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τ-Atomicity and Quotients of Size Four

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Abstract. Given a ring R, an ideal I of R, and an element $a \in I$, we say $a = \lambda b_1 \cdots b_k$ is a τ_I-factorization of a if λ is any unit and $b_1 \equiv \cdots \equiv b_k \pmod{I}$. In this paper, we investigate the τ_I-atomicity of PIDs with ideals where R/I has size four.

1 Introduction

Originally introduced by [1], τ-factorization is a generalized form of factorization in which all factors of the factorization must satisfy a relation. One of the most natural relations is to insist that all factors of a factorization be equivalent modulo a fixed ideal I, which is called τ_I-factorization. The first appearance in the literature in on τ-factorization can be found in [3]. It is further studied in [2, 4, 5, 8, 10]. In this paper, we ask, given any PID, what is the smallest quotient such that R/I fails to be τ_I-atomic. We show that it is possible to construct a PID, R with an ideal I, such that $|R/I| = 4$ and R fails to be τ_I-atomic. We also show that one can find particular PIDs R and ideals $I \subset R$ such R is τ_I-atomic regardless of the size of the quotient. We begin with the more general definition of τ_I-factorization.

Let R is a commutative domain with nonzero identity and I an ideal of R, for $a \in R$ we say $a = \lambda b_1 \cdots b_k$ is a τ_I-*factorization* of a if $b_1 \equiv \cdots \equiv b_k \pmod{I}$ and λ is any unit from the domain. Recall, $\lambda \in R$ is a *unit* if there exists a $\lambda' \in R$ such that $\lambda\lambda' = 1$. We say that a is a τ_I-*atom* if all τ_I-factorizations of a are length one, that is, all τ_I-factorizations of a are of the form $a = \lambda(\lambda^{-1}a)$ where λ is a unit in R. We say R is τ_I-*atomic* if every non-zero non-unit has a τ_I-factorization into a finite product of τ_I-atoms. For example, if we consider $R = \mathbb{Z}$ with the principal ideal $I = (2)$, then $20 = 2 \cdot 10$ is a τ_I-factorization, whereas $20 = 4 \cdot 5$ is not since 4 and 5 are of different parity. Moreover $20 = 2 \cdot 10$ is an τ_I-atomic factorization of 20, as both 2 and 10 are τ_I-atoms. More information about τ_I-factorizations in $\mathbb{Z}$ can be found in [4, 5]. It is in this spirit that we extend the study of τ_I-factorization to any PID.

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On the surface, τ-factorization may seem like just another generalization of factorization. This however, is far from the truth, as most if not all factorization being studied can be thought of in terms of τ-factorization. For example if one takes $I = R$, then τ_I-atomic factorization is just the traditional factorization of elements into atoms or primes. Other forms of factorization, such as comaximal factorization (studied in [6]) is also generalized by τ-factorization.

We are interested to know what effect the size of R/I has, if any, on τ_I-factorization. In particular, we are focusing on the smallest size of R/I such that R is not necessarily τ_I-atomic. This occurs when $|R/I| = 4$. We leave to the reader to verify that if $|R/I| = 2$, or 3, then R is always τ_I-atomic. The only quotient rings that are commutative with unity and are of size 2 and 3 are $\mathbb{Z}_2$ and $\mathbb{Z}_3$ respectively. The arguments are simple variations of the lemmas we present. At the end of the paper we show that R/I can have varying sizes and still be τ_I-atomic. In particular $|R/I|$ can be infinite.

In the problems and solutions section of the October 1964 edition of the *American Mathematical Monthly* [11], it was proven that there are 11 rings with four elements. Of these 11, there are four commutative rings with identity. These rings are $\mathbb{Z}_4$, $\mathbb{F}_4$, $\mathbb{Z}_2[x]/(x^2+x)$, and $\mathbb{Z}_2[x]/(x^2+1)$. These four rings correspond to the four cases that require our scrutiny.

2 Main Results

For this paper, we focus on the instances where R is a PID and I is an ideal of R with $|R/I| = 4$. We show that under very specific circumstances it is possible for R to not be τ_I atomic. The interplay between units of R and R/I requires much attention as we will see. We will also see that having equivalence classes void of primes from R will also play a crucial role. The exact theorem we prove is presented below followed by several lemmas leading to the result.

