z \neq 0$ if and only if f is a rotation of F_λ. Also, If $f \in \mathcal{S}^*_L(\lambda)$, then either f is a rotation of F_λ or

$$\{w \in \mathbb{C}: \quad |w| \le -F_\lambda(-1)\} \subset f(\Delta).$$

Here $-F_\lambda(-1)$ is understood to be the limit of $-F_\lambda(-r)$ as r tends to 1.

For the special case $\lambda = 1/2$, results for functions belonging to the class $\mathcal{S}^*_L = \mathcal{S}^*_L(1/2)$ defined by

$$\mathcal{S}^*_L = \left\{\rho e^{i\varphi}: \quad \rho^2 < 2\cos(2\varphi)\,, \quad -\frac{\pi}{4} < \varphi < \frac{\pi}{4}\right\}$$

and its generalizations can be found in [1, 2, 3, 13, 16, 17, 18, 19, 20].

3 Logarithmic coefficient inequality for the function $f(z)$

Associated with each $f \in \mathcal{S}$ (see [5]) is well defined function

$$\log\frac{f(z)}{z} = \sum_{n=1}^{\infty} 2\gamma_n z^n \qquad (z \in \Delta)\,,$$

and γ_n are called logarithmic coefficients of the function f.

Theorem 3.1. *Let $f \in \mathcal{S}^*_L(\lambda)$. Then the logarithmic coefficients of f satisfy*

$$|\gamma_n| \le \frac{\lambda}{2n} \qquad (n \ge 1)\,.$$

All the inequalities are sharp.

Proof. Let $f \in \mathcal{S}^*_L(\lambda)$. From Definition 5, we have

$$z\left(\log\frac{f(z)}{z}\right)' \prec q(z) - 1, \qquad (z \in \Delta)\,. \tag{3.1}$$

The subordination relation (3.1) implies that

$$\sum_{n=1}^{\infty} 2n\gamma_n z^n \prec \sum_{n=1}^{\infty} B_n z^n,$$

where B_n given by (2.1). Applying Lemma 1.1, we get the inequality $2n\,|\gamma_n| \le |B_1| = \lambda$. To deduce the sharpness, by the definition $F_{\lambda,n}(z)$ and $\mathfrak{q}_\lambda(z)$, we have

$$z\left(\log\frac{F_{\lambda,n}(z)}{z}\right)' = \mathfrak{q}_\lambda(z^n) - 1 \Longleftrightarrow \sum_{k=1}^{\infty} 2k\gamma_k z^k = \sum_{m=1}^{\infty} B_m (z^n)^m, \tag{3.2}$$

where γ_k; $(k = 1, 2, \dots)$ is logarithmic coefficients of $F_{\lambda,n}$ and B_m given in (2.1). Form (3.2), equating coefficients gives $2n\gamma_n = B_1 = \lambda$. □

4 Fekete-Szegö and second Hankel determinant problems for the function class $\mathcal{S}_L^*(\lambda)$

In this section, we find the sharp bounds of Fekete-Szegö functional $a_3 - \mu a_2^2$ and second Hankel determinant functional $a_2a_4 - a_3^2$ defined for $f \in \mathcal{S}_L^*(\lambda)$ given by (1.1).

Theorem 4.1. *let $f \in \mathcal{S}_L^*(\lambda)$ given by (1.1). Then*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{\lambda^2}{4}.$$

The inequalities are sharp.

Proof. Let the function f given by (1.1) be in the class $\mathcal{S}_L^*(\lambda)$. Then there exists a function $\omega \in \mathcal{B}$, such that

$$\frac{zf'(z)}{f(z)} = (1+\omega(z))^{\lambda}. \tag{4.1}$$

Form (4.1), equating coefficients gives, after simplification

$$\begin{cases} a_2 &= \lambda w_1, \\ a_3 &= \frac{\lambda}{2}\left(w_2 + \frac{3\lambda-1}{2}w_1^2\right), \\ a_4 &= \frac{\lambda}{3}\left(w_3 + \left(\frac{5\lambda-2}{2}\right)w_1w_2 + \left(\frac{17\lambda^2-15\lambda+4}{12}\right)w_1^3\right). \end{cases} \tag{4.2}$$

