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BEST L^p-APPROXIMATION OF GENERALIZED BIAXIALLY SYMMETRIC POTENTIALS OVER CARATHÉODORY DOMAINS IN C^N HAVING SLOW GROWTH

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Abstract. Let F be a real valued generalized biaxially symmetric potentials (GBASP) defined on the Carathéodory domain on C^N. Let $L^p_\mu(D)$ be the class of all functions F holomorphic on D such that $\| F \|_{D,p}= [\int_D \mid F \mid^p d\mu]^{\frac{1}{p}}$. Where μ is the positive finite, Boral measure with regular asymptotic distribution on C^N. For $F \in L^p_\mu(D)$, set $E^p_n(F) = \inf\{\| F - P \|_{D,p}: P \in H_n\}$, H_n consist of all real biaxisymmetric harmonic polynomials of degree at most $2n$. The paper deals with the growth of entire function GBASP in terms of approximation error in L^p_μ-norm on D. The analysis utilizes the Bergman and Gilbert integral operator method to extend results from classical function theory on the best polynomial approximation of analytic functions of several complex variables. Finally we prove a generalized decomposition theorem in a new way. The paper is the generalization of the concepts of generalized growth parameters to entire functions on Carathéodory domains on C^N (instead of entire holomorphic functions on C) for slow growth.

1. Introduction

If f has a unique expansion of the form $f = \sum_{n=0}^{\infty} P_n(z)$, where $P_n(z)$ are homogeneous polynomials of degree at most $2n$ on C^N and e^* denotes the vector $(1, 0, \ldots, 0)$ in R^N, then for $u \in C$, there exists a unique entire function f^* on C such that $f^*(u) = f(ue^*) = \sum_{n=0}^{\infty} P_n(ue^*) = \sum_{n=0}^{\infty} P_n(e^*)u^{2n}$. Let $f^*(z) = \sum_{n=0}^{\infty} P_n(e^*)z^{2n}$, this power series converges for all real and hence all complex z so f^* is entire function. The uniqueness is also clear, since entire functions that agree on the real axis are identical. This fact has been used to deduce theorems for entire functions on C^N from classical results about entire functions on C.

Let $F = F(x, y)$ be real valued regular solution to the generalized biaxially symmetric potential equation

$$\Big|\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + \frac{2\beta+1}{y}\frac{\partial}{\partial y}\Big|F = 0, \quad \alpha > \beta > -\frac{1}{2}, \tag{1.1}$$

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subject to the Cauchy data $F_x(0,y) = F_y(x,0) = 0$, along the singular lines in $\sum_R^{\alpha,\beta}$: $|\, x\,|^2 + |\, y\,|^2 < R^2$, the open hypersphere of radius R, where $|\, x\,| = (x_1^2 + x_2^2 + \cdots + x_N^2)^{1/2}$, $|\, y\,| = (y_1^2 + y_2^2 + \cdots + y_N^2)^{1/2}$.

These solutions, called the generalized biaxially symmetric potentials (GBASP) can be expanded in $\sum_R^{\alpha,\beta}$ uniquely as

$$F(x,y) = \sum_{j=0}^{\infty} a_j R_j^{\alpha,\beta}(x,y)$$

in terms of complete set

$$R_j^{\alpha,\beta}(x,y) = (x^2+y^2)^j P_j^{\alpha,\beta}\left[\frac{(x^2-y^2)}{(x^2+y^2)}\right] / P_j^{\alpha,\beta}(1) \quad j = 0,1,2,\ldots,$$

of biaxisymmetric harmonic potentials and $P_j^{\alpha,\beta}$ are Jacobi polynomials ([1],[11]).

Let the operator mapping unique associated even analytic functions

$$f^*(z) = \sum_{n=0}^{\infty} P_n(e^*) z^{2n},$$

on to GBASP

$$F^*(z,0) = \sum_{n=0}^{\infty} P_n(e^*) R_n^{\alpha,\beta}(z,0)$$

where $F^*(z,0) = F(ze^*,0)$ and $P_n(e^*) \equiv a_n$ such that no element of the sequence $\{a_n\}_{n=1}^{\infty}$ is zero.

