TAMKANG JOURNAL OF MATHEMATICS
Volume 33, Number 3, Autumn 2002

S-CLUSTER SETS IN FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper the concept of S-cluster fuzzy sets, of fuzzy functions and fuzzy multifunctions between fuzzy topological spaces is introduced. As an application, characterizations of fuzzy Hausdorff and SQ_α-closed fuzzy topological spaces are achieved via such cluster fuzzy sets.

1. Introduction

The theory of cluster sets was developed long ago, and was initially aimed at the investigations of real and complex function theory. A comprehensive collection of works in this direction can be found in the classical book of Collingwood and Lohwater [4]. Weston [14] was the first to initiate the corresponding theory for functions between topological spaces basically for studyng compactness. The present paper is intended for to introduce the concept of S-cluster fuzzy sets of fuzzy functions and fuzzy multifunctions, which provides a new technique for studying SQ_α-closedness of fuzzy topological spaces. It is shown that such cluster fuzzy sets of suitable fuzzy function can characterize fuzzy Hausdorffness. Finally, we achieve, as our prime motivation, certain characterizations of SQ_α-closed space.

Let X be a set of points and I be the unit interval [0,1]. A fuzzy set μ in X is a mapping from X into I. The class of all fuzzy sets on X denoted by I^X. For $x \in X$ and $\alpha \in (0,1]$, a fuzzy set x_α defined by

$$x_\alpha(y) = \begin{cases} \alpha & : \quad y = x \\ 0 & : \quad y \neq x \end{cases}$$

is called a fuzzy point in X. The class of all fuzzy points of X denoted by $FP(X)$. Let 0_X and 1_X be, respectively, the constant fuzzy sets taking 0 and 1 on X. For $A \subseteq X$, 1_A denotes the characteristic mapping of A. For every $x_\alpha \in FP(X)$ and $\mu \in I^X$, we write $X_\alpha \in \mu$ iff $\alpha \leq \mu(x)$. For every $\mu \in L^X$, denote $supp(\mu) = \{x \in X : \mu(x) > 0\}$, called it the support of μ. For any set $A \subseteq X$, we denote the cardinality of A by $|A|$. If $|A| = 1$,

Received March 30, 2001; revised October 03, 2001.
2000 *Mathematics Subject Classification*. 54D20, 54D30, 54C60, 54C99.
Key words and phrases. Fuzzy topoloty, Q-neighborhood, S-cluster fuzzy set, SQ_α-closedness, θS-closure, semi-open fuzzy sets.

say $A = \{x\}$, then A is called degenerate. A fuzzy set μ is called finite (resp. degenerate) if $|supp(\mu)|$ is finite (resp. $|supp(\mu) = 1$). A fuzzy set μ is called quasi-coincident with a fuzzy set ρ,denoted by $\mu q \rho$ [12], iff there exists $x \in X$ such that $\mu(x) + \rho(x) > 1$. If μ is not quasi-coincident with ρ, then we write $\mu \bar{q} \rho$.

In what follows, we use the concept of a fuzzy topological space (fts, for short) as introduced by Chang [3]. A fuzzy set $\mu \in I^X$ is called semi-open [9] if for some open fuzzy set η, $\eta \le \mu \le c\ell(\eta)$, where $c\ell(\eta)$ denotes the fuzzy closure of η in X. The complements of semi-open fuzzy sets are called semi-closed. Let $x_\alpha \in FP(X)$ and $\mu \in I^X$, by N_μ, $N^Q_{x_\alpha}$, $SON^Q_{x_\alpha}$ and $1_X \setminus \mu = \mu'$, we mean, the open neighbourhood system of μ, open Q-neighbourhood (Q-nbd, for short) system of x_α, the semi-open Q-neighbourhood ($S.Q$-nbd, for short) system of x_α and the pseudo-complement of μ. For any fuzzy set $\mu \in I^X$, the θ-closure [9] (θ-semiclosure [9]) of μ, denoted by $\theta.c\ell(\mu)$ (resp. $\theta S.c\ell(\mu)$), is defined by $x_\alpha \in \theta.c\ell(\mu)$ (resp. $x_\alpha \in \theta S.cl(\mu)$) iff for every $\eta \in N^Q_{x_\alpha}$ (resp. $\eta \in SON^Q_{x_\alpha}$), $c\ell(\eta) q \mu$. The fuzzy set μ is called θ-closed [9] (θ-semiclosed [9]) if $\mu = \theta.c\ell(\mu)$ (resp. $\mu = \theta S.c\ell(\mu)$). It is known [9] that $\theta.c\ell(\mu)$ need not be θ-closed, but it is so if μ is open.