Form (1.5) and (4.2) we have

$$\left|a_2a_4 - a_3^2\right| = \frac{\lambda^2}{12}\left|w_1w_3 - 3w_2^2 + (\lambda-1)\,w_1^2w_2 + \left(\frac{7-13\lambda^2-6\lambda}{12}\right)w_1^4\right|.$$

Using Lemma 1.2, we write the expression w_2 and w_3 in terms of w_1 and without loss of generality assume that $x = w_1$ with $0 \leq x \leq 1$. Then from triangular inequality, we obtain

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| \leq \frac{\lambda^2}{12}\Bigg\{&\left|\frac{13\lambda^2+6\lambda-7}{12}\right|x^4 + |\lambda-1|\,x^2\left(1-x^2\right)|\xi| \\ &+ 3\left(1-x^2\right)^2|\xi|^2 + 4x\left(1-x^2\right)\left(1-|\xi|^2\right) + 4x^2\left(1-x^2\right)|\xi|^2\Bigg\} = g(|\xi|). \end{aligned}$$

A function $g(|\xi|)$ is increasing on the interval $[0,1]$. Thus $g(|\xi|)$ attains its maximum at $|\xi| = 1$, i.e. $g(|\xi|) \leq g(1)$. Consequently

$$\left|a_2a_4 - a_3^2\right| \leq \frac{\lambda^2}{12}\left\{3 - (\lambda+1)\,x^2 + \left(\lambda - 2 + \left|\frac{-13\lambda^2-6\lambda+7}{12}\right|\right)x^4\right\},$$

also,

$$\begin{aligned} \left|a_2a_4 - a_3^2\right| &\leq \begin{cases} \frac{\lambda^2}{12}\left\{3 - (\lambda+1)\,x^2 - \left(\frac{13\lambda^2-6\lambda+17}{12}\right)x^4\right\} & : \ 0 < \lambda \leq \frac{7}{13}, \\ \frac{\lambda^2}{12}\left\{3 - (\lambda+1)\,x^2 - \left(\frac{-13\lambda^2-18\lambda+31}{12}\right)x^4\right\} & : \ \frac{7}{13} \leq \lambda < 1, \end{cases} \\ &\leq \frac{\lambda^2}{4}. \end{aligned}$$

The function $F_{\lambda,2}$ in (2.8), shows that the bound $\lambda^2/4$ is sharp. □

Theorem 4.2. *let $f \in \mathcal{S}_L^*(\lambda)$ given by (1.1). Then we have sharp inequalities*

$$\left|a_3-\delta a_2^2\right| \leq \begin{cases} -\lambda^2\left(\delta+\frac{1-3\lambda}{4\lambda}\right) & : \quad \delta<\frac{3(\lambda-1)}{4\lambda}, \\ \frac{\lambda}{2} & : \quad \frac{3(\lambda-1)}{4\lambda} \leq \delta \leq \frac{1+3\lambda}{4\lambda}, \\ \lambda^2\left(\delta+\frac{1-3\lambda}{4\lambda}\right) & : \quad \delta>\frac{1+3\lambda}{4\lambda}. \end{cases}$$

Proof. Form equations (4.2), we have

$$\left|a_3-\delta a_2^2\right|=\frac{\lambda}{2}\left|w_2-\left(\frac{4\delta\lambda-3\lambda+1}{2}\right)w_1^2\right|.$$

Applying Lemma 1.3 with $\mu=(4\delta\lambda-3\lambda+1)/2$ gives the inequalities. Equality is attained in the second inequality for $f(z)=F_{\lambda,2}(z)$ given by (2.8), and by the function $f(z)=F_\lambda(z)$ given by (2.9) in other cases. □

Let the function F be defined by

$$F(z)=\frac{z}{f(z)}=1+\sum_{n=1}^{\infty} b_n z^n \qquad (z \in \Delta), \tag{4.3}$$

for $f \in \mathcal{A}$ given by (1.1).