The above mapping defined as in ([8], [9]) from Koornwinder's integral for Jacobi polynomials. For details (see [1]). The local function elements F^* and f^* are continued harmonically/analytically by contour deformation using the Envelope Method[2]. The Envelope Method([2], [3]) establishes that the GBASP is regular in the hypersphere if and only if its associate is analytic in the polydisc. On the singular lines $y = 0$, the identity.

$$f(xe^*) = F(xe^*,0), \quad |\, x\,| < R$$

can be analytically continued even associate as

$$f^*(z) = f(ze^*) = F(ze^*,0) = F^*(z,0), \quad |\, z\,| < R.$$

The GBASP are natural extension of harmonic or analytic functions. Hence we anticipate properties similar to those of the harmonic functions found from associated analytic even f on C^N, by taking Ref, the real part of f.

The maximum module of GBASP and associate are defined as in complex function theory [10].

$$M(f^*,r) = \max_{(|z|)=r} |\, f^*(z)\,|,$$
$$M(F,r) = \max_{x^2+y^2=r^2} |\, F(x,y)\,|\,.$$

Let B denote a Carathéodory domain that is a bounded simply connected domain such that the boundary of B coincides with the boundary of the domain lying in the complement of the closure of B and containing the point ∞. In particular, a domain bounded by a Jordan Curve is a Carathéodory domain. Let $L^p_\mu(B)$ and $l^p_\mu(B)$, $1 \leq p \leq \infty$ denote the class of GBASP F and its associate f holomerphic on B such that

$$\| F \|_{B,p} = \Big[\int_B \mid F(z,0) \mid^p d\mu \Big]^{\frac{1}{p}} < \infty,$$
$$\| f \|_{B,p} = \Big[\int_B \mid f(z) \mid^p d\mu \Big]^{\frac{1}{p}} < \infty,$$

where these norms are understood to be $\sup_{z\in B} \mid F(z,0) \mid$, $\sup_{z\in B} \mid f \mid$ for $p=\infty$ and $\| \, . \, \|_{B,p}$ denotes the L^p_μ norm and l^p_μ-norm for F and f, respectively. For $f \in l^p_\mu(B)$, we define $b_n's$ called the Fourier Coefficients of f as

$$b_n = \int_B f(z)\overline{p_n(z,\mu)}d\mu. \tag{1.1}$$

Also $\delta^n_m = \int_B p_n(z,\mu)p_m(z,\mu)d\mu$,
where $\delta^n_m = 1$ for $m=n$ and $\delta^n_m = 0$ otherwise. Since if we consider the monomials $\{z^n\}$ to be ordered lexicographically. By ([7], Prop 1), we may apply the Gram. Schmidt orthogonalization procedure to monomials and obtains orthonormal polynomials denoted by $p_n(z) = p_n(z,\mu)$. So $\{p_n(z,\mu)\}$ is a complete orthonormal sequence of polynomials. $p_n(z)$ being real even polynomial of degree at most $2n$. It is known ([7] Corollary of Lemma 5) that $f \in l^p_\mu(B)$ is entire if and only if

$$\lim_{n\to\infty} \mid b_n \mid^{\frac{1}{n}} = 0.$$

Moreover, $f^*(z) = \sum_{n=0}^\infty b_n p_n(z)$ holds in C^N.

Now for $p=\infty$, the best polynomial approximation error for F (GBASP) and its associate f is defined as

$$e_n(f) \equiv e_n(f,B) = \inf\{\| f-\pi \|, \pi \in h_n\}, \quad n = 0,1,\ldots,$$

where

$$\| f-\pi \| = \sup_{x\in B}\{\mid f(x)-\pi(x) \mid\} x \in R^N,$$

and

$$E_n(F) \equiv E_n(F,B) = \inf\{\| F-P \|, P \in H_n\}, \tag{1.2}$$

where

$$\| F-P \| = \sup_{ze^*\in B} \mid F(ze^*,0) - P(ze^*,0) \mid .$$

The set h_n contains all real homogeneous polynomials of degree at most $2n$ and the set H_n contains all real biasixymmetric harmonic polynomials of degree $2n$. The operators $K_{\alpha,\beta}$ and $K^{-1}_{\alpha,\beta}$[8] establish one-one equivalence of the sets h_n and H_n.