Theorem 1.1.([12]) *Let* $\{\mu_j : j \in J\} \subseteq I^X$ *and* $x_\alpha \in FP(X)$. *Then:*

(i) $x_\alpha q \bigvee_{j \in J} \mu_j$ *iff* $(\exists j_0 \in J)(x_\alpha q \mu_{j0})$.

(ii) *If* $x_\alpha q \bigwedge_{j \in J} \mu_j$, *then* $(\forall_j \in J)(x_\alpha q \mu_j)$. *The converse is true if* J *is finite.*

Definition 1.2.([2]) A fuzzy grill on X is a nonempty subset $\Omega \subseteq I^X$ such that:

(i) $\mu \in \Omega$ and $\eta \le \mu$ implies $\eta \in \Omega$.

(ii) $\mu \vee \eta \in \Omega$ implies $\mu \in \Omega$ or $\eta \in \Omega$.

Definition 1.3.([11]) A fuzzy filterbase on X is a nonempty subset $\beta \subseteq I^X$ such that:

(i) $0_X \notin \beta$.

(ii) If $\mu_1, \mu_2 \in \beta$, then $\exists \mu_3 \in \beta$ such that $\mu_3 \le \mu_1 \wedge \mu_2$.

The fuzzy filter $\mathcal{F}$ generated by β is difined by $\mathcal{F} = \{\mu \in I^X : \eta \le \mu$ for some $\eta \in \beta\}$. A fuzzy filterbase $\mathcal{F}$ on a fts (X, τ) is said to θS-adhere at a fuzzy point $x_\alpha \in FP(X)$, denoted as $x_\alpha \in \theta S.adh(\mathcal{F})$ if $(\forall \eta \in SON^Q_{x\alpha})(\forall \lambda \in \mathcal{F})(c\ell(\eta) q \lambda)$. A fuzzy grill Ω on X is said to θS-converge to a fuzzy point $x_\alpha \in FP(X)$, if for each $\mu \in SON^Q_{x_\alpha}$, there corresponds some $\lambda \in \Omega$ with $\lambda \le c\ell(\mu)$.

Each mapping $f : I^X \to I^Y$ considered in this paper is induced from a crisp mapping $f : X \to Y$ as usual, i.e. for $\mu \in I^X$, $\eta \in I^Y$, $x \in X$ and $y \in Y$, we define $f(\mu)(y) = \bigvee \{\mu(x) : x \in X, f(x) = y\}$ and $f^{-1}(\eta)(x) = \eta(f(x))$.

2. S-cluster Fuzzy Set of Fuzzy Functions

Definition 2.1. Let $f : I^X \to I^Y$ be a function and $x_\alpha \in FP(X)$. The S-cluster fuzzy set of f at x_α, denoted by $S(f, x_\alpha)$ is given by $\wedge\{\theta.c\ell(f(c\ell(\mu))) : \mu \in SON^Q_{x_\alpha}\}$.

In the next theorem, we characterize the S-cluster fuzzy set of a function at some fuzzy point between fuzzy topological spaces.

Theorem 2.2. *For any function $f: I^X \to I^Y$ and $y_v \in FP(Y)$, the following statements are equivalent:*

(i) $y_v \in S(f, x_\alpha)$.

(ii) *The fuzzy filterbase $f^{-1}(c\ell(N^Q_{y_v}))$ θS-adheres at x_α.*

(iii) *There is a fuzzy grill Ω on X such that Ω θS-converges to x_α and $y_v \in \wedge\{\theta.c\ell(f(\mu)) : \mu \in \Omega\}$.*

Proof. (i) $\Longrightarrow$ (ii): Let $y_v \in S(f, x_\alpha)$. Then for each $\mu \in SON^Q_{x_\alpha}$ and each $\eta \in N^Q_{y_v}$, $c\ell(\eta) q f(c\ell(\mu))$. So, for each $\mu \in SON^Q_{x_\alpha}$ and each $\eta \in N^Q_{y_v}$, $f^{-1}(c\ell(\eta)) q\mu$ and so $f^{-1}(c\ell(\eta)) \wedge \mu \neq \emptyset$. It is easy to verify that the family $\{f^{-1}(c\ell(\eta)) : \eta \in N^Q_{y_v}\}$ is a fuzzy filterbase on X θS-adheres at x_α.