Theorem 2.1. *Let R be a PID and I and ideal of R with $|R/I| = 4$. R is τ_I-atomic if and only if R/I fails to satisfy all of the following conditions:*

1. *$R/I \cong \mathbb{F}_4$;*
2. *the only units of R are contained in the $\bar{1}$ class of R/I; and*
3. *R contains a prime in the two classes different from the $\bar{0}$ and $\bar{1}$ classes.*

We begin with a brief note about units.

Remark 1. Let R be a ring with unity and let $I \subset R$ be a proper ideal of R. If $\lambda \in R$ is a unit of R, then $I + \lambda$ is a unit of R/I. Note that the converse of this statement is not true in general (e.g. -3 is a unit in $\mathbb{Z}_4$ but is not a unit in $\mathbb{Z}$).

Care must be taken because we can have an element $a \in R/I$ and multiply by a unit $\lambda \in R$ and have a and λa in different classes of R/I. To tackle this problem in the following lemmas, we apply the following fact. If $I + \lambda$ is not a unit in R/I, then λ is not a unit in R (meaning that there are no units in the $I + \lambda$ class). We show that in the first three lemmas are true regardless of the unit group of R.

Lemma 2.1. *Let R be a PID and I an ideal of R. If $R/I \cong \mathbb{Z}_4$, then R is τ_I-atomic.*

Proof. Since we are assuming R is a PID and $1 \neq 0$, R must have units in $\bar{1}$ and $\bar{3}$ classes. The factorization of a that we write puts no restrictions on k, l, and s, meaning that we are not assuming there is a prime in every nonzero class of the quotient. This will be the same in the next two lemmas. Also we know there are no primes in $\bar{0}$ class, as R/I is not a domain. This will also be the same in the next two lemmas.

Let $a \in R$ and let

$$a = p_1 \cdots p_k q_1 \cdots q_l r_1 \cdots r_s$$

be the unique factorization of a into primes such that we have $p_1, \ldots, p_k \equiv 1 \pmod{I}, q_1, \ldots, q_l \equiv 2 \pmod{I}$, and $r_1, \ldots, r_k \equiv 3 \pmod{I}$. Since I is principal and R/I is not an integral domain, there are no primes congruent to 0 modulo I. This will be true in the next two lemmas as well. Also, as $\bar{2} \in \mathbb{Z}_4$ is a zero divisor, there are no units in R congruent to 2 modulo I.

Then for any $q_i \equiv 2 \pmod{I}$, multiplication by a unit does not change the class, that is, for any $\lambda \in R$, we have $\lambda q_i \equiv 2 \pmod{I}$. Also, note that since there are no units in R congruent to 2 modulo I, it is not possible to multiply p_i or r_i by a unit and change the class to 2 modulo I.

Case 1: Suppose $a \equiv 0, \text{ or } 2 \pmod{I}$, then $l \geq 1$. If $l = 1$, then a is already a τ_I-atom. If $l > 1$, then

$$q_1 \cdots q_{l-1}(q_l p_1 \cdots p_k r_1 \cdots r_s)$$

is a τ_I-atomic factorization of a, with all factors being atoms congruent to 2 modulo I. To see that $q_l p_1 \cdots p_k r_1 \cdots r_s$ is an atom, note that if it factored one of the factors would need to contain q_l, meaning this factor would need to be congruent to 2 modulo I. The fact that we can not factor out a unit $\lambda \in R$ from any p_k or r_k, such that $\lambda^{-1} p_k \equiv 2 \pmod{I}$ or $\lambda^{-1} r_k \equiv 2 \pmod{I}$ shows that whatever remaining factors are not congruent to 2 modulo I, hence it must be the case that $q_l p_1 \cdots p_k r_1 \cdots r_s$ is a τ_I-atom.

Case 2: If $a \equiv 1, \text{ or } 3 \pmod{I}$, then $l = 0$. Thus $a = p_1 \cdots p_k r_1 \cdots r_s$, and

$$a = (-1)^s p_1 \cdots p_k (-r_1) \cdots (-r_s)$$

is a τ_I-atomic factorization of a, with all factors being atoms congruent to 1 modulo I.