Let L^0 denote the class of functions $\phi(x)$ satisfying conditions (i) and (ii):

(i) $\phi(x)$ is defined on $[a,\infty)$; is positive, strictly increasing, differentiable and $\phi(x)\to\infty$ as $x\to\infty$.

(ii) $\lim_{n\to\infty}\frac{\phi[x(1+\varphi(x))]}{\phi(x)}=1$,
for every function $\varphi(x)$ such that $\varphi(x)\to 0$ as $x\to\infty$.
Let Δ be the class of function $\phi(x)$ satisfying (i) and (iii):

(iii) $\lim_{x\to\infty}\frac{\phi(cx)}{\phi(x)}=1$, for every $0<c<\infty$.
Let Ω be the class of functions $\phi(x)$ satisfying (i) and (iv):

(iv) There exists a $\delta(x)\in\Delta$ and x_0, K_1 and K_2 such that

$$0<K_1\leq\frac{d(\phi(x))}{d(\delta(\log x))}\leq K_2<\infty,\quad \forall x>x_0.$$

Also, let $\overline{\Omega}$ be the class of functions $\phi(x)$ satisfying (i) and (v):

(v) $\lim_{x\to\infty}\frac{d(\phi(x))}{d(\log x)}=K$, $0<K<\infty$.
The generalized growth parameters of an entire function GBASP F^* are defined as

$$\lim_{r\to\infty}\begin{matrix}\sup\\ \\ \inf\end{matrix}\quad\frac{\alpha(\log M(F^*,r))}{\alpha(\log r)}=\begin{matrix}\rho(\alpha,\alpha,F^*)\\ \\ \lambda(\alpha,\alpha,F^*)\end{matrix},$$

where $\alpha(x)$ belongs to either Ω or $\overline{\Omega}$.

Definition. An entire function F^* is said to be regular growth if $1<\lambda(\alpha,\alpha,F^*)=\rho(\alpha,\alpha,F^*)<\infty$.

Following the reasoning of McCoy[8], it can be shown that generalized orders of entire GBASP and its associate are same. McCoy([8], [9]) has characterized classical order and type of entire GBASP in terms of approximation error in L^p-norm on [-1,1] in single complex variable. In this paper we extend the results of McCoy to arbitrary domains and generalized growth parameters in C^N. We identify those GBASP, $F^*\in L^p_\mu(B)$ that harmonically continue as an entire function GBASP. The characteristic feature follows from the rate of convergence of a sequence of best GBASP polynomial approximates to F^* in $L^p_\mu(B)$ and sup norms. The generalized growth parameters of an entire GBASP F^* have been characterized in terms of the approximation error $E^p_n(F^*)$ in L^p_μ and sup norms on Carathéodory domains in C^N. In the last we prove the generalized decomposition theorem in a new way. Our results apply satisfactorily to entire GBASP of slow growth and these results are the generalization of the concepts of generalized growth parameters to entire functions on Carathéodory domains in C^N (instead of entire holomorphic functions on C).

We shall use the following notations throughout the paper.

1. $\vartheta_n(z)=\max(1,\xi)$ if $\alpha(x)\in\Omega$,
   $=\eta+\xi$ if $\alpha(x)\in\overline{\Omega}$,
   we shall write $\vartheta(\xi)$ for $\vartheta_1(\xi)$.

2. $G[x,c] = \alpha^{-1}[c\alpha(x)]$, c is a positive constant.