(ii) $\Longrightarrow$ (iii): Let $\mathcal{F}$ be the fuzzy filter on X generated by the fuzzy filterbase $f^{-1}(c\ell(N^Q_{y_v}))$. Then $\Omega = \{\mu \in I^X : \mu q\lambda$ for each $\lambda \in \mathcal{F}\}$ is a fuzzy grill on X. By (ii), for each $\rho \in SON^Q_{x_\alpha}$ and each $\eta \in N^Q_{y_v}$, $c\ell(\rho) q f^{-1}(c\ell(\eta))$. Hence $\lambda q c\ell(\rho)$ for each $\lambda \in \mathcal{F}$ and each $\rho \in SON^Q_{x_\alpha}$. Consequently, $c\ell(\rho) \in \Omega$ for all $\rho \in SON^Q_{x_\alpha}$, which proves that Ω θS-converges to x_α. Now, the definition of Ω yields that $f(\mu) q c\ell(\eta)$ for all $\eta \in N^Q_{y_v}$ and all $\mu \in \Omega$. Then $y_v \in \theta.c\ell(f(\mu))$ for all $\mu \in \Omega$. Hence $y_v \in \wedge\{\theta.c\ell(f(\mu)) : \mu \in \Omega\}$.

(iii) $\Longrightarrow$ (i): Let Ω be a fuzzy grill on X such that Ω θS-converges to x_α, and $y_v \in \wedge\{\theta.c\ell(f(\mu)) : \mu \in \Omega\}$. Then $\{c\ell(\rho) : \rho \in SON^Q_{x_\alpha}\} \subseteq \Omega$ and $y_v \in \theta.c\ell(f(\lambda))$ for each $\lambda \in \Omega$. Hence, in particular, $y_v \in \theta.c\ell(f(c\ell(\rho)))$ for all $\rho \in SON^Q_{x_\alpha}$. So $y_v \in \wedge\{\theta.c\ell(f(c\ell(\rho))) : \rho \in SON^Q_{x_\alpha}\} = S(f, x_\alpha)$.

Definition 2.3. A fts (X, τ) is called fuzzy Hausdorff space (FT_2, for short) iff $(\forall x_\alpha, y_v \in FP(X), x \neq y)(\exists \mu \in N^Q_{x_\alpha})(\exists \eta \in N^Q_{y_v})(\mu \bar{q} \eta)$.

In what follows, we show that S-cluster fuzzy sets of a function at some fuzzy point between fuzzy topological spaces may be used to assertain the fuzzy Hausdorffness of the codomain space.

Theorem 2.4. *Let $f : I^X \to I^Y$ be a function on a fts (X, τ) onto a fts (Y, Δ). Then (Y, Δ) is FT_2 if $S(f, x_\alpha)$ is degenerate for each $x_\alpha \in FP(X)$.*

Proof. Let $y^1_\alpha, y^2_v \in FP(X)$ such that $y^1 \neq y^2$. As f is a surjection, there are x^1_α, $x^2_v \in FP(X)$ such that $x^1 \neq x^2$ and $f(x^i) = y^i$ for $i = 1, 2$. Now, since $S(f, x_\alpha)$ is degenerate for each $x_\alpha \in FP(X)$, $y^2_v = f(x^2_v) \notin S(f, x^1_\alpha)$. Thus, there are $\eta \in N^Q_{y^2_v}$ and $\mu \in SON^Q_{x^1_\alpha}$ such that $c\ell(\eta) \bar{q} f(c\ell(\mu))$ and so $f(c\ell(\mu)) \leq 1_Y \setminus c\ell(\eta)$. Then $\eta \in N^Q_{y^2_\alpha}$, $1_Y \setminus c\ell(\eta) \in N^Q_{y^1_v}$ and $\eta \bar{q} (1_Y \setminus c\ell(\eta))$ which proves that (Y, Δ) is FT_2.

Definition 2.5.([7]) A function $f : I^X \to I^Y$ is called a fuzzy θS-irresolute iff for each $x_\alpha \in FP(X)$ and each $\eta \in SON^Q_{f(x_\alpha)}$, there is $\mu \in SON^Q_{x_\alpha}$ such that $f(c\ell(\mu)) \leq \eta$.