Theorem 4.3. *Let $f \in \mathcal{S}_L^*(\lambda)$ and $F(z)=z/f(z)$ given by (1.1) and (4.3), respectively. Then we have sharp inequalities*

$$\left|b_2-\delta b_1^2\right| \leq \begin{cases} -\lambda^2\left(\delta-\frac{\lambda+1}{4\lambda}\right) & : \quad \delta<\frac{\lambda-1}{4\lambda}, \\ \frac{\lambda}{2} & : \quad \frac{\lambda-1}{4\lambda} \leq \delta \leq \frac{\lambda+3}{4\lambda}, \\ \lambda^2\left(\delta-\frac{\lambda+1}{4\lambda}\right) & : \quad \delta>\frac{\lambda+3}{4\lambda}. \end{cases}$$

Proof. Let $f \in \mathcal{S}_L^*(\lambda)$ given by (1.1) and $F(z)=z/f(z)$ and a computation gives

$$F(z)=\frac{z}{f(z)}=1-a_2 z+\left(a_2^2-a_3\right)z^2+\cdots \qquad (z \in \Delta). \tag{4.4}$$

Form equations (4.3) and (4.4), we have

$$\begin{cases} b_1 & =-a_2, \\ b_2 & =a_2^2-a_3. \end{cases} \tag{4.5}$$

Form equations (4.2) and (4.5), we have

$$\left|b_2-\delta b_1^2\right|=\frac{\lambda}{2}\left|w_2-\frac{w_1^2}{2}(\lambda+1-4\delta\lambda)\right|.$$

Applying Lemma 1.3 with $\mu=(\lambda+1-4\delta\lambda)/2$ gives the inequalities. The function $f(z)=F_{\lambda,2}(z)$ given by (2.8), and function $f(z)=F_\lambda(z)$ given by (2.9), shows that the bounds $\lambda/2$ and $\pm\lambda^2(\delta-(\lambda+1)/(4\lambda))$ are sharps, respectively. □

Let the function f^{-1} be defined by

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n \qquad (|w| < r_0(f)), \tag{4.6}$$

where $r_0(f) \geq -F_\lambda(-1)$ is the radius of the Koebe domain of the function f in the class $\mathcal{S}_L^*(\lambda)$. Then

$$f^{-1}(f(z)) = z; \quad (z \in \Delta) \quad \text{and} \quad f\left(f^{-1}(w)\right) = w; \quad (|w| < r_0(f)).$$

The inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \tag{4.7}$$

Theorem 4.4. *let $f \in \mathcal{S}_L^*(\lambda)$ and $f^{-1}(z)$ given by (1.1) and (4.6), respectively. Then we have sharp inequalities*

$$\left|A_3 - \delta A_2^2\right| \leq \begin{cases} -\lambda^2\left(\delta - \frac{5\lambda+1}{4\lambda}\right) & : \quad \delta < \frac{5\lambda-1}{4\lambda}, \\ \frac{\lambda}{2} & : \quad \frac{5\lambda-1}{4\lambda} \leq \delta \leq \frac{5\lambda+3}{4\lambda}, \\ \lambda^2\left(\delta - \frac{5\lambda+1}{4\lambda}\right) & : \quad \delta > \frac{5\lambda+3}{4\lambda}. \end{cases}$$

Proof. Form (4.6) and (4.7), we have

$$\begin{cases} A_2 & = -a_2, \\ A_3 & = 2a_2^2 - a_3. \end{cases} \tag{4.8}$$

Form (4.2) and (4.8), we have

$$\left|A_3 - \delta A_2^2\right| = \frac{\lambda}{2}\left|w_2 - \frac{w_1^2}{2}(5\lambda + 1 - 4\delta\lambda)\right|.$$

Applying Lemma 1.3 with $\mu = (5\lambda + 1 - 4\delta\lambda)/2$ gives the inequalities. The inequality is sharp for the function

$$f(z) = \begin{cases} F_{\lambda,2}(z) & : \quad \frac{5\lambda-1}{4\lambda} \leq \delta \leq \frac{5\lambda+3}{4\lambda}, \\ F_\lambda(z) & : \quad \delta \in \left(-\infty, \frac{5\lambda-1}{4\lambda}\right) \cup \left(\frac{5\lambda+3}{4\lambda}, \infty\right). \end{cases}$$

□

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