2. Auxiliary Results

Let B^* be the component of the compliment of the closure of the Carathéodory domain B that contains the point ∞. Set $B_r = \{z : \mid \varphi(z) \mid = r, r > 1\}$, where the function $w = \varphi(z)$ maps B^* conformally on to $\mid w \mid > 1$ such that $\varphi(\infty) = \infty$ and $\varphi'(\infty) > 0$.

Lemma 1. *If F^* be entire function GBASP having generalized growth parameters $\rho(\alpha, \alpha, F^*)$ and $\lambda(\alpha, \alpha, F^*)$. Then*

$$\limsup_{r\to\infty} \frac{\alpha(\log \overline{M}(F^*, r))}{\alpha(\log r)} = \rho(\alpha, \alpha, F^*)$$
$$\text{and} \quad \liminf_{r\to\infty} \frac{\alpha(\log \overline{M}(F^*, r))}{\alpha(\log r)} = \lambda(\alpha, \alpha, F^*)^{\alpha},$$

where $\overline{M}(F^*, r) = \max_{z\in B_r} \mid F^*(z, 0) \mid$.

Proof. By ([6], Lemma 1) the lemma follows for the associate f^*. Using the reasoning of McCoy[8], it can be easily seen that generalized orders of entire GBASP F^* are same as f^*. Hence the proof is completed.

Lemma 2. *If $F^*(z, 0)$ be a real valued entire function GBASP defined as earlier. Then*

$$\rho = (\alpha, \alpha, F^*) = \vartheta(H) \text{ and } \lambda(\alpha, \alpha, F^*) = (\vartheta), \text{ where} \tag{2.1}$$

$$H = \limsup_{r\to\infty} \frac{\alpha(v(r))}{\alpha(\log r)}$$
$$\text{and} \quad \theta = \liminf_{r\to\infty} \frac{\alpha(v(r))}{\alpha(\log r)},$$

and $v(r)$ denote the rank of maximum term of F^* on B.

Proof of this lemma follows on the lines of ([5], Thm.3).

Lemma 3. *If $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$ be the restriction to B of an entire function GBASP having generalized growth parameters $\rho(\alpha, \alpha, F^*)$ and $\lambda(\alpha, \alpha, F^*)$. Then, $g(z) = \sum_{n=0}^{\infty} b_n z^{2n}$, $b_{n,s}$ are given by (1.1) is also an entire function.*
Further, we have

$$\rho(\alpha, \alpha, F^*) = \rho(\alpha, \alpha, g) \text{ and } \lambda(\alpha, \alpha, F^*) = \lambda(\alpha, \alpha, g).$$

Lemma 4. *If $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$ be the restriction to B of an entire function GBASP having generalized growth parameters $\rho(\alpha, \alpha, F^*)$ and $\lambda(\alpha, \alpha, F^*)$. Then, $g'(z) = \sum_{n=0}^{\infty} E^p_n(F^*) z^{2n}$ is also an entire function.*
Further, we have

$$\rho(\alpha, \alpha, F^*) = \rho(\alpha, \alpha, g') \text{ and } \lambda(\alpha, \alpha, F^*) = \lambda(\alpha, \alpha, g').$$

Proof of Lemmas 3 and 4. By the application of ([8], Thm.1) we can see that F^* is entire if and only if its associate f^* is entire. These lemmas follows in the same manner as ([6], Lemma 2, 3) for the associate f^* and hence holds for F^*.

3. Main Results

Theorem 1. *If $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$ be the restriction to B of an entire function GBASP having generalized order $\rho(\alpha, \alpha, F^*)$ and generalized lower order $\lambda(\alpha, \alpha, F^*)$. Then*

(i) $\rho(\alpha, \alpha, F^*) = \vartheta(L)$
(ii) $\rho(\alpha, \alpha, F^*) = \vartheta(L^*)$,

$$\text{where} \qquad L = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\Big(\log(E^p_{n-1}(F^*)/E^p_n(F^*))\Big)}.$$

$$\text{and} \qquad L^* = \limsup_{n\to\infty} \frac{\alpha(n)}{\alpha\Big(-\frac{1}{n}\log E^p_n(F^*)\Big)}$$