Theorem 2.6. *Let $f : I^X \to I^Y$ be a fuzzy θS-irresolute function with (Y, Δ) a FT_2-space. Then $S(f, x_\alpha)$ is degenerate for each $x_\alpha \in FP(X)$.*

Proof. Let $x_\alpha \in FP(X)$. As f is fuzzy θS-irresolute, for any $\eta \in SON^Q_{f(x_\alpha)}$, there is $\mu \in SON^Q_{x_\alpha}$ such that $f(c\ell(\mu)) \leq \eta$. Then $S(f, x_\alpha) = \wedge\{\theta.c\ell(f(c\ell(\mu))) : \mu \in SON^Q_{x_\alpha}\} \subseteq \wedge\{\theta.c\ell(\eta) : \eta \in SON^Q_{f(x_\alpha)}\}$. Let $y_\alpha \in FP(Y)$ with $y \neq f(x)$. As (Y, Δ) is FT_2, then there are $\rho_1 \in N^Q_{y_\alpha}$ and $\rho_2 \in N^Q_{f(x_\alpha)}$ such that $\rho_1 \bar{q} \rho_2$. Obviously, as $\rho_1 \bar{q} c\ell(\rho_2)$, $y_\alpha \notin c\ell(\rho_2) = \theta.c\ell(\rho_2)$. As $\rho_2 \in N^Q_{f(x_\alpha)} \subseteq SON^Q_{f(x_\alpha)}$, $y_\alpha \notin \wedge\{\theta.c\ell(\eta) : \eta \in SON^Q_{f(x_\alpha)}\}$ and hence $y_\alpha \notin S(f, x_\alpha)$. Thus $S(f, x_\alpha) = \{f(x_\alpha)\}$.

Combining the lase two results, we get the following characterization for the fuzzy Hausdorffness of the codomain space of a kind of function in terms of the degeneracy of its S-cluster fuzzy set.

Theorem 2.7. *$f : I^X \to I^Y$ be a fuzzy θS-irresolute function on a fts (X, τ) onto a fts (Y, Δ). Then (Y, Δ) is FT_2 iff $S(f, x_\alpha)$ is degenerate for each $x_\alpha \in FP(X)$.*

We have just seen that degeneracy of the S-cluster fuzzy set of an arbitrary fuzzy function is a sufficient condition for the fuzzy Hausdorffness of the codomain space. We thus like to examine some other situations when the S-cluster fuzzy sets are degenerate, thereby ensuring the fuzzy Hausdorffness of the codomain space of the fuzzy function concerned. To this end, we recall the following definition.

Definition 2.8.([9]) A fts (X, τ) is called fuzzy almost regular (FAR_2, for short) iff $(\forall x_\alpha \in FP(X))(\forall \lambda \in RCF(X, \tau))(x_\varepsilon \notin \lambda)(\exists \eta \in N^Q_{x_\alpha})(\exists \rho \in N_\lambda)(\eta \bar{q} \rho)$, where $RCF(X, \tau)$ denotes the class of all regular closed fuzzy sets in (X, τ).

Theorem 2.9.([9]) *In any FAR_2-space (X, τ), $\theta.c\ell(\mu)$ is θ-closed fuzzy set for each $\mu \in I^X$.*

Definition 2.10.([7]) A funciton $f : I^X \to I^Y$ is called fuzzy θ-closed if the image of each θ-closed fuzzy set of a fts (X, τ) is a θ-closed fuzzy set of a fts (Y, Δ).

Theorem 2.11. *Let $f : I^X \to I^Y$ be a fuzzy θ-closed function from a FAR_2-space (X, τ) into a fts (Y, Δ). If $f^{-1}(y_v)$ is θ-closed in (X, τ) for all $y_v \in FP(Y)$, then $S(f, x_\alpha)$ is degenerate for each $x_\alpha \in FP(X)$.*