Note here that we are using the tools from Remark 1 here. In particular, we observe that $r_i \equiv 3 \pmod{I}$ and multiplication by the unit -1 yields $-r_i \equiv 1 \pmod{I}$. Based on the fact that all units in R/I live in the equivalence classes of 1 or 3, we observe that for any $a \equiv 1,$ or $3 \pmod{I}$, multiplication by any unit $\lambda \in R/I$ yields $\lambda a \equiv 1,$ or $3 \pmod{I}$. Lastly, it should be clear that all factors in the factorization are τ_I-atoms as they are prime in R and cannot be factored. □

Lemma 2.2. *Let R be a PID and I an ideal of R. If $R/I \cong \mathbb{Z}_2[x]/(x^2+x)$, then R is τ_I-atomic.*

Proof. For the benefit of the reader, we provide a Cayley table describing multiplication in R/I.

	1	x	$1+x$
1	1	x	$1+x$
x	x	x	0
$1+x$	$1+x$	0	$1+x$

Since the x and $1+x$ are zero divisors, all units of R/I must live in the equivalence class of 1; see Remark 1. Therefore, in this case, we need not worry about multiplication by a unit causing an element to change class. We are also aided by the fact that x and $1+x$ are idempotent.

Let a be a non-zero element in the ring. We write the factorization of a as follows:

$$a = p_1 \cdots p_k q_1 \cdots q_l r_1 \cdots r_s$$

where $p_1, \ldots, p_k \equiv 1 \pmod{I}$, $q_1, \ldots, q_l \equiv x \pmod{I}$, and $r_1, \ldots, r_s \equiv 1+x \pmod{I}$. There are two main cases to consider.

Case 1: Suppose $a \in I$. Then $l > 0$ and $s > 0$. If $l > s$, we have the τ_I-atomic factorization $a = (q_1 r_1) \cdots (q_{s-1} r_{s-1})(q_s r_s p_1 \cdots p_k q_{s+1} \cdots q_l)$. Note here that if $s = 1$, then $a = q_1 r_1 p_1 \cdots p_k q_2 \cdots q_l$ is already a τ_I-atom. If $l < s$, we have the τ_I-atomic factorization

$$a = (q_1 r_1) \cdots (q_{l-1} r_{l-1})(q_l r_l p_1 \cdots p_k r_{l+1} \cdots r_s).$$

Note here that if $l = 1$, then $a = q_1 r_1 p_1 \cdots p_k r_2 \cdots r_s$ is already a τ_I-atom. And if $l = s$, we have the τ_I-atomic factorization

$$a = (q_1 r_1) \cdots (q_{s-1} r_{s-1})(q_s r_s p_1 \cdots p_k).$$

In all three cases, the atoms in the factorizations are equivalent to $0 \pmod{I}$.

To see that $(q_s r_s p_1 \cdots p_k q_{s+1} \cdots q_l)$ is an atom, note that any factorization would have to contain a factor congruent to 0 modulo I (if it contained both a factor from the $\bar{x}$ and $\overline{x+1}$ classes. But then it would be impossible for the other factors to be equivalent to 0 modulo I. Similarly, any other factorization would necessarily contain a factor in the $\bar{x}$ class and a factor in the $\overline{x+1}$ class, which would not be a τ_I-factorization. The lack of units outside the $\overline{1}$ class is important in these arguments. Similar observations allows us to see the other "longer" factors are indeed τ_I-atoms.

Case 2: Suppose $a \notin I$. Then either $s = 0$ or $l = 0$. If $s = 0$, we have the τ_I-atomic factorization

$$\begin{aligned} a = (p_1q_1)\cdots(p_kq_k)(q_{k+1})\cdots(q_l) & \quad \text{if } k < l; \\ a = (p_1q_1)\cdots(p_{l-1}q_{l-1})(q_lp_1\cdots p_k) & \quad \text{if } k > l; \text{ and} \\ a = (p_1q_1)\cdots(p_lq_l) & \quad \text{if } k = l. \end{aligned}$$

The case where $l = 0$ is similar. Thus R is τ_I-atomic. □

Lemma 2.3. *Let R be a PID and I an ideal of R. If $R/I \cong \mathbb{Z}_2[x]/(x^2+1)$, then R is τ_I atomic.*

Proof. For the benefit of the reader, we provide a Cayley table describing multiplication in R/I.