(iii) $\lambda(\alpha, \alpha, F^*) \ge \vartheta(l')$, *where*

$$l' = \liminf_{n\to\infty} \frac{\alpha(n)}{\alpha\Big(-\frac{1}{n}\log E^p_n(F^*)\Big)}.$$

(iv) *If we take $\alpha(x) = \alpha(a)$ on $(-\infty, a)$, then $\lambda(\alpha, \alpha, F^*) \ge \vartheta(l^*)$,*

$$l^* = \liminf_{n\to\infty} \frac{\alpha(n)}{\alpha\Big(\log(E^p_{n-1}(F^*)/E^p_n(F^*))\Big)}.$$

Theorem 2. *If $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$ be the restriction to B of an entire function GBASP having generalized order $\rho(\alpha, \alpha, F^*)$ and generalized lower order $\lambda(\alpha, \alpha, F^*)$. If $E^p_n(F^*)/E^p_{n-1}(F^*)$ is nondecreasing. Then*

(i) $\rho(\alpha, \alpha, F^*) = \vartheta(L) = \vartheta(L^*)$
(ii) $\lambda(\alpha, \alpha, F^*) = \vartheta(l') = \vartheta(l^*)$.

Theorem 3. *If $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$ be the restriction to B of an entire function GBASP having generalized lower order $\lambda(\alpha, \alpha, F^*)$. Then, (i) if $\alpha(x) \in \Omega$, we have*

$$\lambda(\alpha,\alpha,F^*) = \max_{\{n_k\}}[\vartheta_\xi(l'')] \tag{3.1}$$

and, further, if we take $\alpha(x) = \alpha(a)$ on $(-\infty, a)$, then

$$\lambda(\alpha,\alpha,F^*) = \max_{\{n_k\}}[\vartheta_\xi(l'^*)], \tag{3.2}$$

where

$$\begin{aligned} \xi \equiv \xi(n_k) &= \liminf_{k\to\infty} \frac{\alpha(n_{k-1})}{\alpha(n_k)}, \\ l'' \equiv l''(n_k) &= \liminf_{n\to\infty} \frac{\alpha(n_{k-1})}{\alpha\Big(-\frac{1}{n_k}\log E^p_{n_k}(F^*)\Big)} \\ \textit{and} \qquad l'^* &= \liminf_{n\to\infty} \frac{\alpha(n_{k-1})}{\alpha\Big(\frac{1}{n_k-n_{k-1}}\log(E^p_{n_{k-1}}(F^*)/E^p_{n_k}(F^*))\Big)}. \end{aligned}$$

The maximum in (3.1) *and* (3.2) *is taken over all increasing sequence* $\{n_k\}$ *of positive integers.*

Further, if $\{n_m\}$ is the sequence of principal indices of the entire function $g'(z) = \sum_{n=0}^{\infty} E^p_n(F^*) z^{2n}$ and $\alpha(n_m) \approx \alpha(n_{m+1})$ as $\to \infty$, then (3.1) and (3.2) also hold for $\alpha(x) \in \overline{\Omega}$.

Proof of Theorems 1, 2, 3. These theorems follows easily from ([5], Thms 4-6, Lemma 1) and Lemma 4 of this paper.

For $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$, let $\{n_k\}$, $n_0 = 0$ be a sequence of positive integers such that

$$E^p_{n_{k-1}}(F^*) > E^p_{n_k}(F^*) \text{ and } E^p_n(F^*) = E^p_{n_{k-1}}(F^*), \text{ for } n_{k-1} \le n \le n_k. \quad k = 1, 2, \ldots \tag{3.3}$$

We now prove a theorem that shows how this sequence influences the growth of an entire GBASP on C^N.