Proof. Since, $c\ell(\mu) \leq \theta.c\ell(\mu)$ for each $\mu \in I^X$, then $S(f, x_\alpha) = \wedge\{\theta.c\ell(f(c\ell(\mu))) : \mu \in SON^Q_{x_\alpha}\} \leq \wedge\{\theta.c\ell(f(\theta.c\ell(\mu))) : \mu \in SON^Q_{x_\alpha}\}$. As (X, τ) is FAR_2, $\theta.c\ell(\mu)$ is θ-closed for all $\mu \in SON^Q_{x_\alpha}$. Since f is a fuzzy θ-closed function, $\theta.c\ell(f(\theta.c\ell(\mu))) = f(\theta.c\ell(\mu))$ for each $\mu \in SON^Q_{x_\alpha}$. Thus $S(f, x_\alpha) \leq \wedge\{f(\theta.c\ell(\mu)) : \mu \in SON^Q_{x_\alpha}\}$. Now, let $y_v \in FP(Y)$ such that $y \neq f(x)$. Then since $f^{-1}(y_v)$ is θ-closed and $x_\alpha \notin f^{-1}(y_v)$, there is some $\rho \in N^Q_{x_\alpha}$ such that $c\ell(\rho) \bar{q} f^{-1}(y_v)$. So, $y_v \notin f(c\ell(\rho)) = f(\theta.c\ell(\rho))$ (as ρ is an open fuzzy set) and hence $y_v \notin \wedge\{f(\theta.c\ell(\mu)) : \mu \in SON^Q_{x_\alpha}\}$. Thus, we conclude that $y_v \notin S(f, x_\alpha)$, which proves that $S(f, x_\alpha)$ is degenerate.

Theorem 2.12. *Let $f : I^X \to I^Y$ be a fuzzy θ-closed injection funciton, where (X,τ) is a FAR_2 and FT_2-space. Then $S(f,x_\alpha)$ is degenerate for each $x_\alpha \in FP(X)$.*

Proof. Since the fts (X,τ) is FAR_2 and the mapping f is a fuzzy θ-closed, we have $\theta.c\ell(f(\theta.c\ell(\mu))) = f(\theta.c\ell(\mu))$ for any $\mu \in SON^Q_{x_\alpha}$ and, hence

$$\begin{aligned} S(f,x_\alpha) &= \wedge\{\theta.c\ell(f(c\ell(\mu))) : \mu \in SON^Q_{x_\alpha}\} \\ &\le \wedge\{\theta.c\ell(f(\theta.c\ell(\mu))) : \mu \in SON^Q_{x_\alpha}\} \\ &= \wedge\{f(\theta.c\ell(\mu)) : \mu \in SON^Q_{x_\alpha}\}. \qquad (*) \end{aligned}$$

For each $x^1_\alpha \in FP(X)$ with $x \neq x^1,\ f(x_\alpha) \neq f(x^1_\alpha)$ as f is injective. By the fuzzy Hausdorffness of (X,τ), there are $\mu \in N^Q_{x_\alpha}$ and $\eta \in N^Q_{x^1_\alpha}$ such that $\mu\bar{q}\eta$. Obviously, $\mu\bar{q}c\ell(\eta)$. So $x^1_\alpha \notin \theta.c\ell(\mu)$ and hence $f(x^1_\alpha) \notin f(\theta.c\ell(\mu))$. Since $N^Q_{x_\alpha} \subseteq SON^Q_{x_\alpha}$, then $\mu \in SON^Q_{x_\alpha}$ and hence by equation $(*)$, we have $f(x^1_\alpha) \notin S(f,x_\alpha)$. Thus, $S(f,x_\alpha)$ is degenerate for each $x_\alpha \in FP(X)$.

Definition 2.13.([9]) A fts (X,τ) is called fuzzy regular (FR_2, for short) iff $(\forall x_\alpha \in FP(X))(\forall\lambda \in \tau')(x_\alpha \notin \lambda)(\exists\eta \in N^Q_{x_\alpha})(\exists\rho \in N_\lambda)(\eta\bar{q}\rho)$, where τ' represents the class of all closed fuzzy sets in (X,τ).

Now, Theorem 2.12., is equivalent to the following apparently weaker result when (X,τ) is FR_2.

Theorem 2.14. *If $f : I^X \to I^Y$ is a fuzzy θ-closed injection, where (X,τ) is a FR_2-space, then $S(f,x_\alpha)$ is degenerate for each $x_\alpha \in FP(X)$.*

Proof. It is known that in a FR_2-space (X,τ), $\theta.c\ell(\mu) = c\ell(\mu)$ for any $\mu \in I^X$. Since (X,τ) is FT_3 and f is a fuzzy θ-colsed injection, $\{f(x_\alpha)\} \le S(f,x_\alpha) = \wedge\{f(c\ell(\mu)) : \mu \in SON^Q_{x_\alpha}\} \le \wedge\{f(c\ell(\mu)) : \mu \in N^Q_{x_\alpha}\} = \{f(x_\alpha)\}$. Thus $S(f,x_\alpha) = \{f(x_\alpha)\}$.