	1	x	$1+x$
1	1	x	$1+x$
x	x	1	$1+x$
$1+x$	$1+x$	$1+x$	0

Since $1+x$ is zero divisors, all units of R/I must live in the equivalence classes of 1 or x; see Remark 1. Then for any element in the class of $1+x$ modulo I, multiplication by a unit does not change the class.

Let $a = p_1\cdots p_kq_1\cdots q_lr_1\cdots r_s$ be the unique factorization of a into primes such that $p_1,\ldots,p_k \equiv 1 \pmod I$, $q_1,\ldots,q_l \equiv x \pmod I$, and $r_1,\ldots,r_s \equiv 1+x \pmod I$.

Case 1: Now if $a \equiv 0$, we must have $s \geq 2$, as $1+x$ is the only zero divisor in the quotient. Then $a = r_1\cdots r_{s-1}(r_sp_1\cdots p_kq_1\cdots q_l)$ is a τ_I-atomic factorization of a with all elements in the product atoms congruent to $1+x \pmod I$. Note anything times something in the $\overline{x+1}$ class is in the $\overline{x+1}$ class. And two elements outside the $\overline{x+1}$ class cannot multiply to be in the $\overline{x+1}$ class.

Case 2: If $a \equiv 1+x \pmod I$, then $s = 1$ and hence a is a τ_I-atom. If $a \equiv 1$ or $x \pmod I$, then $s = 0$ and $a = q_1\cdots q_{l-1}(q_lp_1\cdots p_k)$ is a τ_I-atomic factorization of a with every element in the product an atom equivalent to x modulo I, provided that there are no units $\lambda \equiv x \pmod I$. If there exists a unit $\lambda \equiv x \pmod I$, then we can multiply each element q_i by λ and we have a τ_I-factorization where every element is equivalent to 1 modulo I. That is $a = \lambda^{-s}p_1\cdots p_k(\lambda q_1)\cdots(\lambda q_l)$ is a τ_I-factorization of a. Note we could have also multiplied the p_i by λ to make them congruent to the q_i.

□

The following theorem shows that if R/I has a unit in every nonzero class, then R is always τ_I-atomic.

Theorem 2.2. *Let R be a PID, and $M \subset R$ be a maximal ideal such that R/M has a unit in every non-zero class. Then R is τ_M-atomic.*

Proof. First note that we are not assuming anything special about M be maximal, this is forced by the condition that there is a unit in every nonzero class, hence R/M is a field implying that M is maximal.

Now, since R is a PID, we have $M = (p)$ for some prime $p \in R$. Suppose $a \equiv 0 \pmod{M}$. Then $a = p^k l$ for some positive integer k and some $l \notin M$. Now a has the τ_M-atomic factorization $a = p \cdots p(pl)$ which is a τ_M-atomic factorization of length k with all factors congruent to zero modulo M. Note it is important that M is a maximal (hence prime) ideal so that we can be assured that l cannot factor into a product that is in M.

Now suppose $a \not\equiv 0 \pmod{M}$. Then we can write $a = p_1 p_2 \cdots p_k$ as a product of primes, none of which are zero modulo M. Now, as R/M is a field with a unit in every nonzero class, we can find a unit λ_i in the p_i^{-1} class. Hence $\lambda_i p_i \equiv 1 \pmod{M}$. Now we can have the following τ_M-atomic factorization of $a = (\lambda_1 \cdots \lambda_k)^{-1}(\lambda_1 p_1)(\lambda_2 p_2) \cdots (\lambda_k p_k)$, where all factors are equivalent to 1 modulo M. □

This situation can happen with the quotient isomorphic to $\mathbb{F}_4$. To see this, set $\alpha = \frac{1+\sqrt{5}}{2}$. Now the ring $\mathbb{Z}[\alpha]$ is a PID with a unit group generated by -1 and α. Thus, its unit group is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}$. Note $-1 + \alpha = \alpha^{-1}$. Now the prime 2 remains inert in this extension, thus $\mathbb{Z}[\alpha]/2\mathbb{Z}[\alpha] \cong \mathbb{F}_4$. Thinking of $\mathbb{Z}[\alpha]/2\mathbb{Z}[\alpha] = \{\bar{a} + \bar{b}\alpha : \bar{a}, \bar{b} \in \mathbb{Z}_2\}$, it is easy to see that $-1, \alpha$, and α^{-1} are all units in R with each one in a different non-zero class.