Theorem 4. *If $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$ be the restriction to B of an entire function GBASP and θ and H be defined as in* (2.1)*. Then*

$$\theta \le H \liminf_{k\to\infty} \frac{\alpha(n_k)}{\alpha(n_{k+1})}$$

Proof. Let $R = \liminf_{k\to\infty} \frac{\alpha(n_k)}{\alpha(n_{k+1})}$. If $P > R$, there exists a sequence $\{c(k)\}$ such that $\alpha(n_{c(k)}) < P\alpha(n_{c(k)+1})$. Let r_t be a value of r at which $v(r)$ jumps from a value less than or equal to $n_{c(t)}$ to a value greater than or equal to $n_{c(t)+1}$. Since $\alpha(v(r_t-0)) \le \alpha(n_{c(t)}) \le P\alpha(n_{c(t)+1}) \le P\alpha(v(r_t+0))$, we have $\theta \le \limsup_{t\to\infty} \frac{\alpha(v(r_t-0))}{\alpha(\log r)} < P \limsup_{t\to\infty} \frac{\alpha(v(r_t+0))}{\alpha(\log r)} \le PH$.

Since this inequality is true for every $P > R$, so we have $\theta \le PH$. This proves the theorem.

The following Corollaries follows easily from Theorem 4.

Corollary 1. *If $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$ be the restriction to B of an entire function GBASP having generalized growth parameters $\rho(\alpha,\alpha,F^*)$ and $\lambda(\alpha,\alpha,F^*)$. Then for $\alpha \in \Omega$,*

$$\lambda(\alpha,\alpha,F^*) \le \rho(\alpha,\alpha,F^*) \liminf_{k\to\infty} \frac{\alpha(n_k)}{\alpha(n_{k+1})}, \tag{3.4}$$

and for $\alpha \in \overline{\Omega}$

$$(\lambda(\alpha,\alpha,F^*) - 1) \le (\rho(\alpha,\alpha,F^*) - 1) \liminf_{k\to\infty} \frac{\alpha(n_k)}{\alpha(n_{k+1})}. \tag{3.5}$$

where n_k is defined by (3.3).

Corollary 2. *If $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$ be the restriction to B of an entire function GBASP with generalized regular growth having generalized order $\rho (b < \rho < \infty)$ then $\alpha(n_k) \sim \alpha(n_{k+1})$ as $k \to \infty$, b being defined as $b = 0$, if $\alpha \in \Omega$ and $b = 1$, if $\alpha \in \overline{\Omega}$.*

Remark. (3.4) generalizes a result of Juneja, Kapoor and Bajpai [4]

Corollary 3. *$F^* \in L^p_\mu(B)$, $1 \le p \le \infty$ be the restriction to B of an entire function GBASP having generalized order ρ. Let $\{n_k\}$ be the sequence of principal indices and $\{\xi(n_k)\}$ be the jump points of the rank $v(r)$.*

Then $\rho(\alpha,\alpha,F^*) = \vartheta(U)$, where

$$U = \limsup_{k\to\infty} \frac{\alpha(n_k)}{\alpha\Big(\log(\xi(n_k))\Big)}.$$

Now we prove a decomposition theorem.

Theorem 5. Decomposition Theorem. *If $F^* \in L^p_\mu(B)$, $1 \le p \le \infty$, be the restriction to B of an entire function GBASP having generalized growth parameters $\rho(\alpha,\alpha,F^*)$ and $\lambda(\alpha,\alpha,F^*)$ and μ^* be a number such that $\lambda(\alpha,\alpha,F^*) < \mu^* < \rho(\alpha,\alpha,F^*)$, then $F^*(z,0) = g^*(z,0) + h^*(z,0)$ where generalized order of $g^*(z,0)$ in less than or equal to μ^* and $h^*(z,0) = \sum_{k=0}^{\infty} P_k(e^*) z^{m_k} (P_k(e^*) \ne 0$ for all $k)$ satisfies*

$$\lambda(\alpha,\alpha,F^*) \ge \mu^* \liminf_{k\to\infty} \frac{\alpha(m_k)}{\alpha(m_{k+1})}. \tag{3.6}$$