Note that the above result is indeed equivalent to that of Theorem 2.9 follows from the following considerations: For any fuzzy set $\mu \in I^X$ in a fts (X,τ), θ-closure of μ in (X,τ) is the same as that in (X,τ_S), where (X,τ_S) denotes the fuzzy semiregularization space [9] of (X,τ). Moreover, it is known [9] that (X,τ) is FT_2 (FAR_2) iff (X,τ_S) is FT_2 (FR_2). Now, since $SO(X,\tau_S) \le SO(X,\tau)$, it follows that $S(f,x_\alpha) = S(f : (X,\tau) \to Y, x_\alpha) \le S(f : (X,\tau_S) \to Y, x_\alpha)$. So, $S(f,x_\alpha)$ is degenerate for each $x_\alpha \in FP(X)$ if (X,τ) is an FAR_2 and FT_2 space and $f : I^X \to I^Y$ is a nonempty fuzzy θ-closed injection.

3. S-cluster Sets of Fuzzy Multifunctions and SQ_α-closedness

Definition 3.1.([10]) Let (X,T) be a topological space in the classical sense and (Y,Δ) be a fts. A map $F : X \to I^Y$ is called a fuzzy multifunction iff for each $x \in X,\ F(x)$ is a nonempty fuzzy set in Y.

In the following, unless otherewise is stated, by $F : X \to I^Y$ we will mean that F is a fuzzy multifunction from a classical topological space (X,T) to a fts (Y,Δ). Let

(X, T) be a classical topological space, $x \in X$ and $A \subseteq X$, by N_x (resp. N_A) and SON_x (resp. SON_A), we mean, the open neighbourhood system of x (resp. of A) and the semi-open neighbourhood system of x (resp. of A).

Definition 3.2.([10]) For a fuzzy multifunction $F : X \to I^Y$, the lower inverse $F^-(\eta)$ of a fuzzy set η in Y is defined as: $F^-(\eta) = \{x \in X : F(x)q\eta\}$.

Definition 3.3. Let $F : X \to I^Y$ be a fuzzy multifunction and $x \in X$. Then the S-cluster fuzzy set of F at x, denoted by $S(F, x)$ is defined to be the set $\wedge\{\theta.c\ell(F(c\ell(U))) : U \in SON_x\}$.

Definition 3.4.([8]) Let (X, τ) be a fts, $\alpha \in (0, 1]$ and $\mu \in I^X$. Then

(i) The familly $\mathcal{U} = \{\eta_j : j \in J\} \subseteq SO(X, \tau)$ is called a semi-open Q_α-cover of μ iff $(\forall x \in X$ with $\mu(x) \geq \alpha)(\exists j \in J)(x_\alpha q \eta_j)$.
(ii) A subfamilly $\mathcal{U}_0$ of an Q_α-cover $\mathcal{U}$ of μ, which is also a Q_α-cover of μ, is called an Q_α-subcover of μ.
(iii) A fuzzy set μ is called SQ_α-closed if each semi-open Q_α-cover $\mathcal{U}$ of μ there exists a finite subfamily $\mathcal{U}_0$ of $\mathcal{U}$ such that $\{c\ell(\eta) : \eta \in \mathcal{U}_0\}$ is an Q_α-cover of μ.
(iv) A fts (X, τ) is called SQ_α-closed iff 1_X is SQ_α-closed.

We now turn our attention to the characterizations of SQ_α-closedness via S-cluster fuzzy sets. We need the following two lemmas for this purpose.

Lemma 3.5. *A fuzzy set μ in a fts (X, τ) is an SQ_α-closed iff for every fuzzy filterbase $\mathcal{F}$ on X such that $\lambda \wedge \eta(x) \geq \alpha$ for all $\lambda \in \mathcal{F}$ and for all $\eta \in SON_\mu^{Q_\alpha}$, then $\mu \wedge \theta S.adh(\mathcal{F})(x) \geq \alpha$ for some $x \in X$.*