Lemma 2.4. *Let R be a PID with I an ideal of R such that R/I has a prime in every class. If $R/I \cong \mathbb{F}_4$ with all units of R in the $\bar{1}$ class, then R is not τ_I-atomic.*

Proof. If we label the four elements of $\mathbb{F}_4$ as $0, 1, a, b$, then we have the following Cayley table with respect to multiplication.

	1	a	b
1	1	a	b
a	a	b	1
b	b	1	a

Now there exists primes p and q with $p \equiv a \pmod{I}$ and $q \equiv b \pmod{I}$. Consider $d = p^2q$. There does not exist a unit $\lambda \in R$ with $\lambda p \equiv q \pmod{I}$ or equivalently $\lambda q \equiv p \pmod{I}$ as this would imply $\lambda \not\equiv 1 \pmod{I}$. Thus the only τ_I-factorization is $d = (p^2)q$. But p^2 is not a τ_I-atom, as $p^2 = pp$ which is a nontrivial τ_I factorization of p^2. Thus R is not τ_I-atomic. It should be pointed out that if there was a unit outside of the $\bar{1}$ class, then there is a unit in every nonzero class of R/I allowing us to apply the previous theorem. □

An example of such a ring is $R = \mathbb{Z}_2[x]$ with $I = (x^2 + x + 1)$. This has the quotient $R/I \cong \mathbb{F}_4$ and fails to be τ_I-atomic. R has a prime in every class, namely x^2+x+1, x^4+x+1, x, and $1 + x$ are primes in the $0, 1, x$, and $1 + x$ classes respectively. Also, it is straight forward to see that 1 is the only unit in R

To finish the proof of the theorem we need one final lemma

Lemma 2.5. *Suppose that $R/I \cong \mathbb{F}_4$ with all units of R in the $\bar{1}$ class. Suppose further R does not contain a prime in both the $\bar{a}$ and $\bar{b}$ classes, then R is τ_I-atomic.*

Proof. Let $w \in R$ with $w = p^k q_1 q_2 \cdots q_s r_1 r_2 \cdots r_t$ be a prime factorization with $I = (p)$, $q_i \equiv 1 \pmod{I}$, and all $r_i \equiv a \pmod{I}$ or all $r_i \equiv b \pmod{I}$. If $k \neq 0$, it is clear that $w = pp \cdots (pq_1 q_2 \cdots q_s r_1 r_2 \cdots r_t)$ is a τ_I-atomic factorization of w regardless of the values of s and t. If $k = 0$, then $w = (r_1 q_1)(r_2 q_2) \cdots (r_t q_t \cdots q_s)$ is a τ_I-atomic factorization of w when $t \leq s$ and $w = (r_1 q_1)(r2q_2) \cdots (r_s q_s) r_{s+1} \cdots r_t$ is a τ_I-atomic factorization of w when $t > s$. Note that these factorizations work if $t = 0$ or $s = 0$, hence we are not insisting there be a prime outside the $\bar{0}$ class. Thus R is τ_I-atomic in this case. □

This completes the proof of the main result and shows all conditions are necessary and sufficient.

To see the previous lemma in action, consider the ring $R = \mathbb{F}_4[[x]]$, the power series ring over $\mathbb{F}_4$. This is a PID with only one prime x. If we let $I = (x)$, then $R/I \cong \mathbb{F}_4$. Moreover, since any power series with a non-zero constant term is a unit, we have for $f \in R$, $f = x^n g$ where g has a non-zero constant term. Since g is a unit, this is a τ_I-atomic factorization of length n with every atom congruent 0 modulo I. This example can be generalized to any field. For example let $R = \mathbb{Q}[[x]]$ and $I = (x)$, then $R/I \cong \mathbb{Q}$ and is τ_I-atomic.

One application of this result is that the Gaussian integers are $\tau_{(2)}$-atomic. Similar questions could be explored in rings with larger quotients, although the number of cases to consider would grow dramatically. Studying how the complexity increases as the size of the quotient grows would make for an interesting project. The factorizations in this paper can be highly non-unique. This indicates that another natural area of study is τ_I-elasticity which could produce some interesting results.

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