Proof. Let $g^*(z,0) = \sum_{k=0}^{\infty} d_k z^k$, and further $d_k = P_k(e^*)$ if $\log \mid P_k(e^*) \mid < -kG[k,c]$ where $c = \frac{1}{(\mu^*-A+\varepsilon)}$, A being equal to 1 if $\alpha \in \overline{\Omega}$ and 0 if $\alpha \in \Omega$. Then $g^*(z,0)$ is an entire function. It follows easily that the generalized order ρ^* of $g^*(z,0)$ satisfies $\rho^* = \vartheta(L) \leq \vartheta(\mu^* - A) = \mu^*$. Now, let $h^*(z,0) = F^*(z,0) - g^*(z,0) = \sum_{k=0}^{\infty} P_k(e^*) z^{m_k}$ and put $A_k = \mid P_k(e^*) \mid$. Then,

$$\log A_k > -m_k G[m_k, c], \tag{3.7}$$

Let $\alpha \in \Omega$, then for $r_k \leq r < r_{k+1}$, by Cauchy inequality,

$$\log M(F^*, r) \geq \log A_k + m_k \log r_k, \tag{3.8}$$

Choose $\log r_k = 1 + G[m_k, c]$. By (3.7) and (3.8) we get

$$\log M(F^*, r) > -m_k G[m_k, c] + m_k(1 + G[m_k, c]) = m_k.$$

Hence, $$\log M(F^*, r) > \alpha^{-1}\left[\frac{1}{c}\alpha\Big\{\log\left(\frac{r_k}{e}\right)\Big\}\right],$$

or $$\frac{\alpha(\log M(F^*, r))}{\alpha(\log r)} > \frac{\frac{1}{c}\alpha\{\log(\frac{r_k}{e})\}}{\alpha(\log(r_{k+1}))},$$

or, in view of (iii), $\frac{\alpha(\log M(F^*,r))}{\alpha(\log r)} > \frac{1}{c}\frac{\alpha(m_k)}{\alpha(m_{k+1})}$.

Proceeding to limits we get $\lambda(\alpha, \alpha, F^*) \geq \mu^* \liminf_{k\to\infty} \frac{\alpha(m_k)}{\alpha(m_{k+1})}$.

Now let $\alpha \in \overline{\Omega}$. Then for $r_k \leq r < r_{k+1}$, choose $\log r_k = c\ \exp\{\alpha(G[m_k, c])\} + G[m_k, c]$. By (3.7) and (3.8) we have

$$\begin{aligned}\log M(F^*, r) &\geq -m_k G[m_k, c] + c\ m_k \exp\{\alpha(G[m_k, c])\} + m_k G[m_k, c].\\ &= c\ m_k \exp\{\alpha(G[m_k, c])\}.\end{aligned}$$

$\Rightarrow$

$$\alpha(\log M(F^*, r)) \geq \alpha(c\ m_k \exp\{G[m_k, c]\}).$$

Since $\alpha \in \overline{\Omega}$ we get $\alpha(\log M(F^*, r)) \geq \alpha(c\ m_k) + \alpha\{G[m_k, c]\}x$

$$\frac{d\alpha(x)}{d(\log x)}\ \Big|_{x=x^*(m_k)}$$

where $c(m_k) < x^* < c\ m_k \exp\{\alpha(G[m_k, c])\}$,which gives

$$\frac{\alpha(\log M(F^*, r))}{\alpha(\log r)} > \frac{\alpha(m_k)(1+c)}{\alpha(\log r_{k+1})}.$$

Since $\alpha(\log(r_{k+1})) \approx c\alpha(m_{k+1})$. Hence

$$\frac{\alpha(\log M(F^*, r))}{\alpha(\log r)} > \frac{\alpha(m_k)(1+c)}{c\alpha(m_{k+1})} = (\mu^* + \varepsilon)\frac{\alpha(m_k)}{\alpha(m_{k+1})}.$$

Proceeding to limits, we get $\lambda(\alpha, \alpha, F^*) \geq \mu^* \liminf_{k\to\infty} \frac{\alpha(m_k)}{\alpha(m_{k+1})}$. Hence the proof is completed.

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