Proof. Let $\mu \in I^X$ be an SQ_α-closed and $\mathcal{F}$ be a fuzzy filterbase on X and assume that $\mu \wedge \theta S.adh(\mathcal{F})(x) < \alpha$ for each $x \in X$. Then, for all $x_\alpha \in \mu$, we have $x_\alpha \notin \theta S.adh(\mathcal{F})$ and so $(\exists \eta_{x_\alpha} \in SON_{x_\alpha}^Q)(\exists \lambda_{x_\alpha} \in \mathcal{F})(c\ell(\eta_{x_\alpha})\bar{q}\lambda_{x_\alpha})$. The family $\mathcal{U} = \{\eta_{x_\alpha} : x_\alpha \in \mu\}$ is a semi-open Q_α-cover of μ. By the SQ_α-closedness of μ, there exists a finite subset μ^* of μ such that the family $\mathcal{U}_0 = \{c\ell(\eta_{x_\alpha}) : X_\alpha \in \mu^*\}$ is an Q_α-cover of μ. Choose $\lambda \in \mathcal{F}$ with $\lambda \leq \wedge\{\lambda_{x_\alpha} \in \mathcal{F} : x_\alpha \in \mu^*\}$. Put $\eta = \bigvee\{\eta_{x_\alpha} \in SON_{x_\alpha}^Q : x_\alpha \in \mu^*\}$. Then $\eta \in SON_{x_\alpha}^Q$ and $c\ell(\eta)\bar{q}\lambda$. Since $x_\alpha \in \mu$, $x_\alpha q \eta$ and $c\ell(\eta)\bar{q}\lambda$, then $\lambda(x) \leq 1_X \setminus c\ell(\eta)(x) < \alpha \leq \mu(x)$ for each $x \in X$. Hence $(\mu \wedge \lambda)(x) < \alpha$ for each $x \in X$, a contradiction. Conversely, suppose that μ is not SQ_α-closed. Then there exists a semi-open Q_α-cover $\mathcal{U} = \{\eta_j : j \in J\}$ of μ such that for every finite subset J_0 of J, the family $\mathcal{U}_0 = \{c\ell(\eta_j) : j \in J_0\}$ is not Q_α-cover of μ. Then, there exists $x_\alpha \in \mu$ such that for all $c\ell(\eta_j) \in \mathcal{U}_0\}$ we have $x_\alpha \bar{q} c\ell(\eta_j)$ and so $x_\alpha \bar{q} \bigvee_{j \in J_0} c\ell(\eta_j)$. Hence $x_\alpha \in \bigwedge_{j \in J_0} (1_X \setminus c\ell(\eta_j))$ and so $\bigwedge_{j \in J_0} ((1_X \setminus c\ell(\eta_j)) \wedge \mu)(x) \geq \alpha$ for some $x \in X$. So, $\mathcal{F} = \{\mu \wedge (1_X \setminus c\ell(\eta_j)) : j \in J_0\}$ is a fuzzy filterbase on μ. By hypothesis, we have $x_\alpha \in \mu \wedge \theta S.adh(\mathcal{F})$. Assume, $\eta_j \in SON_{x_\alpha}^Q$ and let $J_0 = \{j\}$. Since, $x_\alpha \in \theta S.adh(\mathcal{F})$, then $c\ell(\eta_j)q(\mu \wedge (1_X \setminus \eta_j))$ and so $c\ell(\eta_j)q(1_X) \setminus c\ell(\eta_j))$ which is impossible.

Lemma 3.6.([8]) *For any fts* (X, τ) *and* $\alpha \in (0, 1]$ *we have:*

(i) *A fts* (X, τ) *is* SQ_α*-closed iff every fuzzy filterbase* θS*-adheres in* X.
(ii) *Any* θ*-semiclosed fuzzy set of an* SQ_α*-closed space is* SQ_α*-closed.*

Definition 3.7. For a fuzzy multifunction $F : X \to I^Y$ and a subset A of X, the notation $S(F, A)$ stands for the set $\vee\{S(F, x) : x \in A\}$.

Theorem 3.8. *For any topological space* (X, T) *and* $\alpha \in (0, 1]$, *the following statements are equivalent:*

(i) 1_X *is* SQ_α*-closed.*
(ii) $S(F, A) \supseteq \wedge\{\theta.c\ell(F(U)) : U \in SON_A\}$ *for each* θ*-semiclosed subset* A *of* X, *for each fts* (Y, Δ) *and each fuzzy multifunction* $F : X \to I^Y$.
(iii) $S(F, A) \supseteq \wedge\{\theta S.c\ell(F(U)) : U \in SON_A\}$ *for each* θ*-semiclosed subset* A *of* X, *for each fts* (Y, Δ) *and each fuzzy multifunction* $F : X \to I^Y$.

Proof. (i) $\Longrightarrow$ (ii): Let A be any θ-semiclosed subset of X. Since, 1_X is SQ_α-closed, then by Lemma 3.6(ii), 1_A is SQ_α-closed. Now, let $z_\alpha \in \wedge\{\theta.c\ell(F(W)) : W \in SON_A\}$. Then for all $\eta \in N^Q_{z_\alpha}$ and for each $U \in SON_A$, $c\ell(\eta)qF(U)$ and so $F^-(c\ell(\eta)) \cap U \neq \emptyset$. Thus $\mathcal{F} = \{F^-(c\ell(\eta)) : \eta \in N^Q_{z_\alpha}\}$ is clearly a fuzzy filterbase on X, satisfying the condition of Lemma 3.5. Hence $(1_A \wedge \theta S.adh(\mathcal{F}))(x) \geq \alpha$. Then $x \in A$, and for all $U \in SON_x$ and each $\eta \in N^Q_{z_\alpha}$, $c\ell(1_U)qF^-(c\ell(\eta))$, i.e., $F(c\ell(U))qc\ell(\eta)$ and so $z_\alpha \in S(F, x) \subseteq S(F, A)$.

(ii) $\Longrightarrow$ (iii): Obvious.

(iii) $\Longrightarrow$ (i): In order to show that 1_X is SQ_α-closed, it is enough to prove, by virtue of Lemma 3.6(i), that every fuzzy filterbase $\mathcal{F}$ on X θS-adheres at some $x_\alpha \in FP(X)$. Let $\mathcal{F}$ be a fuzzy filterbase on X. Take $y^0 \notin X$, and construct $Y = X \cup \{y^0\}$. Define $\Delta = \{\eta \in I^Y : y^0_\alpha \bar{q}\eta\} \cup \{\eta \in I^Y : y^0_\alpha q\eta,\ \lambda \leq \eta \text{ for some } \lambda \in \mathcal{F}\}$. By Theorem 1.1, it is easy to verify that Δ is a fuzzy topoloty on Y. Consider the function $\Psi : X \to Y$ by $\Psi(x) = x$. In order to avoid possible confusion, let us denote the closure and θS-closure of a fuzzy set μ in $X(Y)$, respectively, by $X.c\ell(\mu)(Y.c\ell(\mu))$ and $X.\theta S.c\ell(\mu)(Y.\theta S.c\ell(\mu))$. As X is θ-semiclosed in X, by the given codition, $S(\Psi, x) \supseteq \wedge\{Y.\theta S.c\ell(\Psi(U)) : U \in SON_X\} = \wedge\{Y.\theta S.c\ell(U) : U \in SON_X\} = Y.\theta S.c\ell(X)$. We consider $y^0_\alpha \in FP(Y)$ and $\rho_0 \in SON^Q_{y_\alpha}$. There is some $\eta \in \Delta$ such that $\eta \leq \rho_0 \leq Y.c\ell(\eta)$. If $y^0_\alpha \bar{q}\eta$, then $\eta \leq 1_X$ and hence $Y.c\ell(\eta)q1_X$. If on the other hand, $y^0_\alpha q\eta$, then there is some $\lambda \in \mathcal{F}$ such that $\lambda \leq \eta$ and hence $Y.c\ell(\lambda) \leq Y.c\ell(\eta)$. So $1_X qY.c\ell(\eta)$. So, in any case, $1_X qY.c\ell(\eta)$ and, consequently, as $Y.c\ell(\eta) = Y.c\ell(\rho_0)$, $1_X qY.c\ell(\rho_0)$. Thus $y^0_\alpha \in Y.\theta S.c\ell(X)$. So, $y^0_\alpha \in S(\Psi, x)$ for some $x \in X$. Consider any $V \in SON_x$ and $\lambda \in \mathcal{F}$. Then $\lambda \vee \{y^0_\alpha\} \in \Delta$ and $y^0_\alpha qY \setminus (\lambda \wedge \{y^0_\alpha\})$. Thus $Y \setminus (\lambda \vee \{y^0_\alpha\}) \in \Delta$, which proves that $Y.c\ell(\lambda \vee \{y^0_\alpha\}) = \lambda \vee \{y^0_\alpha\}$. Now, since $Y.c\ell(\lambda \vee y^0_\alpha)q\Psi(X.c\ell(V))$, then $X.c\ell(V)q\lambda \vee y^0_\alpha$ which implies that $X.c\ell(V)q\lambda$. Thus, $x_\alpha \in \theta S.adh(\mathcal{F})$.

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